

ON REFINED YOUNG INEQUALITIES AND REVERSE INEQUALITIES

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Abstract. In this paper, we show refined Young inequalities for two positive operators. Our results refine the ordering relations among the arithmetic mean, the geometric mean and the harmonic mean for two positive operators. In addition, we give two different reverse inequalities for the refined Young inequality for two positive operators.

1. Introduction

We start from the famous arithmetic-geometric mean inequality which is often called Young inequality:

$$(1 - v)a + vb \geq a^{1-v}b^v \quad (1)$$

for nonnegative real numbers a, b and $v \in [0, 1]$.

Recently, the inequality (1) was refined by F.Kittaneh and Y.Manasrah in the following form, for the purpose of the study on matrix norm inequalities.

PROPOSITION 1.1. ([1]) *For $a, b \geq 0$ and $v \in [0, 1]$, we have*

$$(1 - v)a + vb \geq a^{1-v}b^v + r(\sqrt{a} - \sqrt{b})^2, \quad (2)$$

where $r \equiv \min\{v, 1 - v\}$.

In the section 2 of this paper, we give refined Young inequalities for two positive operators based on the scalar inequality (2).

As for the reverse inequalities of the Young inequality, M.Tominaga gave the following interesting operator inequalities. He called them *converse* inequalities, however we use the term *reverse* for such inequalities, throughout this paper.

PROPOSITION 1.2. ([2]) *Let $v \in [0, 1]$, positive operators A and B such that $0 < mI \leq A, B \leq MI$ with $h \equiv \frac{M}{m} > 1$. Then we have the following inequalities for every $v \in [0, 1]$.*

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(i) (Reverse ratio inequality)

$$S(h)A\sharp_{\nu}B \geq (1-\nu)A + \nu B,$$

where the constant $S(h)$ is called Specht's ratio [3, 4] and defined by

$$S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \quad (h \neq 1)$$

for positive real number h .

(ii) (Reverse difference inequality)

$$hL(m, M) \log S(h) + A\sharp_{\nu}B \geq (1-\nu)A + \nu B, \quad (3)$$

where the logarithmic mean L is defined by

$$L(x, y) \equiv \frac{y-x}{\log y - \log x}, \quad (x \neq y) \quad L(x, x) \equiv x$$

for two positive real numbers x and y .

In the section 3 of this paper, we give reverse ratio type inequalities of the refined Young inequality for positive operators. In the section 4 of this paper, we also give reverse difference type inequalities of the refined Young inequality for positive operators.

2. Refined Young inequalities for positive operators

Let \mathcal{H} be a complex Hilbert space. We also represent the set of all bounded operators on \mathcal{H} by $B(\mathcal{H})$. If $A \in B(\mathcal{H})$ satisfies $A^* = A$, then A is called a self-adjoint operator. A self-adjoint operator A satisfies $\langle x|A|x \rangle \geq 0$ for any $|x \rangle \in \mathcal{H}$, then A is called a positive operator. For two self-adjoint operators A and B , $A \geq B$ means $A - B \geq 0$.

It is well-known that we have the following Young inequalities for invertible positive operators A and B :

$$(1-\nu)A + \nu B \geq A\sharp_{\nu}B \geq \{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1}, \quad (4)$$

where $A\sharp_{\nu}B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^{\nu}A^{1/2}$ defined for $\nu \in [0, 1]$. The power mean was originally introduced in the paper [5]. The simplified and elegant proof for the inequalities (4) was given in [6]. See also [7] for the reader having interests in operator inequalities.

As a refinement of the inequalities (4), we have the following refined Young inequality for positive operators.

THEOREM 2.1. For $v \in [0, 1]$ and positive operators A and B , we have

$$(1 - v)A + vB \geq A\sharp_v B + 2r \left(\frac{A+B}{2} - A\sharp_{1/2} B \right) \quad (5)$$

$$\geq A\sharp_v B \quad (6)$$

$$\geq \left\{ A^{-1}\sharp_v B^{-1} + 2r \left(\frac{A^{-1} + B^{-1}}{2} - A^{-1}\sharp_{1/2} B^{-1} \right) \right\}^{-1} \quad (7)$$

$$\geq \{(1 - v)A^{-1} + vB^{-1}\}^{-1} \quad (8)$$

where $r \equiv \min\{v, 1 - v\}$ and $A\sharp_v B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}$ defined for $v \in [0, 1]$.

Proof. The second inequality (6) is trivial. We prove the first inequality. From the inequality (2), we have for $v \in [0, 1]$ and $x \geq 0$

$$(1 - v + vx) - x^v - 2r \left(\frac{1+x}{2} - \sqrt{x} \right) \geq 0. \quad (9)$$

From here, we suppose that A and B are invertible. (For a general case, we consider the invertible positive operator $A_\varepsilon \equiv A + \varepsilon I$ for positive real number ε . If we take a limit $\varepsilon \rightarrow 0$, the following result also holds. Throughout this paper, we apply this continuity argument, however, from now on, we omit such descriptions for simplicity.) In general, by using the notion of the representing function $f_m(x) = mx$ for operator mean m , it is well-known [5] that $f_m(x) \leq f_n(x)$ holds for $x > 0$ if and only if $AmB \leq AnB$ holds for all positive operators A and B . Therefore the inequality (9) implies the inequality (5). Since we have $(A^{-1}\sharp_v B^{-1})^{-1} = A\sharp_v B$, we also have the third inequality (7) and the last inequality (8). \square

In the paper [8], the equivalent relation between the Young inequality and the Hölder-McCarthy inequality [9] was shown by a simplified elegant proof. Here we show a kind of the refinement of the Hölder-McCarthy inequality applying Theorem 2.1.

COROLLARY 2.2. For $v \in [0, 1]$ and any positive operator A on the Hilbert space \mathcal{H} and any unit vector $|x\rangle \in \mathcal{H}$, if $\langle x|A|x\rangle \neq 0$, then we have

$$1 - \langle x|A|x\rangle^{-v} \langle x|A^v|x\rangle \geq r \left(1 - \langle x|A|x\rangle^{-1/2} \langle x|A^{1/2}|x\rangle \right)^2, \quad (10)$$

where $r \equiv \min\{v, 1 - v\}$.

Proof. If $v = 0$, then the inequality (10) is trivial. It is sufficient that we prove it for the case of $v \in (0, 1]$. From the inequality (9), we have for any positive real number k , the unit vector $|x\rangle \in \mathcal{H}$ and the positive operator A ,

$$vk^{\frac{1}{v}} \langle x|A|x\rangle + 1 - v \geq k \langle x|A^v|x\rangle + r \left(k^{\frac{1}{2v}} \langle x|A^{1/2}|x\rangle - 1 \right)^2. \quad (11)$$

In the inequality (11), if we put $k = \langle x|A|x\rangle^{-v}$, then we obtain the inequality (10). \square

REMARK 2.3. From Hölder-McCarthy inequality [9]:

$$\langle x|A|x \rangle^v \geq \langle x|A^v|x \rangle \quad (12)$$

for any unit vector $|x\rangle \in \mathcal{H}$, if $\langle x|A|x \rangle \neq 0$, then we have

$$1 - \langle x|A|x \rangle^{-v} \langle x|A^v|x \rangle \geq 0.$$

The inequality (10) gives a refined one for the above inequality which is equivalent to the inequality (12) in the case of $\langle x|A|x \rangle \neq 0$.

3. A reverse ratio inequality for a refined Young inequality

For positive real numbers a, b and $v \in [0, 1]$, M.Tominaga showed the following inequality [2]:

$$S\left(\frac{a}{b}\right) a^{1-v} b^v \geq (1-v)a + vb, \quad (13)$$

which is called the converse ratio inequality for the Young inequality in [2]. In this section, we show the reverse ratio inequality of the refined Young inequality (2).

LEMMA 3.1. For positive real numbers a, b and $v \in [0, 1]$, we have

$$S\left(\sqrt{\frac{a}{b}}\right) a^{1-v} b^v \geq (1-v)a + vb - r(\sqrt{a} - \sqrt{b})^2, \quad (14)$$

where $r \equiv \min\{v, 1-v\}$.

Proof. It is sufficient to prove the case of $v \leq 1/2$, since we have $S(h) = S(1/h)$ for all $h > 0$ and $Am_v B = Bm_{1-v} A$ for all parametrized means m_v in this paper. We consider the following function.

$$g_b(v) \equiv \frac{vb + (1-v) - v(\sqrt{b}-1)^2}{b^v}, \quad \left(0 \leq v \leq \frac{1}{2}\right).$$

Then we have

$$g'_b(v) = \frac{2(\sqrt{b}-1) - \{2(\sqrt{b}-1)v + 1\} \log b}{b^v}$$

so that the equation $g'_b(v) = 0$ implies

$$v = \frac{1}{\log b} - \frac{1}{2(\sqrt{b}-1)} \equiv v_b.$$

From the Klein inequality:

$$1 - \frac{1}{\sqrt{b}} \leq \log \sqrt{b} \leq \sqrt{b} - 1, \quad (b > 0)$$

we have $v_b \in [0, \frac{1}{2}]$. We also find that $g'_b(v) > 0$ for $v < v_b$ and $g'_b(v) < 0$ for $v > v_b$. Thus the function $g_b(v)$ takes a maximum value when $v = v_b$, ($b \neq 1$) and it is calculated as follows.

$$\begin{aligned} \max_{0 \leq v \leq \frac{1}{2}} g_b(v) = g_b(v_b) &= \frac{2(\sqrt{b}-1) \left(\frac{1}{\log b} - \frac{1}{2(\sqrt{b}-1)} \right) + 1}{b^{\frac{1}{\log b}} b^{\frac{-1}{2(\sqrt{b}-1)}}} \\ &= \frac{\frac{2(\sqrt{b}-1)}{\log b}}{e b^{\frac{-1}{2(\sqrt{b}-1)}}} = \frac{\left(\sqrt{b} \right)^{\frac{1}{\sqrt{b}-1}}}{e \log \left(\sqrt{b} \right)^{\frac{1}{\sqrt{b}-1}}} = S(\sqrt{b}). \end{aligned}$$

Thus we have the following inequality.

$$S(\sqrt{b})b^v \geq vb + (1-v) - v(\sqrt{b}-1)^2.$$

In the case of $b = 1$, we have the equality in the above inequality, since we have $S(1) = 1$. Replacing b by $\frac{b}{a}$ and then multiplying a to both sides, we have

$$S\left(\sqrt{\frac{a}{b}}\right)a^{1-v}b^v \geq (1-v)a + vb - v(\sqrt{a}-\sqrt{b})^2,$$

since we have $S(x) = S(1/x)$ for $x > 0$. \square

REMARK 3.2. We easily find that both sides in the inequality (14) is less than or equal to those in the inequality (13) so that neither the inequality (14) nor the inequality (13) is uniformly better than the other.

In addition, our next interest moves to the ordering between $S\left(\sqrt{\frac{a}{b}}\right)a^{1-v}b^v$ and $(1-v)a + vb$. However we have no ordering between them, because we have the following examples. For example, let $a = 1$ and $b = 10$. If $v = 0.9$, then $(1-v)a + vb - S\left(\sqrt{\frac{a}{b}}\right)a^{1-v}b^v \simeq -0.246929$. And if $v = 0.6$, then $(1-v)a + vb - S\left(\sqrt{\frac{a}{b}}\right)a^{1-v}b^v \simeq 1.71544$.

Applying Lemma 3.1, we have the reverse ratio inequality of the refined Young inequality for positive operators.

THEOREM 3.3. We suppose two invertible positive operators A and B satisfy $0 < mI \leq A, B \leq MI$, where I represents an identity operator and $m, M \in \mathbb{R}$. For any $v \in [0, 1]$, we then have

$$S(\sqrt{h})A\sharp_v B \geq (1-v)A + vB - 2r \left(\frac{A+B}{2} - A\sharp_{1/2}B \right),$$

where $h \equiv \frac{M}{m} > 1$ and $r \equiv \min\{v, 1-v\}$.

Proof. In Lemma 3.1, we put $a = 1$, then we have for all $b > 0$,

$$S(\sqrt{b})b^v \geq vb + (1 - v) - r(\sqrt{b} - 1)^2$$

We consider the invertible positive operator T such that $0 < mI \leq T \leq MI$. Then we have the following inequality

$$\max_{m \leq t \leq M} S(\sqrt{t})T^v \geq vT + (1 - v) - r(T - 2T^{1/2} + 1),$$

for any $v \in [0, 1]$. We put $T = A^{-1/2}BA^{-1/2}$. Since we then have $\frac{1}{h} = \frac{m}{M} \leq A^{-1/2}BA^{-1/2} \leq \frac{M}{m} = h$, we have

$$\begin{aligned} & \max_{\frac{1}{h} \leq t \leq h} S(\sqrt{t}) \left(A^{-1/2}BA^{-1/2} \right)^v \\ & \geq vA^{-1/2}BA^{-1/2} + (1 - v) - r \left\{ A^{-1/2}BA^{-1/2} - 2 \left(A^{-1/2}BA^{-1/2} \right)^{1/2} + 1 \right\}. \end{aligned}$$

Note that $h > 1$ and $S(x)$ is monotone decreasing for $0 < x < 1$ and monotone increasing for $x > 1$ [2]. Thus we have

$$\begin{aligned} & S(\sqrt{h}) \left(A^{-1/2}BA^{-1/2} \right)^v \\ & \geq vA^{-1/2}BA^{-1/2} + (1 - v) - r \left\{ A^{-1/2}BA^{-1/2} - 2 \left(A^{-1/2}BA^{-1/2} \right)^{1/2} + 1 \right\}. \end{aligned}$$

Multiplying $A^{1/2}$ to the above inequality from both sides, we have the present theorem. \square

4. A reverse difference inequality for a refined Young inequality

For the classical Young inequality, the following reverse inequality is known. For positive real numbers a, b and $v \in [0, 1]$, M.Tominaga showed the following inequality [2]:

$$L(a, b) \log S \left(\frac{a}{b} \right) \geq (1 - v)a + vb - a^{1-v}b^v, \quad (15)$$

which is called the converse difference inequality for the Young inequality in [2]

In this section, we show the reverse difference inequality of the refined Young inequality (2).

LEMMA 4.1. *For positive real numbers a, b and $v \in [0, 1]$, we have*

$$\omega L(\sqrt{a}, \sqrt{b}) \log S \left(\sqrt{\frac{a}{b}} \right) \geq (1 - v)a + vb - a^{1-v}b^v - r \left(\sqrt{a} - \sqrt{b} \right)^2, \quad (16)$$

where $\omega \equiv \max \left\{ \sqrt{a}, \sqrt{b} \right\}$.

Proof. We firstly consider the following function.

$$g_b(v) \equiv vb + (1 - v) - b^v - v(\sqrt{b} - 1)^2, \quad \left(0 \leq v \leq \frac{1}{2}\right).$$

From $g'_b(v) = 2(\sqrt{b} - 1) - b^v \log b$, we have

$$g'_b(v) = 0 \Leftrightarrow v = \frac{\log \frac{\sqrt{b}-1}{\log \sqrt{b}}}{\log b} \equiv v_b.$$

We also find that $v_b \in [0, \frac{1}{2}]$ by elementally calculations with the following inequalities:

$$1 - \frac{1}{\sqrt{b}} \leq \log \sqrt{b} \leq \sqrt{b} - 1, \quad (b > 0).$$

In addition, we have $g''_b(v) = -b^v (\log b)^2 < 0$. Therefore g_b takes a maximum value when $v = v_b$, and it is calculated as $g_b(v_b) = L(1, \sqrt{b}) \log S(\sqrt{b})$ by simple but slightly complicated calculations. Thus we have

$$L(1, \sqrt{b}) \log S(\sqrt{b}) \geq vb + (1 - v) - b^v - v(\sqrt{b} - 1)^2.$$

We put $\frac{b}{a}$ instead of b in the above inequality, and then multiplying a to both sides, we have

$$\sqrt{a}L(\sqrt{a}, \sqrt{b}) \log S\left(\sqrt{\frac{a}{b}}\right) \geq (1 - v)a + vb - a^{1-v}b^v - v(\sqrt{a} - \sqrt{b})^2, \quad (17)$$

since $xL\left(1, \frac{y}{x}\right) = L(x, y) = L(y, x)$ and $S(x) = S(1/x)$ for $x > 0$.

For the case of $v \geq 1/2$, by the similar way, we also have the following inequality:

$$\sqrt{b}L(\sqrt{b}, \sqrt{a}) \log S\left(\sqrt{\frac{a}{b}}\right) \geq (1 - v)a + vb - a^{1-v}b^v - (1 - v)(\sqrt{a} - \sqrt{b})^2, \quad (18)$$

From the inequalities (17) and (18), we have the present theorem. \square

REMARK 4.2. We easily find that the right hand side of the inequality (15) is greater than that of the inequality (16). Therefore, if the left hand side of the inequality (16) is greater than that of the inequality (15), then Theorem 4.1 is trivial one. However, we have not yet found any counter-example such that

$$L(a, b) \log S\left(\frac{a}{b}\right) \geq \omega L(\sqrt{a}, \sqrt{b}) \log S\left(\sqrt{\frac{a}{b}}\right), \quad (19)$$

where $\omega = \max\{\sqrt{a}, \sqrt{b}\}$ for any $a, b > 0$ by the computer calculations. Here we give a remark that we have the following inequalities:

$$L(a, b) \leq \omega L(\sqrt{a}, \sqrt{b}), \quad \text{and} \quad \log S\left(\frac{a}{b}\right) \geq \log S\left(\sqrt{\frac{a}{b}}\right)$$

for any $a, b > 0$. At least, we actually have many examples satisfying the inequality (19) so that we claim that Theorem 4.1 is nontrivial as a refinement of the inequality (15).

In addition, it is remarkable that we have no ordering between

$$L(a, b) \log S\left(\frac{a}{b}\right)$$

and

$$\omega L(\sqrt{a}, \sqrt{b}) \log S\left(\sqrt{\frac{a}{b}}\right) + r\left(\sqrt{a} - \sqrt{b}\right)^2$$

for any $a, b > 0$ and $v \in [0, 1]$. Therefore we may claim that Theorem 4.1 is also nontrivial from the sense of finding a tighter upper bound of $(1 - v)a + vb - a^{1-v}b^v$.

Moreover, we have not yet found any counter-example such that

$$\omega L(\sqrt{a}, \sqrt{b}) \log S\left(\sqrt{\frac{a}{b}}\right) \geq (1 - v)a + vb - a^{1-v}b^v,$$

for any positive numbers a and b , and $v \in [0, 1]$ by the computer calculations. Therefore we may have a conjecture such that the left hand side of the inequality (16) exists between the left hand side of the inequality (15) and the right hand side of the inequality (15).

Finally we prove the following theorem. It can be proven by the similar method in [2].

THEOREM 4.3. *We suppose two invertible positive operators satisfy $0 < mI \leq A, B \leq MI$, where I represents an identity operator and $m, M \in \mathbb{R}$. For any $v \in [0, 1]$, we then have*

$$h\sqrt{ML}(\sqrt{M}, \sqrt{m}) \log S(\sqrt{h}) \geq (1 - v)A + vB - A\sharp_v B - 2r\left(\frac{A+B}{2} - A\sharp_{1/2} B\right), \quad (20)$$

where $h \equiv \frac{M}{m} > 1$ and $r \equiv \min\{v, 1 - v\}$.

Proof. From the inequality (16), we have

$$\omega L(\sqrt{b}, 1) \log S(\sqrt{b}) \geq vb + (1 - v) - b^v - r(b - 2\sqrt{b} + 1),$$

for all $v \in [0, 1]$, putting $b = 1$. We consider the invertible positive operator T such that $0 < mI \leq T \leq MI$. Then we have the following inequality

$$\max_{m \leq t \leq M} \max\{\sqrt{t}, 1\} L(\sqrt{t}, 1) \log S(\sqrt{t}) \geq vT + (1 - v) - T^v - r(T - 2T^{1/2} + 1),$$

for any $v \in [0, 1]$. We put $T = A^{-1/2}BA^{-1/2}$. Since we then have $\frac{1}{h} = \frac{m}{M} \leq A^{-1/2}BA^{-1/2} \leq \frac{M}{m} = h$, we have

$$\begin{aligned} \max_{\frac{1}{h} \leq t \leq h} \max\{\sqrt{t}, 1\} L(\sqrt{t}, 1) \log S(\sqrt{t}) &\geq vA^{-1/2}BA^{-1/2} + (1 - v) - \left(A^{-1/2}BA^{-1/2}\right)^v \\ &\quad - r\left\{A^{-1/2}BA^{-1/2} - 2\left(A^{-1/2}BA^{-1/2}\right)^{1/2} + 1\right\}. \end{aligned}$$

Note that $h > 1$ and $L(u, 1)$ is monotone increasing function for $u > 0$. In addition, we note that $S(x)$ is monotone decreasing for $0 < x < 1$ and monotone increasing for $x > 1$ [2]. Thus we have

$$\begin{aligned} \sqrt{h}L(\sqrt{h}, 1) \log S(\sqrt{h}) &\geq vA^{-1/2}BA^{-1/2} + (1 - v) - \left(A^{-1/2}BA^{-1/2}\right)^v \\ &\quad - r \left\{ A^{-1/2}BA^{-1/2} - 2 \left(A^{-1/2}BA^{-1/2}\right)^{1/2} + 1 \right\}. \end{aligned}$$

Multiplying $A^{1/2}$ to the above inequality from both sides, we have

$$\sqrt{h}L(\sqrt{h}, 1) \log S(\sqrt{h})A \geq (1 - v)A + vB - A\sharp_v B - 2r \left(\frac{A+B}{2} - A\sharp_{1/2} B \right).$$

Since the left hand side in the above inequality is less than

$$\sqrt{h}L(\sqrt{h}, 1) \log S(\sqrt{h})M = h\sqrt{ML}(\sqrt{M}, \sqrt{m}) \log S(\sqrt{h}),$$

the proof is completed. \square

REMARK 4.4. As mentioned in Remark 4.2, we have not yet found the ordering between the left hand side of the inequality (20) and that of the inequality (3). Therefore Theorem 4.3 is not a trivial result.

5. Concluding remarks

As we have seen, we gave refined Young inequalities for two positive operators. In addition, we gave reverse ratio type inequalities and reverse difference type inequalities for the refined Young inequality for positive operators. Closing this paper, we shall give a refinement of the weighted arithmetic-geometric mean inequality for n real numbers by a simple proof.

PROPOSITION 5.1. For $a_1, \dots, a_n \geq 0$ and $p_1, \dots, p_n \geq 0$ with $\sum_{j=1}^n p_j = 1$, we have

$$\sum_{i=1}^n p_i a_i - \prod_{i=1}^n a_i^{p_i} \geq n\lambda \left(\frac{1}{n} \sum_{i=1}^n a_i - \prod_{i=1}^n a_i^{1/n} \right), \tag{21}$$

with equality if and only if $a_1 = \dots = a_n$, where $\lambda \equiv \min \{p_1, \dots, p_n\}$.

Proof. We suppose $\lambda = p_j$. For any $j = 1, \dots, n$, we then have

$$\begin{aligned} \sum_{i=1}^n p_i a_i - p_j \left(\sum_{i=1}^n a_i - n \prod_{i=1}^n a_i^{1/n} \right) &= np_j \left(\prod_{i=1}^n a_i^{1/n} \right) + \sum_{i=1, i \neq j}^n (p_i - p_j) a_i \\ &\geq \prod_{i=1, i \neq j}^n \left(a_1^{1/n} \dots a_n^{1/n} \right)^{np_j} a_i^{p_i - p_j} \\ &= a_1^{p_1} \dots a_n^{p_n}. \end{aligned}$$

In the above process, the classical weighted arithmetic-geometric mean inequality [10, 11];

$$\sum_{j=1}^n p_j a_j \geq \prod_{j=1}^n a_j^{p_j}, \quad (22)$$

with equality if and only if $a_1 = \cdots = a_n$, was used. The equality in the inequality (21) holds if and only if

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} = a_1 = a_2 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_n$$

by the equality condition of the classical weighted arithmetic-geometric mean inequality. Therefore $a_1 = a_2 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_n \equiv a$, then we have $a_j^{\frac{1}{n}} a^{\frac{n-1}{n}} = a$ from the first equality. Thus we have $a_j = a$, which completes the proof. \square

Proposition 5.1 gives a refinement of the classical weighted arithmetic-geometric mean inequality (22). At the same time, it gives a natural generalization of the inequality (2) proved in [1]. It is also notable that Proposition 5.1 can be proven by using the bounds for the normalized Jensen functional, which were obtained by S. S. Dragomir in [12].

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