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A COROLLARY OF ALEXANDER'S INEQUALITY

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Abstract. A simple corollary of Alexander's inequality is shown for a uniform algebra. As a result, we give an inequality between the L^∞-type norm of a given analytic function and the L^1-type norm of the derivative.

Let A be a uniform algebra on a compact Hausdorff space X. The spectrum $\sigma(f)$ of $f \in A$ is the compact set of complex members λ such that $1/(\lambda - f)$ does not belong to A. $A^\perp$ denotes the set of all orthogonal measures on X for A.

For f in A f^* denotes the complex conjugate of f and for μ in $A^\perp$ μ^* denotes the complex measure on X such that

$$\int_X g d\mu^* = \left(\int_X g^* d\mu\right)^*.$$

For a continuous function ϕ on X, $\|\phi\| = \sup_{x\in X} |\phi(x)|$ and for a complex measure μ on X, $\|\mu\| = |\mu|(X)$.

The following theorem is a simple consequence of Alexander's inequality. The Corollary follows from the theorem below and F. and M. Riesz theorem.

THEOREM. *Let f be in A and μ be in $A^\perp$. If $\left|\int_X f^* d\mu\right| / \text{Area}\,(\sigma(f)) = \rho$ then*

$$\text{dist}(\mu^*, A^\perp) \geqslant \frac{\rho\pi}{\|f^*\|}\,\text{dist}(f^*, A)^2$$

Proof. Let $S = \text{Area}\,(\sigma(f))$, then

$$\begin{aligned}\rho S &= \left|\int_X f^* d\mu\right| \\ &= \left|\int_X f^*(d\mu - d\lambda^*)\right| \\ &\leqslant \|f^*\|\|\mu - \lambda^*\|\end{aligned}$$

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Since $\text{dist}(f^*,A)^2 \leqslant S/\pi$ by Alexander's inequality (see [5], [6]),

$$\text{dist}(f^*,A)^2 \times \rho\pi \leqslant \|f^*\|\text{dist}(\mu^*,A^\perp). \quad \Box$$

Let X be the boundary of a compact subset Y of the plane whose complement has only finitely many components. Let A be the algebra of all functions on X that can be uniformly approximated by rational functions with poles off Y. Let m be the harmonic measure on X evaluated at a point in Y and $A_0 = \{f \in A : \int_X f dm = 0\}$. H_0^1 denotes the closure of A_0 in $L^1(X,m)$ and N denotes the set $\{u \in L^1(X,m) : \int_X u(f+\bar{g})dm = 0$ for all $f,g \in A\}$. Then the following holds.

COROLLARY. *Suppose f and f' belong to A. Then*

$$\text{dist}_{L^1}\left(\left(\zeta f' \frac{ds}{dm}\right)^*, H_0^1 + N\right) \geqslant \frac{1}{\|f^*\|}\,\text{dist}(f^*,A)^2$$

where ds is the arc length measure on X.

Proof. Assuming that f is analytic in a neighborhood of $X \bigcup Y$. According to Stokes' Theorem,

$$\begin{aligned} \left|\int_X f^* f' d\zeta\right| &= \left|2i \iint_Y |f'|^2 dxdy\right| \\ &= |2i(\text{Area } f(Y) \text{ with multiplicity})| \geqslant 2\text{Area }(f(Y)) \end{aligned}$$

(see [7]). Put $d\mu = f'd\zeta$ then μ belongs to $A^\perp$. By Theorem,

$$\text{dist}(\mu^*,A^\perp) \geqslant \frac{\rho\pi}{\|f^*\|}\text{dist}(f^*,A)^2.$$

Then $\rho \geqslant 2$ and $A^\perp = H_0^1 + N$ (see [4]). $\Box$

For $1 \leqslant p \leqslant \infty$, H^p denotes a Hardy space on the unit circle $\partial\Delta$ where $\partial\Delta$ is the boundary of the open unit disc Δ in $\mathbb{C}$ (see [3]). The following proposition follows from the above corollary.

PROPOSITION. *Suppose $1 \leqslant p \leqslant q \leqslant \infty$ with $1/p + 1/q = 1$. If f is a function in H^q then*

$$\text{dist}_{L^p}(f'^*, z^2H^p) \geqslant \frac{1}{\|f^*\|_q}\text{dist}_{L^q}(f^*,H^q)^2$$

Proof. Let A be the disc algebra on $\partial\Delta$. Assuming that f is analytic in a neighborhood of the closed unit disc $\bar{\Delta}$, by the proof of Corollary

$$
\begin{aligned}
\operatorname{dist}_q(f^*,A)^2 &\leqslant \operatorname{dist}_\infty(f^*,A)^2 \leqslant \frac{1}{\pi}\text{Area }(f(\Delta)) \\
&\leqslant \left|\frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})f'(e^{i\theta})e^{i\theta}d\theta\right| \\
&= \left|\frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})(e^{i\theta}f'(e^{i\theta})-g^*(e^{i\theta}))d\theta\right| \\
&\leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}|f^*(e^{i\theta})|^q d\theta\right)^{1/q}\left(\frac{1}{2\pi}\int_0^{2\pi}|e^{i\theta}f'(e^{i\theta})-g^*(e^{i\theta})|^p d\theta\right)^{1/p}
\end{aligned}
$$

where $g \in e^{i\theta}A$. This proves the Proposition. □

REMARKS.

(1) In this paper as in [2] and [8], the following inequality is called Alexander's inequality.

$$\operatorname{dist}(f^*,A)^2 \leqslant \text{ Area }(\sigma(f)/\pi) \quad (f \in A).$$

However it may not be a correct terminology. In fact, the inequality was shown by Gamelin [5], and Gamelin and Khavinson [6]. Using this, they showed Alexander's spectral area estimate [1], that is,

$$\int_X |f - \int_X f dm|^2 dm \leqslant \text{ Area }(\sigma(f)/\pi) \quad (f \in A)$$

where m is a probability measure on X which is multiplicative on A.

(2) The main idea of comparing the L^∞ norm of an analytic function with the H^1 norm of its derivative is not exactly new. We should note Theorem 1 and Corollary 3 in [7].

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