Journal of Mathematical Inequalities
Volume 5, Number 1 (2011), 13–20

GENERALIZATIONS AND REFINEMENTS FOR NESBITT'S INEQUALITY

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(*Communicated by J. Pečarić*)

Abstract. In this paper we present a new generalizations and refinements for Nesbitt's inequality (see [5]). In every section, we give example of inequalities by particularization.

1. Introduction

We consider the set $\mathbb{N} = \{1,2,\dots\}$. In this paper we prove a general inequalities. By particularization we obtain the Nesbitt's inequality and refinements of this inequality.

2. A general inequality obtained by using the inequality of convex functions

THEOREM 2.1. *If* $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha \in (-\infty,0] \cup [1,\infty)$, $x_k > 0$, $p_k \in [0,1]$, $k \in \{1,2,\dots,n\}$ *such that* $\sum\limits_{k=1}^{n} p_k = 1$, *then*

$$\frac{\left(\sum\limits_{k=1}^{n} p_k x_k\right)^{\alpha}}{\sum\limits_{k=1}^{n} p_k(x_1+x_2+\dots+x_{k-1}+x_{k+1}+\dots+x_n)} \leqslant \sum_{k=1}^{n} \frac{p_k x_k^{\alpha}}{x_1+x_2+\dots+x_{k-1}+x_{k+1}+\dots+x_n}. \tag{2.1}$$

Proof. Let $f:(0,1)\to\mathbb{R}$ be a function defined by $f(x) = \frac{x^{\alpha}}{1-x}$ for any $x \in (0,1)$. By calculus we obtain that

$$f''(x) = \frac{x^{\alpha-2}((\alpha-1)(\alpha-2)x^2 - 2\alpha(\alpha-2)x + \alpha(\alpha-1))}{(1-x)^3},$$

for any $x \in (0,1)$ and let $g_{\alpha}:(0,1)\to\mathbb{R}$ be a function defined by $g_{\alpha}(x) = (\alpha-1)(\alpha-2)x^2 - 2\alpha(\alpha-2)x + \alpha(\alpha-1)$ for any $x \in (0,1)$. On verifies immediately that

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Nesbitt's inequality.

Paper JMI-05-02

for $\alpha \in \{0,1,2\}$ we have $g_\alpha(x) > 0$ for any $x \in (0,1)$. If $\alpha < 0$, then the function g_α have a minimum in the point $x_v = \frac{\alpha}{\alpha-1} \in (0,1)$ and $g_\alpha(x_v) = \frac{\alpha}{\alpha-1} > 0$, so $g_\alpha(x) > 0$ for any $x \in (0,1)$. If $\alpha \in (1,2)$, then g_α is an increasing function on $(0,1)$, so $g(x) \geqslant \lim\limits_{\substack{x\to 0\\ x>0}} g_\alpha(x) = \alpha(\alpha-1) > 0$ for any $x \in (0,1)$. If $\alpha > 2$, then g_α is a decreasing function on $(0,1)$, so $g_\alpha(x) \geqslant \lim\limits_{\substack{x\to 1\\ x<1}} g_\alpha(x) = 2 > 0$ for any $x \in (0,1)$. From the remarks above it results that $g_\alpha(x) > 0$ for any $x \in (0,1)$, so $f''(x) > 0$ for $x \in (0,1)$. Then f is a convex function on $(0,1)$ and the inequality $f\left(\sum\limits_{k=1}^{n} p_k x_k\right) \leqslant \sum\limits_{k=1}^{n} p_k f(x_k)$ holds. Choosing x_k by $\frac{x_k}{x_1+x_2+\ldots+x_n}$, $k \in \{1,2,\ldots,n\}$, we obtain the inequality (2.1). □

COROLLARY 2.1. *If $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha \in (-\infty,0] \cup [1,\infty)$, $x_k > 0$, $k \in \{1,2,\ldots,n\}$, then*

$$\frac{n^{2-\alpha}}{n-1}\left(\sum_{k=1}^{n} x_k\right)^{\alpha-1} \leqslant \sum_{k=1}^{n} \frac{x_k^\alpha}{x_1+x_2+\ldots+x_{k-1}+x_{k+1}+\ldots+x_k}. \tag{2.2}$$

Proof. In Theorem 2.1 we take $p_1 = p_2 = \ldots = p_n = \frac{1}{n}$. □

COROLLARY 2.2. *If $n \in \mathbb{N}$, $x_k > 0$, $k \in \{1,2,\ldots,n\}$, then*

$$\frac{n}{n-1} \leqslant \sum_{k=1}^{n} \frac{x_k}{x_1+x_2+\ldots+x_{k-1}+x_{k+1}+\ldots+x_n}. \tag{2.3}$$

Proof. In Corollary 2.1 we take $\alpha = 1$. □

REMARK 2.1. For $n = 3$ in Corollary 2.2, we obtain the "classical" Nesbitt's inequality

$$\frac{3}{2} \leqslant \frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_1} + \frac{x_3}{x_1+x_2} \tag{2.4}$$

for any $x_1, x_2, x_3 > 0$, so the results from Theorem 2.1, Corollary 2.1 and Corollary 2.2 are the generalizations of Nesbitt's inequality.

3. The refinement of Nesbitt's inequality and applications

THEOREM 3.1. *If $x,y,z > 0$ and we note $m = \min A$, $M = \max A$, where*

$$A = \left\{ \frac{x}{y+z} + \frac{2(y+z)}{2x+y+z}, \frac{y}{z+x} + \frac{2(z+x)}{2y+z+x}, \frac{z}{x+y} + \frac{2(x+y)}{2z+x+y} \right\},$$

then

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geqslant M \geqslant m \geqslant \frac{3}{2}. \tag{3.1}$$

Proof. We prove that $\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$ is last great or equal then any element of A and any element of A is last great or equal then $\frac{3}{2}$. By example the inequality $\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geqslant\frac{x}{y+z}+\frac{2(y+z)}{2x+y+z}$ is equivalent after calculus with the following true inequality $(y-z)^2(x+y+z)\geqslant 0$. The inequality $\frac{x}{y+z}+\frac{2(y+z)}{2x+y+z}\geqslant\frac{3}{2}$ is equivalent with $(2x-y-z)^2\geqslant 0$, which is a true inequality. □

REMARK 3.1. The inequalities from (3.1) are the refinement of Nesbitt's inequality.

THEOREM 3.2. *If $x,y,z>0$, then*

$$\frac{3}{2}+2\sum_{cyclic}\left(\frac{x-y}{x+y+2z}\right)^2\leqslant\sum_{cyclic}\frac{x}{y+z}\leqslant\frac{3}{2}+\frac{1}{8}\sum_{cyclic}\frac{(x-y)^2}{z\sqrt{xy}}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}E&=\sum_{cyclic}\left(\frac{x}{y+z}-\frac{1}{2}\right)\\&=\sum_{cyclic}\left(\frac{x-y}{2(y+z)}+\frac{x-z}{2(z+y)}\right)\\&=\left(\frac{x-y}{2(y+z)}+\frac{y-x}{2(x+z)}\right)+\left(\frac{y-z}{2(z+x)}+\frac{z-y}{2(y+x)}\right)\\&\quad+\left(\frac{z-x}{2(x+y)}+\frac{x-z}{2(z+y)}\right)\\&=\sum_{cyclic}\frac{(x-y)^2}{2(z+x)(z+y)}.\end{aligned}$$

But $2(z+x)(z+y)\geqslant 2\cdot 2\sqrt{zx}\cdot 2\sqrt{zy}=8z\sqrt{xy}$, $2(z+x)(z+y)\leqslant 2\left(\frac{(z+x)+(z+y)}{2}\right)^2=\frac{(x+y+2z)^2}{2}$ and then, from remarks above it results the inequalities from (3.2). □

REMARK 3.2. The inequalities from (3.2) are the refinement of Nesbitt's inequality.

Let the triangle ABC with the sides $AB=c$, $BC=a$, $CA=b$, the measure of angles A,B,C, s the semiperimeter, R the circumradius, r the inradius and T the area.

COROLLARY 3.1. *The following inequalities are true*

$$\begin{aligned}\frac{3}{2}+2\sum_{cyclic}\left(\frac{a-b}{a+b+2c}\right)^2&\leqslant\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}\\&\leqslant\frac{3}{2}+\frac{1}{8}\sum_{cyclic}\frac{(a-b)^2}{c\sqrt{ab}}\end{aligned} \tag{3.3}$$

and

$$\frac{3}{2}+2\sum_{cyclic}\left(\frac{a-b}{a+b}\right)^2 \leqslant \frac{s^2+r^2-8Rr}{4Rr} \leqslant \frac{3}{2}+\frac{1}{8T}\sum_{cyclic}\frac{(a-b)^2}{\sqrt{s-c}}. \qquad (3.4)$$

Proof. In Theorem 3.2 we consider respectively $(x,y,z)\in\{(a,b,c),\ (s-a,s-b,s-c)\}$ and taking into account that $\sum\limits_{cyclic}\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}$. □

4. A general inequality obtained by using Chebyshev's inequality

THEOREM 4.1. *If $n\in\mathbb{N}$, $\alpha\geqslant 0$, $a,b,c\in\mathbb{R}$ such that $an+c-b>0$, $x_k>0$ and $\left(a+\frac{c}{n}\right)\sum\limits_{k=1}^{n}x_k-bx_k>0$ for any $k\in\{1,2,\ldots,n\}$, then*

$$\sum_{k=1}^{n}\frac{x_k^{\alpha+1}}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_k}\geqslant\frac{1}{n}\left(\sum_{k=1}^{n}x_k^{\alpha+1}\right)\left(\sum_{k=1}^{n}\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_k}\right)$$
$$\geqslant\frac{n^{1-\alpha}\left(\sum\limits_{k=1}^{n}x_k\right)^{\alpha}}{an+c-b}. \qquad (4.1)$$

Proof. We suppose that $x_1\leqslant x_2\leqslant\ldots\leqslant x_n$. Then $x_1^{\alpha+1}\leqslant x_2^{\alpha+1}\leqslant\ldots\leqslant x_n^{\alpha+1}$ and

$$\sum_{k=1}^{n}\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_1}\leqslant\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_2}\leqslant\ldots$$
$$\leqslant\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_n}.$$

Using Chebyshev's inequality we have

$$\sum_{k=1}^{n}\frac{x_k^{\alpha+1}}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_k}\geqslant\frac{1}{n}\left(\sum_{k=1}^{n}x_k^{\alpha+1}\right)\left(\sum_{k=1}^{n}\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_k}\right),$$

so we obtain the first inequality from (4.1). Applying Jensen's inequality we have $\frac{1}{n}\sum\limits_{k=1}^{n}x_k^{\alpha+1}\geqslant\left(\frac{\sum\limits_{k=1}^{n}x_k}{n}\right)^{\alpha+1}$ and applying Cauchy-Schwarz's inequality we have $\sum\limits_{k=1}^{n}\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_k}\geqslant\frac{n^2}{\sum\limits_{k=1}^{n}\left(\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n}x_k\right)-bx_k\right)}=\frac{n^2}{(an+c-b)\sum\limits_{k=1}^{n}x_k}$.

Then $\frac{1}{n}\left(\sum\limits_{k=1}^{n} x_k^{\alpha+1}\right)\sum\limits_{k=1}^{n}\frac{1}{\left(a+\frac{c}{n}\right)\left(\sum\limits_{k=1}^{n} x_k\right)-bx_k} \geqslant \left(\frac{\sum\limits_{k=1}^{n} x_k}{n}\right)^{\alpha+1} \frac{n^2}{(a+c-b)\sum\limits_{k=1}^{n} x_k}$ from where, the second inequality from (4.1) results. □

COROLLARY 4.1. *If $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha \geqslant 0$ and $x_k > 0$, $k \in \{1,2,\ldots,n\}$, then*

$$\sum_{k=1}^{n}\frac{x_k^{\alpha+1}}{x_1+x_2+\ldots+x_{k-1}+x_{k+1}+\ldots+x_n}$$
$$\geqslant \frac{1}{n}\left(\sum_{k=1}^{n} x_k^{\alpha+1}\right)\left(\sum_{k=1}^{n}\frac{1}{x_1+x_2+\ldots+x_{k-1}+x_{k+1}+\ldots+x_n}\right)$$
$$\geqslant \frac{n^{1-\alpha}}{n-1}\left(\sum_{k=1}^{n} x_k\right)^{\alpha}. \tag{4.2}$$

Proof. For $a=b=1$ and $c=0$ in Theorem 4.1, the inequality (4.2) results. □

REMARK 4.1. The inequality from (4.2) is a refinement of inequality (2.2), so the inequality (4.1) is a generalization and a refinement of Nesbitt's inequality.

COROLLARY 4.2. *If $x,y,z>0$, then*

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geqslant \frac{1}{3}(x+y+z)\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right) \geqslant \frac{3}{2}. \tag{4.3}$$

Proof. We take $n=3$ and $\alpha=0$ in Corollary 4.1. □

REMARK 4.2. The inequality (4.3) is a refinement of Nesbitt's inequality.

COROLLARY 4.3. *The following inequalities are true*

$$2 > \frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr} \geqslant \frac{5s^2+r^2+4Rr}{3(s^2+r^2+2Rr)} \geqslant \frac{3}{2} \tag{4.4}$$

and

$$\frac{s^2+r^2-8Rr}{4Rr} \geqslant \frac{s^2+r^2+4Rr}{12Rr} \geqslant \frac{3}{2}. \tag{4.5}$$

Proof. From $a<b+c$, it results that $a+b+c<2(b+c)$, equivalent with $\frac{a}{b+c} < \frac{2a}{a+b+c}$ and then $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2$.

On the other hand, in Corollary 4.2 we consider respectively $(x,y,z)\in\{(a,b,c),(s-a,s-b,s-c)\}$. □

5. "Linear" inequalities

THEOREM 5.1. *If* $\alpha_1 \in \mathbb{R}$, $\alpha_k \geqslant 0$, $k \in \{2,3,\ldots,n\}$ *and* $y_k > 0$, $k \in \{1,2,\ldots,n\}$, *then*

$$\frac{\alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n}{y_1} + \frac{\alpha_1 y_2 + \alpha_2 y_3 + \ldots + \alpha_n y_1}{y_2} + \ldots + \frac{\alpha_1 y_n + \alpha_2 y_1 + \ldots + \alpha_n y_{n-1}}{y_n} \geqslant n(\alpha_1 + \alpha_2 + \ldots + \alpha_n). \tag{5.1}$$

Proof. We have

$$\begin{aligned} &\frac{\alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n}{y_1} + \frac{\alpha_1 y_2 + \alpha_2 y_3 + \ldots + \alpha_n y_1}{y_2} + \ldots \\ &+ \frac{\alpha_1 y_n + \alpha_2 y_1 + \ldots + \alpha_n y_{n-1}}{y_n} \\ &= n\alpha_1 + \left(\frac{y_2}{y_1} + \frac{y_3}{y_2} + \ldots + \frac{y_1}{y_n}\right)\alpha_2 + \left(\frac{y_3}{y_1} + \frac{y_4}{y_2} + \ldots + \frac{y_2}{y_n}\right)\alpha_3 + \ldots \\ &+ \left(\frac{y_n}{y_1} + \frac{y_1}{y_2} + \ldots + \frac{y_{n-1}}{y_n}\right)\alpha_n \end{aligned}$$

and because $\frac{y_2}{y_1} + \frac{y_3}{y_2} + \ldots + \frac{y_1}{y_n} \geqslant n$, $\frac{y_3}{y_1} + \frac{y_4}{y_2} + \ldots + \frac{y_2}{y_n} \geqslant n, \ldots, \frac{y_n}{y_1} + \frac{y_1}{y_2} + \ldots + \frac{y_{n-1}}{y_n} \geqslant n$ the inequality from (5.1) results. □

REMARK 5.1. In Theorem 5.1 we take $y_k = x_1 + x_2 + \ldots + x_{k-1} + x_{k+1} + \ldots + x_n$, where $x_k > 0$, $k \in \{1,2,\ldots,n\}$ and $n \in \mathbb{N}$, $n \geqslant 2$. Then we have

$$\sum_{cyclic} \frac{(\alpha_2+\alpha_3+\ldots+\alpha_n)x_1 + (\alpha_1+\alpha_3+\ldots+\alpha_n)x_2 + \ldots + (\alpha_1+\alpha_2+\ldots+\alpha_{n-1})x_n}{x_2 + x_3 + \ldots + x_n} \geqslant n(\alpha_1 + \alpha_2 + \ldots + \alpha_n).$$

Now, we put $\alpha_2 + \alpha_3 + \ldots + \alpha_n = 1$, $\alpha_1 + \alpha_3 + \ldots + \alpha_n = 0, \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = 0$, from where $\alpha_1 = \frac{2-n}{n-1}$, $\alpha_2 = \alpha_3 = \ldots = \alpha_n = \frac{1}{n-1}$. Then, from the inequality above we obtain the inequality (2.3), so the inequality (5.1) is a generalization a Nesbitt's inequality.

THEOREM 5.2. *If* $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha_1 \in \mathbb{R}$, $\alpha_k \geqslant 0$, $k \in \{2,3,\ldots,n\}$, $x_k, b_k \in \mathbb{R}$, $y_k = \beta_1 x_k + \beta_2 x_{k+1} + \beta_3 x_{k+2} + \ldots + \beta_{n-k+1} x_n + \beta_{n-k+2} x_1 + \beta_{n-k+3} x_2 + \ldots + \beta_n x_{k-1} > 0$, *where* $k \in \{1,2,\ldots,n\}$ *and* $\gamma_1 = \alpha_1\beta_1 + \alpha_2\beta_n + \alpha_3\beta_{n-1} + \ldots + \alpha_n\beta_2$, $\gamma_2 = \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_n + \ldots + \alpha_n\beta_3, \ldots, \gamma_n = \alpha_1\beta_n + \alpha_2\beta_{n-1} + \alpha_3\beta_{n-2} + \ldots + \alpha_n\beta_1$, *then*

$$\sum_{k=1}^{n} \frac{\gamma_1 x_k + \gamma_2 x_{k+1} + \ldots + \gamma_n x_{k-1}}{y_k} \geqslant n(\alpha_1 + \alpha_2 + \ldots + \alpha_n). \tag{5.2}$$

Proof. In Theorem 5.1 we take $y_k = \beta_1 x_k + \beta_2 x_{k+1} + \ldots + \beta_n x_{k-1}$, $k \in \{1,2,\ldots,n\}$. □

In the following we give an example.

COROLLARY 5.1. *If* $\gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3, x, y, z \in \mathbb{R}$ *such that* $\beta_1 x + \beta_2 y + \beta_3 z > 0,\ \beta_1 y + \beta_2 z + \beta_3 x > 0,\ \beta_1 z + \beta_2 x + \beta_3 y > 0,\ \Delta \neq 0,\ \frac{\Delta_2}{\Delta} > 0,\ \frac{\Delta_3}{\Delta} > 0$, *where* $\Delta_2 = \begin{vmatrix} \beta_1 & \gamma_1 & \beta_2 \\ \beta_2 & \gamma_2 & \beta_3 \\ \beta_3 & \gamma_3 & \beta_1 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} \beta_1 & \beta_3 & \gamma_1 \\ \beta_2 & \beta_1 & \gamma_2 \\ \beta_3 & \beta_2 & \gamma_3 \end{vmatrix}$ *and* $\Delta = \begin{vmatrix} \beta_1 & \beta_3 & \beta_2 \\ \beta_2 & \beta_1 & \beta_3 \\ \beta_3 & \beta_2 & \beta_1 \end{vmatrix}$, *then*

$$\frac{\gamma_1 x + \gamma_2 y + \gamma_3 z}{\beta_1 x + \beta_2 y + \beta_3 z} + \frac{\gamma_1 y + \gamma_2 z + \gamma_3 x}{\beta_1 y + \beta_2 z + \beta_3 x} + \frac{\gamma_1 z + \gamma_2 x + \gamma_3 y}{\beta_1 z + \beta_2 x + \beta_3 y} \geqslant \frac{3(\gamma_1 + \gamma_2 + \gamma_3)}{\beta_1 + \beta_2 + \beta_3}. \tag{5.3}$$

Proof. Because $\Delta = \beta_1^3 + \beta_2^3 + \beta_3^3 - 3\beta_1\beta_2\beta_3 = (\beta_1 + \beta_2 + \beta_3)(\beta_1^2 + \beta_3^2 + \beta_3^2 - \beta_1\beta_2 - \beta_2\beta_3 - \beta_3\beta_1)$ and $\Delta \neq 0$, it results that $\beta_1 + \beta_2 + \beta_3 \neq 0$ or $\beta_1^2 + \beta_2^2 + \beta_3^2 - \beta_1\beta_2 - \beta_2\beta_3 - \beta_3\beta_1 \neq 0$. From the last relation, it results that $\beta_1, \beta_2, \beta_3$ cannot be simultaneously equal.

In Theorem 5.2 we consider $n = 3$ and we determine $\alpha_1, \alpha_2, \alpha_3$ from the system of equations $\begin{cases} \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 = \gamma_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3 = \gamma_2 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 = \gamma_3. \end{cases}$

The solution of this system is $\begin{cases} \alpha_1 = \frac{\Delta_1}{\Delta} \\ \alpha_2 = \frac{\Delta_2}{\Delta} \\ \alpha_3 = \frac{\Delta_3}{\Delta} \end{cases}$ and summing the equations of the system we have $(\alpha_1 + \alpha_2 + \alpha_3)(\beta_1 + \beta_2 + \beta_3) = \gamma_1 + \gamma_2 + \gamma_3$, from where $\alpha_1 + \alpha_2 + \alpha_3 = \frac{\gamma_1 + \gamma_2 + \gamma_3}{\beta_1 + \beta_2 + \beta_3}$. Because the conditions from Theorem 5.2 are verified, it results that inequality (5.3) is true. □

COROLLARY 5.2. *If* a, b, c *are the sides of a triangle, then*

$$\frac{a + 2b + 3c}{-a + b + c} + \frac{b + 2c + 3a}{-b + c + a} + \frac{c + 2a + 3b}{-c + a + b} \geqslant 18. \tag{5.4}$$

Proof. We take $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 3$, $\beta_1 = -1$, $\beta_2 = \beta_3 = 1$ and $x = a$, $y = b$, $z = c$ in Corollary 5.1. □

REMARK 5.2. If $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = 0$, $\beta_1 = 0$, $\beta_2 = \beta_3 = 1$, then the conditions from Corollary 5.1 are verified. In this case the inequality (5.4) becomes the Nesbitt's inequality.

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(Received June 14, 2010)

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