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LOWER BOUND FOR THE NORM OF LOWER TRIANGULAR MATRICES ON BLOCK WEIGHTED SEQUENCE SPACES

R. LASHKARIPOUR AND G. TALEBI

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Abstract. Let $1 < p < \infty$ and $A = (a_{n,k})_{n,k\geqslant 1}$ be a non-negative matrix. Denote by $\|A\|_{w,p,F}$, the infimum of those U satisfying the following inequality:

$$\|Ax\|_{w,p,F} \leqslant U\,\|x\|_{w,p,I},$$

where $x \geqslant 0$ and $x \in l_p(w,I)$ and also $w = (w_n)_{n=1}^{\infty}$ is a decreasing, non-negative sequence of real numbers. The purpose of this paper is to give a lower bound for $\|A\|_{w,p,F}$, where A is a lower triangular matrix. In particular, we apply our results to Weighted mean matrices and Nörlund matrices which recently considered in [2,3,6] on the usual sequence spaces. Our results generalize some work of Jameson, Lashkaripour, Frotannia and Chen in [4,7,8].

1. Introduction

Let $p \geqslant 1$ and $(w_n)_{n=1}^{\infty}$ be a decreasing, non-negative sequence of real numbers. We define the weighted sequence space $l_p(w)$ as

$$l_p(w) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} w_k |x_k|^p < \infty \right\},$$

with a norm $\|.\|_{w,p}$ which is defined in the following way:

$$\|x\|_{w,p} := \left(\sum_{k=1}^{\infty} w_k |x_k|^p \right)^{\frac{1}{p}}.$$

Next, assume that F is a partition of positive integers. If $F = (F_n)$, where each (F_n) is a finite interval of positive integers and

$$\max F_n < \min F_{n+1} \qquad (n = 1,2,3,\ldots),$$

we define the weighted sequence space $l_p(w,F)$ as

$$l_p(w,F) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} w_k \,|\langle x, F_k\rangle|^p < \infty \right\},$$

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where $\langle x, F_k \rangle = \sum_{j\in F_k} x_j$ (see [1] for more details). The norm on $l_p(w,F)$, denoted by $\|.\|_{w,p,F}$, is defined as follows:

$$\|x\|_{w,p,F} = \left(\sum_{k=1}^{\infty} w_k \left| \langle x, F_k \rangle \right|^p \right)^{\frac{1}{p}}.$$

For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$, is a partition of positive integers, $l_p(w,I) = l_p(w)$ and also $\|x\|_{w,p,F} = \|x\|_{w,p}$.

It is known that any bounded linear operator T from $l_p(w,I)$ into $l_p(w,F)$ is uniquely determine by a matrix $A = (a_{n,k})_{n,k\geqslant 1}$ which satisfies $Tx = Ax$ for all $x \in l_p(w,I)$. On the other hand, given any real matrix $A = (a_{n,k})_{n,k\geqslant 1}$, define Tx by $Tx = Ax$. For suitable A, T may define a bounded linear operator.

We consider the upper bounds U of the form

$$\|Tx\|_{w,p,F} \leqslant U \|x\|_{w,p,I}, \tag{1.1}$$

for all non-negative sequence x. The constant U not depending on x. We seek the smallest possible value of U, and denote the best upper bound by $\|T\|$ for operators from $l_p(w,I)$ into $l_p(w,F)$.

In this paper, we shall relax the conditions on A (e.g., A is a lower triangular matrix) such that (1.1) can be investigated for all real sequence x. Note that for such A, U may be infinite and $||T||$ may not be defined. Due to these facts, we write $\|A\|_{w,p,F}$ in the place of $||T||$; also we write $\|A\|_{w,p,I}$ or $\|A\|_{w,p}$ when T define from $l_p(w)$ to itself.

We give a lower bound for $\|A\|_{w,p,F}$, where A is a lower triangular matrix (see Theorem 2.1). Also, we apply our results to Weighted mean matrices , $M_a = (m_{n,k})_{n,k\geqslant 1}$, and Nörlund matrices, $N_a = (b_{n,k})_{n,k\geqslant 1}$, where the Weighted mean matrices and the Nörlund matrices are define as below:

$$m_{n,k} = \begin{cases} \frac{a_k}{A_n} & 1 \leqslant k \leqslant n \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and

$$b_{n,k} = \begin{cases} \frac{a_{n-k+1}}{A_n} & 1 \leqslant k \leqslant n \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Here $A_n = \sum_{k=1}^{n} a_k$ and $a = (a_n)_{n=1}^{\infty}$ is a non negative sequence with $a_1 > 0$.

Throughout this paper, for the lower triangular matrix $A = (a_{n,k})_{n,k\geqslant 1}$ we set

$$m_A := \sup_{N\geqslant 1} \inf_{n\geqslant N} \left\{ n a_{n,N} + \frac{n}{n-N+1} \sum_{k=N+1}^{n} (n-k+1) \left(a_{n,k} - a_{n,k-1} \right)^{-} \right\},$$

where $\eta^{-} = \min(\eta, 0)$ and $a_{n,0} = 0$ for all $n \geqslant 1$.

2. Main result

THEOREM 2.1. *Suppose that* $p>1$ *and* $A=(a_{n,k})$ *is a lower triangular matrix with non-negative entries. Let* $w=(w_n)$ *be a decreasing, non-negative sequence such that* $\sum_{n=1}^{\infty}\frac{w_n}{n}$ *is divergent and* $\left(\frac{w_n}{w_{n+1}}\right)$ *is decreasing, then*

$$\|A\|_{w,p,F}\geqslant m_A\left(\frac{p}{p-1}\right).$$

Theorem 2.1, generalizes ([5], Theorem 5.2.5(ii)). For other applications, we refer the readers to the next two sections.

Note that in the following two sections, we assume $w=(w_n)$ is a decreasing sequence with non-negative entries and $\left(\frac{w_n}{w_{n+1}}\right)$ is decreasing and also $\sum_{n=1}^{\infty}\frac{w_n}{n}=\infty$. These conditions are satisfied by $w_n={}^1/_{(\log(n+1))^\theta}$ where $0<\theta\leqslant 1$, so that we have at least one example of $w=(w_n)$ before all the results depending on it.

3. Matrices with increasing rows

For $A\geqslant 0$ and $a_{n,k}\leqslant a_{n,k+1}$ for $0\leqslant k<n$, we know that $m_A=\left(\sup_{N\geqslant 1}\inf_{n\geqslant N} na_{n,N}\right)$. Applying Theorem 2.1, we have the following corollaries.

COROLLARY 3.1. *Suppose that* $p>1$ *and* $A=(a_{n,k})$ *is a lower triangular matrix with non-negative entries. If* $a_{n,k}\leqslant a_{n,k+1}$ *for* $0\leqslant k<n$, *then*

$$\|A\|_{w,p,F}\geqslant\left(\sup_{N\geqslant 1}\inf_{n\geqslant N} na_{n,N}\right)\left(\frac{p}{p-1}\right).$$

COROLLARY 3.2. *Suppose that* $p>1$ *and* $A=(a_{n,k})$ *is a lower triangular matrix with non-negative entries. If* $a_{n,k}\leqslant a_{n,k+1}$ *for* $0\leqslant k<n$ *and also* $(na_{n,k})$ *is an increasing sequence for each* k, *then*

$$\|A\|_{w,p,F}\geqslant\left(\sup_{n\geqslant 1} na_{n,n}\right)\left(\frac{p}{p-1}\right).$$

In particular, $\|C_N\|_{w,p,F}\geqslant\left(\frac{p}{p-1}\right)$, *where* $C_N=(C^N_{n,k})$, *the generalized Cesaro matrix, defined as*

$$C^N_{n,k}=\begin{cases}\dfrac{1}{n+N-1} & n\geqslant k\\ 0 & n<k.\end{cases}\tag{3.1}$$

(Here $N\geqslant 1$.*)*

We apply the above corollary to the following two special cases.

COROLLARY 3.3. *Suppose that $p>1$ and $N_a=(b_{n,k})$ is the Nörlund matrix defined by (1.3). If $a_n \downarrow \alpha$, where $\alpha>0$, then*

$$\|N_a\|_{w,p,F} \geqslant \left(\frac{p}{p-1}\right).$$

COROLLARY 3.4. *Suppose that $p>1$ and $M_a=(m_{n,k})$ is the Weighted mean matrix defined by (1.2). If If $a_n \uparrow \alpha$, where $\alpha<\infty$, then*

$$\|M_a\|_{w,p,F} \geqslant \left(\frac{p}{p-1}\right).$$

4. Matrices with decreasing rows

For $A \geqslant 0$ and $a_{n,k} \geqslant a_{n,k+1}$ for $0 \leqslant k<n$, we know that $m_A \geqslant \left(\inf_{n\geqslant 1} \sum_{k=1}^{n} a_{n,k}\right)$. Applying Theorem 2.1, we have the following corollaries.

COROLLARY 4.1. *Suppose that $p>1$ and $A=(a_{n,k})$ is a lower triangular matrix with non-negative entries. If $a_{n,k} \geqslant a_{n,k+1}$ for $0 \leqslant k<n$, then*

$$\|A\|_{w,p,F} \geqslant \left(\inf_{n\geqslant 1} \sum_{k=1}^{n} a_{n,k}\right)\left(\frac{p}{p-1}\right)$$

We apply the above Corollary to the following two special cases.

COROLLARY 4.2. *Suppose that $p>1$ and $N_a=(b_{n,k})$ is the Nörlund matrix defined by (1.3). If (a_n) is an increasing non-negative sequence, then*

$$\|N_a\|_{w,p,F} \geqslant \left(\frac{p}{p-1}\right).$$

COROLLARY 4.3. *Suppose that $p>1$ and $M_a=(m_{n,k})$ be defined by (1.2). If (a_n) is a decreasing non-negative sequence, then*

$$\|M_a\|_{w,p,F} \geqslant \left(\frac{p}{p-1}\right).$$

5. Proof of Theorem 2.1

To prove Theorem 2.1 we need the following statements.

LEMMA 5.1. ([4], Lemma 2.2) *Suppose that $N \geqslant 1$ and a, x are non-negative sequences with $x_N \geqslant x_{N+1} \geqslant \ldots \geqslant 0$ and also $x_n=0$ for $n<N$. Then*

$$\sum_{k=1}^{n} a_k x_k \geqslant \left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\left\{n a_N + \frac{n}{n-N+1}\sum_{k=N+1}^{n}(n-k+1)(a_k-a_{k-1})^{-}\right\}.$$

LEMMA 5.2. ([9], Lemma 2.3) *Let* $p > 1$*, and let* $w = (w_n)$ *be a decreasing sequence with non-negative entries and* $\sum_{n=1}^{\infty} \frac{w_n}{n}$ *is divergent. Then*

$$\|C_N\|_{w,p} = \frac{p}{p-1},$$

where C_N *is defined by (3.1).*

Proof of Theorem 2.1. We have $m_A = \sup_{N \geqslant 1} \delta_N$ where

$$\delta_N = \inf_{n \geqslant N} \left\{ n a_{n,N} + \frac{n}{n-N+1} \sum_{k=N+1}^{n} (n-k+1) \left(a_{n,k} - a_{n,k-1}\right)^{-} \right\}.$$

Let $N \geqslant 1$, so that $\delta_N \geqslant 0$. Suppose that $y = (y_n)$ is a decreasing sequence with non-negative entries such that $||y||_{w,p} = 1$. We set $x_1 = x_2 = ... = x_{N-1} = 0$ and

$$x_{n+N-1} = \left(\frac{w_n}{w_{n+N-1}}\right)^{\frac{1}{p}} y_n,$$

for all $n \geqslant 1$. Clearly $||x||_{w,p} = ||y||_{w,p} = 1$. Applying Lemma 5.1, we have

$$\begin{aligned}
\|Ax\|_{w,p,F}^{p} &= \sum_{n=1}^{\infty} w_n \left|< Ax, F_n >\right|^p \\
&= \sum_{n=1}^{\infty} w_n \left(\sum_{j \in F_n} \sum_{k=1}^{j} a_{j,k} x_k \right)^p \\
&\geqslant \sum_{n=1}^{\infty} w_n \sum_{j \in F_n} \left(\sum_{k=1}^{j} a_{j,k} x_k \right)^p \\
&\geqslant \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{n} a_{n,k} x_k \right)^p \\
&\geqslant \delta_N^p \sum_{n=1}^{\infty} w_n \left(\frac{1}{n} \sum_{j=1}^{n} x_j \right)^p \\
&= \delta_N^p \sum_{n=1}^{\infty} w_{N+n-1} \left(\frac{1}{N+n-1} \sum_{j=1}^{n} x_{N+j-1} \right)^p \\
&= \delta_N^p \sum_{n=1}^{\infty} w_{N+n-1} \left(\frac{1}{N+n-1} \sum_{j=1}^{n} \left(\frac{w_j}{w_{N+j-1}} \right)^{\frac{1}{p}} y_j \right)^p \\
&\geqslant \delta_N^p \left\|C^N y\right\|_{w,p}^p,
\end{aligned}$$

Applying Lemma 5.2, we conclude that $\|A\|_{w,p,F} \geqslant \delta_N \left(\frac{p}{p-1}\right)$ and so

$$\|A\|_{w,p,F} \geqslant m_A \left(\frac{p}{p-1}\right).$$

This completes the proof of statement. □

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R. Lashkaripour
Department of Mathematics
Faculty of Mathematics
University of Sistan and Baluchestan
e-mail: lashkari@hamoon.usb.ac.ir

G. Talebi
University of Sistan and Baluchestan
Zahedan, Iran
e-mail: gh11talebi@gmail.com