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GENERALIZATIONS OF CONVERSE JENSEN'S INEQUALITY AND RELATED RESULTS

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Abstract. In this paper we prove generalizations of Converse Jensen's inequality for convex functions defined on convex hulls. As consequences we get generalizations of the Hermite-Hadamard inequality for convex functions defined on k-simplices in $\mathbb{R}^k$. We also present some related results which generalize results in [8].

1. Introduction

Let U be a convex subset of $\mathbb{R}^k$ and $n \in \mathbb{N}$. If $f : U \to \mathbb{R}$ is a convex function, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ and $p_1, \ldots, p_n$ nonnegative real numbers with $P_n = \sum_{i=1}^n p_i$, then the well known Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i \boldsymbol{x}_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i f(\boldsymbol{x}_i) \tag{1.1}$$

holds.

If the following conditions are satisfied

$$p_1 > 0, \quad p_i \leqslant 0 \quad (i = 2, \ldots, n) \quad P_n > 0,$$

$$\frac{1}{P_n}\sum_{i=1}^n p_i \boldsymbol{x}_i \in U,$$

then Reversed Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i \boldsymbol{x}_i\right) \geqslant \frac{1}{P_n}\sum_{i=1}^n p_i f(\boldsymbol{x}_i) \tag{1.2}$$

holds (see [14]).

The convex hull of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ is represented by $K = \operatorname{conv}(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$.

Barycentric coordinates over K are continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_n$ on K with following properties:

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(1) $\lambda_i(\boldsymbol{x}) \geqslant 0, \quad i = 1, \dots, n,$

(2) $\sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) = 1,$

(3) $\boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) \boldsymbol{x}_i \,.$

If $\boldsymbol{x}_2 - \boldsymbol{x}_1, \dots, \boldsymbol{x}_n - \boldsymbol{x}_1$ are linearly independent vectors, then each $\boldsymbol{x} \in K$ can be written in unique way as convex combination of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ in the form (3).

We also consider k-simplex $S = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ which is convex hull of its vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1} \in \mathbb{R}^k$. Barycentric coordinates $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$ over S are non-negative linear polynomials on S and have special form (see the third section).

The next variant of Jensen's inequality was proved by A. Matković and J. Pečarić [8].

THEOREM A. *Let U be a convex subset in $\mathbb{R}^k$, $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in U$ and $\boldsymbol{y}_1, \dots, \boldsymbol{y}_m \in$ conv$(\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\})$. If f is a convex function on U, then the inequality*

$$f\left(\frac{\sum_{i=1}^{n} p_i \boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{P_n - W_m}\right) \leqslant \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{P_n - W_m} \tag{1.3}$$

holds for all positive real numbers $p_1, \dots, p_n$ and $w_1, \dots, w_m$ satisfying the condition

$$p_i \geqslant W_m \quad \textit{for all } i = 1, \dots, n,$$

where $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$.

In the following, let E be a nonempty set and L be a linear class of functions $f : E \to \mathbb{R}$ having the properties:

(L1) if $f, g \in L$ then $(af + bg) \in L$ for all $a, b \in \mathbb{R}$

(L2) $1 \in L$ where $1(t) = 1$ for all $t \in E$.

We consider positive linear functionals $A : L \to \mathbb{R}$. That is, we assume:

(A1) $A(af + bg) = aA(f) + bA(g)$ for all $f, g \in L$, $a, b \in \mathbb{R}$ (linearity)

(A2) if $f \in L$, $f(t) \geqslant 0$ for all $t \in E$ then $A(f) \geqslant 0$ (positivity).

From (A1) we obtain

(A1') $A\left(\sum_{i=1}^{k} a_i g_i\right) = \sum_{i=1}^{k} a_i A(g_i)$ for $g_1, \dots, g_k \in L$, $a_1, \dots, a_k \in \mathbb{R}$ (linearity).

If in addition $A(1) = 1$ is satisfied, we say that A is a positive normalized linear functional.

With L^k we denote a linear class of functions $\boldsymbol{g} : E \to \mathbb{R}^k$ defined by

$$\boldsymbol{g}(t) = (g_1(t), ..., g_k(t)),\ g_i \in L\ (i = 1, ..., k).$$

We also consider linear operators $\widetilde{A} : L^k \to \mathbb{R}^k$ defined by

$$\widetilde{A}(\boldsymbol{g}) = (A(g_1), ..., A(g_k)).$$

If $A(1) = 1$ is satisfied, then using (A1) we also have

(A3) $A(f(\boldsymbol{g})) = f(\widetilde{A}(\boldsymbol{g}))$ for every linear function f on $\mathbb{R}^k$.

Next we introduce the functional versions of Jensen's inequality and some related results which we generalize in sequel.

B. Jessen [14, p. 47] gave the following generalization of Jensen's inequality for positive linear functionals.

THEOREM B. (Jessen's inequality) *Let L satisfy properties L1, L2 on nonempty set E and A be a positive normalized linear functional on L. Let f be a continuous convex function on an interval $I \subset \mathbb{R}$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have $A(g) \in I$ and*

$$f(A(g)) \leqslant A(f(g)). \tag{1.4}$$

The next theorem, proved by J. Pečarić and P. R. Beesack, presents generalization of Theorem Lah-Ribarić (see [10, p. 98], [14, p. 98]).

THEOREM C. (Converse Jessen's inequality) *Let L satisfy properties L1, L2 and A be a positive normalized linear functional on L. Let f be a convex function on an interval $I = [m, M] \subset \mathbb{R}$ $(-\infty < m < M < \infty)$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have*

$$A(f(g)) \leqslant \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M). \tag{1.5}$$

Using Theorem C, Beesack and Pečarić also proved the next result [14, p. 101].

THEOREM D. *Let L, A and f be as in Theorem C. Let J be an interval in $\mathbb{R}$ such that $f(I) \subset J$. If $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have*

$$\begin{aligned} F(A(f(g)), f(A(g))) &\leqslant \max_{x \in [m,M]} F\left(\frac{M - x}{M - m} f(m) + \frac{x - m}{M - m} f(M), f(x)\right) \\ &= \max_{\theta \in [0,1]} F\left(\theta f(m) + (1 - \theta) f(M), f(\theta m + (1 - \theta) M)\right). \end{aligned} \tag{1.6}$$

REMARK 1. If we choose $F(x,y) = x - y$, as a simple consequence of Theorem D it follows

$$A(f(g)) - f(A(g)) \leqslant \max_{\theta\in[0,1]} [\theta f(m) + (1-\theta)f(M) - f(\theta m + (1-\theta)M)]. \quad (1.7)$$

Choosing $F(x,y) = \frac{x}{y}$, it follows

$$\frac{A(f(g))}{f(A(g))} \leqslant \max_{\theta\in[0,1]} \left[\frac{\theta f(m) + (1-\theta)f(M)}{f(\theta m + (1-\theta)M)}\right]. \quad (1.8)$$

It is obviously that the main results in [15], [16] and [17] can be obtained as direct consequences of Theorem D published many years earlier.

Additional generalization of Jessen's inequality (1.4) is proved by E. J. McShane (see [9], [14, p. 48]).

THEOREM E. (McShane's inequality) *Let L satisfy properties L1, L2, A be a positive normalized linear functional on L and $\widetilde{A} = (A,...,A) : L^k \to \mathbb{R}^k$ a linear operator. Let f be a continuous convex function on a closed convex set $U \subset \mathbb{R}^k$. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset U$ and $f(\boldsymbol{g}) \in L$, we have that $\widetilde{A}(\boldsymbol{g}) \in U$ and*

$$f(\widetilde{A}(\boldsymbol{g})) \leqslant A(f(\boldsymbol{g})). \quad (1.9)$$

It is known that for a convex function $f : [a,b] \to \mathbb{R}$ the Hermite -Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} \quad (1.10)$$

holds.

In this paper, as our main results we present generalizations of Theorem C and Theorem D for convex functions defined on convex hulls. As consequences, we obtain generalizations of the Hermite-Hadamard inequality (1.10) for convex functions defined on k-simplices in $\mathbb{R}^k$. Some related results can be found in [5], [6], [7]. We also present related results which generalize results in [8].

2. Main results

For $n \in \mathbb{N}$ we denote

$$\Delta^n = \left\{(\Lambda_1,...,\Lambda_n) : \Lambda_i \geqslant 0, \forall i \in \{1,...,n\}, \sum_{i=1}^n \Lambda_i = 1\right\}.$$

The next theorem presents generalization of Theorem C.

THEOREM 1. *Let L satisfy properties L1, L2 on nonempty set E and A be a positive normalized linear functional on L. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in \mathbb{R}^k$ and $K = conv(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L$ $(i=1,\dots,n)$ we have*

$$A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\, f(\boldsymbol{x}_i). \tag{2.1}$$

Proof. For each $t \in E$ we have $\boldsymbol{g}(t) \in K$. Then there exist barycentric coordinates $\lambda_i(\boldsymbol{g}(t)) \geqslant 0$ $(i=1,\dots,n)$ such that $\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t)) = 1$ and

$$\boldsymbol{g}(t) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Since f is convex on K, then

$$f(\boldsymbol{g}(t)) = f\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i\right) \leqslant \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t)) f(\boldsymbol{x}_i).$$

Now, applying a functional A on the last inequality we get

$$A(f(\boldsymbol{g})) \leqslant A\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}) f(\boldsymbol{x}_i)\right) = \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\, f(\boldsymbol{x}_i). \qquad \square$$

REMARK 2. If all the assumptions of Theorem 1 are satisfied and in addition f is continuous, then

$$f(\widetilde{A}(\boldsymbol{g})) \leqslant A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i)$$

The first inequality is consequence of Theorem E and the second of Theorem 1.

Using Theorem 1 we prove generalization of Theorem D.

THEOREM 2. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A} = (A,\dots,A) : L^k \to \mathbb{R}^k$ a linear operator. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in \mathbb{R}^k$ and $K = conv(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L$ $(i=1,\dots,n)$ we have*

$$\begin{aligned} F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) &\leqslant F\left(\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\, f(\boldsymbol{x}_i), f(\widetilde{A}(\boldsymbol{g}))\right) \\ &\leqslant \max_{(\Lambda_1,\dots,\Lambda_n)\in\Delta^n} F\left(\sum_{i=1}^{n} \Lambda_i f(\boldsymbol{x}_i), f\left(\sum_{i=1}^{n} \Lambda_i \boldsymbol{x}_i\right)\right). \end{aligned} \tag{2.2}$$

Proof. For each $t \in E$ we have $\boldsymbol{g}(t) \in K$. Then there exist barycentric coordinates $\lambda_i(\boldsymbol{g}(t)) \geqslant 0$ $(i = 1,...,n)$ such that $\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t)) = 1$ and

$$\boldsymbol{g}(t) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Since A is a positive normalized linear functional on L and $\widetilde{A} = (A,...,A)$ a linear operator on L^k, we have

$$\widetilde{A}(\boldsymbol{g}) = (A(g_1),...,A(g_k)) = \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i,$$

where

$$A(\lambda_i(\boldsymbol{g})) \geqslant 0, \; i = 1,...,n$$

and

$$\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g})) = A\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g})\right) = A(1) = 1.$$

Therefore, $\widetilde{A}(\boldsymbol{g}) \in K$.

Since $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, using (2.1) we have

$$F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) \leqslant F\left(\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i), f(\widetilde{A}(\boldsymbol{g}))\right). \tag{2.3}$$

By substitutions

$$A(\lambda_i(\boldsymbol{g})) = \Lambda_i \; (i = 1,...,n),$$

it follows

$$\widetilde{A}(\boldsymbol{g}) = \sum_{i=1}^{n} \Lambda_i \boldsymbol{x}_i.$$

Now we have

$$\begin{aligned} F\left(\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i), f(\widetilde{A}(\boldsymbol{g}))\right) &= F\left(\sum_{i=1}^{n} \Lambda_i f(\boldsymbol{x}_i), f\left(\sum_{i=1}^{n} \Lambda_i \boldsymbol{x}_i\right)\right) \\ &\leqslant \max_{(\Lambda_1,...,\Lambda_n) \in \Delta^n} F\left(\sum_{i=1}^{n} \Lambda_i f(\boldsymbol{x}_i), f\left(\sum_{i=1}^{n} \Lambda_i \boldsymbol{x}_i\right)\right). \end{aligned} \tag{2.4}$$

By combining (2.3) and (2.4) we get (2.2). □

REMARK 3. If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 2 it follows

$$A(f(\boldsymbol{g})) - f(\widetilde{A}(\boldsymbol{g})) \leqslant \max_{(\Lambda_1,...,\Lambda_n) \in \Delta^n} \left(\sum_{i=1}^{n} \Lambda_i f(\boldsymbol{x}_i) - f\left(\sum_{i=1}^{n} \Lambda_i \boldsymbol{x}_i\right)\right). \tag{2.5}$$

Choosing $F(x,y)=\dfrac{x}{y}$, it follows

$$\frac{A(f(\boldsymbol{g}))}{f(\widetilde{A}(\boldsymbol{g}))}\leqslant \max_{(\Lambda_1,\dots,\Lambda_n)\in\Delta^n}\left(\frac{\sum_{i=1}^{n}\Lambda_i f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^{n}\Lambda_i\boldsymbol{x}_i\right)}\right). \tag{2.6}$$

The inequalities (2.5) and (2.6) present generalizations of (1.7) and (1.8).

Replacing F by $-F$ in Theorem 2 we get the next theorem.

THEOREM 3. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}=(A,\dots,A):L^k\to\mathbb{R}^k$ a linear operator. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\in\mathbb{R}^k$ and $K=conv(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K)\subset J$ and $F:J\times J\to\mathbb{R}$ is an decreasing function in the first variable, then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset K$ and $f(\boldsymbol{g}),\lambda_i(\boldsymbol{g})\in L$ $(i=1,\dots,n)$ we have*

$$\begin{aligned}F\left(A(f(\boldsymbol{g})),f(\widetilde{A}(\boldsymbol{g}))\right)&\geqslant F\left(\sum_{i=1}^{n}A(\lambda_i(\boldsymbol{g}))f(\boldsymbol{x}_i),f(\widetilde{A}(\boldsymbol{g}))\right)\\&\geqslant \min_{(\Lambda_1,\dots,\Lambda_n)\in\Delta^n}F\left(\sum_{i=1}^{n}\Lambda_i f(\boldsymbol{x}_i),f\left(\sum_{i=1}^{n}\Lambda_i\boldsymbol{x}_i\right)\right).\end{aligned}$$

3. Convex functions on k-simplices in $\mathbb{R}^k$

In this section we give analogs to Theorem 1 and Theorem 2 for convex functions defined on k-simplices in $\mathbb{R}^k$. As a consequence we obtain generalizations of the Hermite-Hadamard inequality (1.10).

Let $S=[\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_{k+1}]$ be k-simplex in $\mathbb{R}^k$ with vertices $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_{k+1}\in\mathbb{R}^k$. The barycentric coordinates $\lambda_1,\dots,\lambda_{k+1}$ over S are nonnegative linear polynomials that satisfy Lagrange's property:

$$\lambda_i(\boldsymbol{v}_j)=\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}.$$

Therefore, it is known that for each $\boldsymbol{x}\in S$ the barycentric coordinates $\lambda_1(\boldsymbol{x}),\dots,$

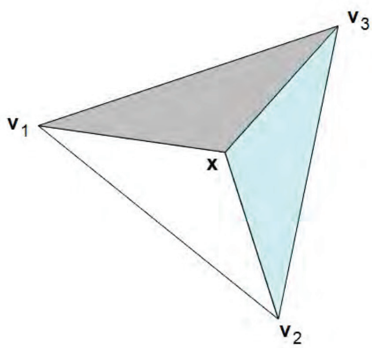
$\lambda_{k+1}(\boldsymbol{x})$ have the form

$$\lambda_1(\boldsymbol{x}) = \frac{\mathrm{Vol}_k\left([\boldsymbol{x},\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)},$$
$$\lambda_2(\boldsymbol{x}) = \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1,\boldsymbol{x},\boldsymbol{v}_3,\ldots,\boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)},$$
$$\vdots$$
$$\lambda_{k+1}(\boldsymbol{x}) = \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)}, \tag{3.1}$$

where Vol_k denotes k-dimensional Lebesgue measure on S.

Here, for example, $[\boldsymbol{v}_1,\boldsymbol{x},\ldots,\boldsymbol{v}_{k+1}]$ denotes the subsimplex obtained by replacing $\boldsymbol{v}_2$ by $\boldsymbol{x}$, i.e. the subsimplex opposite to $\boldsymbol{v}_2$, when adding $\boldsymbol{x}$ as a new vertex.

In other words, we see that the barycentric coordinates $\lambda_1,\ldots,\lambda_{k+1}$ for each $\boldsymbol{x}\in S$ can be presented as the ratios of the volume of subsimplex with one vertex in $\boldsymbol{x}$ and the volume of S (see Picture 1).



Picture 1. 2-simplex $S=[\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3]$ in $\mathbb{R}^2$ divided into 3 subsimplices.

The signed volume $\mathrm{Vol}_k(S)$ is given by $(k+1)\times(k+1)$ determinant

$$\mathrm{Vol}_k(S) = \frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ v_{12} & v_{22} & & v_{k+12} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix},$$

where $\boldsymbol{v}_1=(v_{11},v_{12},\ldots,v_{1k}),\ldots,\boldsymbol{v}_{k+1}=(v_{k+11},v_{k+12},\ldots,v_{k+1k})$ (see [18]).

Since vectors $\boldsymbol{v}_2-\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}-\boldsymbol{v}_1$ are linearly independent, then each $\boldsymbol{x}\in S$ can be written in unique way as convex combination of $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}$ in the form

$$\boldsymbol{x} = \frac{\mathrm{Vol}_k\left([\boldsymbol{x},\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)}\boldsymbol{v}_1+\ldots+\frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)}\boldsymbol{v}_{k+1}. \tag{3.2}$$

Now we present an analog of Theorem 1 for convex functions defines on k-simplices in $\mathbb{R}^k$.

THEOREM 4. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}=(A,...,A):L^k\to\mathbb{R}^k$ a linear operator. Let f be a convex function on k-simplex $S=[\boldsymbol{v}_1,\boldsymbol{v}_2,...,\boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1,...,\lambda_{k+1}$ barycentric coordinates over S. Then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset S$ and $f(\boldsymbol{g})\in L$ we have*

$$\begin{aligned} A(f(\boldsymbol{g})) &\leqslant \sum_{i=1}^{k+1} A(\lambda_i(\boldsymbol{g}))\, f(\boldsymbol{v}_i) \qquad (3.3)\\ &= \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}),\boldsymbol{v}_2,...,\boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1)+...+\frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1,\boldsymbol{v}_2,...,\widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}). \end{aligned}$$

Proof. For each $t\in E$ we have $\boldsymbol{g}(t)\in S$. Then there exist the barycentric coordinates

$$\lambda_1(\boldsymbol{g}(t))=\frac{\mathrm{Vol}_k([\boldsymbol{g}(t),\boldsymbol{v}_2,...,\boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)}=\frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1\\ g_1(t) & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ g_k(t) & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1\\ v_{11} & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}},$$

$$\vdots$$

$$\lambda_{k+1}(\boldsymbol{g}(t))=\frac{\mathrm{Vol}_k([\boldsymbol{v}_1,...,\boldsymbol{v}_k,\boldsymbol{g}(t)])}{\mathrm{Vol}_k(S)}=\frac{\frac{1}{k!}\begin{vmatrix} 1 & \cdots & 1 & 1\\ v_{11} & & v_{k1} & g_1(t)\\ \vdots & & \vdots & \vdots\\ v_{1k} & \cdots & v_{kk} & g_k(t)\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1\\ v_{11} & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}$$

such that $\sum_{i=1}^{k+1}\lambda_i(\boldsymbol{g}(t))=1$ and $\boldsymbol{g}(t)=\sum_{i=1}^{k+1}\lambda_i(\boldsymbol{g}(t))\boldsymbol{v}_i$.

Since f is convex on S, then

$$f(\boldsymbol{g}(t))\leqslant\sum_{i=1}^{k+1}\lambda_i(\boldsymbol{g}(t))f(\boldsymbol{v}_i).$$

Using the Laplace expansion of the determinant we can easily check that $\lambda_i(\boldsymbol{g}) \in L$ for all $i = 1,...,k+1$.

Now, applying A on the last inequality we have

$$A(f(\boldsymbol{g})) \leqslant A\left(\sum_{i=1}^{k+1}\lambda_i(\boldsymbol{g})f(\boldsymbol{v}_i)\right) = \sum_{i=1}^{k+1}A(\lambda_i(\boldsymbol{g}))\,f(\boldsymbol{v}_i), \tag{3.4}$$

where

$$A(\lambda_1(\boldsymbol{g})) = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ A(g_1) & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ A(g_k) & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}} = \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, ..., \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)},$$

$$\vdots \tag{3.5}$$

$$A(\lambda_{k+1}(\boldsymbol{g})) = \frac{\frac{1}{k!}\begin{vmatrix} 1 & \cdots & 1 & 1 \\ v_{11} & & v_{k1} & A(g_1) \\ \vdots & & \vdots & \vdots \\ v_{1k} & \cdots & v_{kk} & A(g_k)\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}} = \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, ..., \boldsymbol{v}_k, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)},$$

By combining (3.4) and (3.5) we obtain (3.3). □

Using Theorem 4 we prove an analog of Theorem 2.

THEOREM 5. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A} = (A,...,A) : L^k \to \mathbb{R}^k$ a linear operator. Let f be a convex function on k-simplex $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1,...,\lambda_{k+1}$ barycentric coordinates over S. If J is an interval in $\mathbb{R}$ such that $f(S) \subset J$ and $F : J \times J \to \mathbb{R}$ an increasing function in the first variable, then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$ we have*

$$\begin{aligned} &F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) \\ &\leqslant \max_{\boldsymbol{x}\in S} F\left(\frac{\mathrm{Vol}_k([\boldsymbol{x}, \boldsymbol{v}_2, ..., \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + ... + \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, ..., \boldsymbol{v}_k, \boldsymbol{x}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}), f(\boldsymbol{x})\right) \\ &= \max_{(\Lambda_1,...,\Lambda_{k+1})\in\Delta^{k+1}} F\left(\sum_{i=1}^{k+1}\Lambda_i f(\boldsymbol{v}_i), f\left(\sum_{i=1}^{k+1}\Lambda_i \boldsymbol{v}_i\right)\right). \end{aligned} \tag{3.6}$$

Proof. Since for each $t \in E$ we have $\boldsymbol{g}(t) \in S$, then it follows $\widetilde{A}(\boldsymbol{g}) \in S$ (see the first part of proof of Theorem 2).

Since $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, by Theorem 4 we have

$$\begin{aligned} & F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) \\ & \leqslant F\left(\frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \ldots + \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}), f(\widetilde{A}(\boldsymbol{g}))\right) \\ & \leqslant \max_{\boldsymbol{x} \in S} F\left(\frac{\mathrm{Vol}_k([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \ldots + \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}), f(\boldsymbol{x})\right). \end{aligned}$$

The equality in (3.6) is simple consequence of substitutions

$$\Lambda_1 = \frac{\mathrm{Vol}_k([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)}, \ldots, \Lambda_{k+1} = \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}])}{\mathrm{Vol}_k(S)},$$

and

$$\boldsymbol{x} = \sum_{i=1}^{k+1} \Lambda_i \boldsymbol{v}_i. \qquad \square$$

REMARK 4. Replacing F by $-F$ in Theorem 5 we can get an analog of Theorem 3 for convex functions defines on k-simplices in $\mathbb{R}^k$.

REMARK 5. If all the assumptions of Theorem 4 are satisfied and in addition f is continuous, then

$$\begin{aligned} f(\widetilde{A}(\boldsymbol{g})) & \leqslant A(f(\boldsymbol{g})) \\ & \leqslant \sum_{i=1}^{k+1} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{v}_i) \\ & = \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \ldots + \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}). \end{aligned} \tag{3.7}$$

The first inequality is consequence of Theorem E and the second of Theorem 4.

EXAMPLE 1. Let $p_1, \ldots, p_{k+1} \geqslant 0$ such that $\sum_{i=1}^{k+1} p_i = 1$. We define the functional $A : L \to \mathbb{R}$ by

$$A(g) = \sum_{i=1}^{k+1} p_i g(t_i).$$

It is obviously that A is positive normalized linear functional on L. Then the linear operator $\widetilde{A} = (A, \ldots, A) : L^k \to \mathbb{R}^k$ is defined by

$$\widetilde{A}(\boldsymbol{g}) = \sum_{i=1}^{k+1} p_i \boldsymbol{g}(t_i).$$

We set $\boldsymbol{g}(t_i) = \boldsymbol{v}_i$ for all $i = 1,...,k+1$. Let $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_{k+1}]$ be k-simplex in $\mathbb{R}^k$ and f be a continuous convex function on S such that $f(\boldsymbol{g}) \in L$. Then as a simple consequence of (3.7) it follows

$$f\left(\sum_{i=1}^{k+1} p_i \boldsymbol{v}_i\right) \leqslant A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^{k+1} p_i f(\boldsymbol{v}_i).$$

Setting $p_1 = ... = p_{k+1} = \frac{1}{k+1}$ we get

$$f\left(\frac{1}{k+1}\sum_{i=1}^{k+1} \boldsymbol{v}_i\right) \leqslant A(f(\boldsymbol{g})) \leqslant \frac{1}{k+1}\sum_{i=1}^{k+1} f(\boldsymbol{v}_i).$$

Related results are obtained in [1], [20].

EXAMPLE 2. Let $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_{k+1}]$ be k-simplex in $\mathbb{R}^k$ and f a continuous convex function on S. Let $L = (E, \mathscr{A}, \lambda)$ be a measure space with positive measure λ. We define the functional $A : L \to \mathbb{R}$ by

$$A(g) = \frac{1}{\lambda(E)}\int_E g(t)\mathrm{d}\lambda(t).$$

It is obviously that A is positive normalized linear functional on L. Then the linear operator $\widetilde{A} = (A, ..., A) : L^k \to \mathbb{R}^k$ is defined by

$$\widetilde{A}(\boldsymbol{g}) = \frac{1}{\lambda(E)}\int_E \boldsymbol{g}(t)\mathrm{d}\lambda(t).$$

We denote $\overline{\boldsymbol{g}} = \frac{1}{\lambda(E)}\int_E \boldsymbol{g}(t)\mathrm{d}\lambda(t)$. If $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$, then from (3.7) it follows

$$\begin{aligned} f(\overline{\boldsymbol{g}}) &\leqslant A(f(\boldsymbol{g})) \\ &\leqslant \frac{\mathrm{Vol}_k([\overline{\boldsymbol{g}}, \boldsymbol{v}_2, ..., \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + ... + \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, ..., \boldsymbol{v}_k, \overline{\boldsymbol{g}}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}), \end{aligned} \tag{3.8}$$

Related results are obtained as consequences of Choquet's theory (see [4], [11], [12], [13], [19]).

4. Related results

In this section we present generalizations of results in [8].

The next theorem generalizes Theorem A.

THEOREM 6. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive linear functional on L and $\widetilde{A} = (A, ..., A) : L^k \to \mathbb{R}^k$ a linear operator. Let $\boldsymbol{x}_1, ..., \boldsymbol{x}_n \in \mathbb{R}^k$ and $K = conv(\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1, ..., \lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L$*

$(i=1,\ldots,n)$ and positive real numbers $p_1,\ldots,p_n$, with $P_n=\sum_{i=1}^n p_i$, satisfying the condition

$$p_i \geqslant A(1) \text{ for all } i=1,\ldots,n, \tag{4.1}$$

we have

$$f\left(\frac{\sum_{i=1}^n p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(1)}\right) \leqslant \frac{\sum_{i=1}^n p_i f(\boldsymbol{x}_i)-\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))f(\boldsymbol{x}_i)}{P_n-A(1)}$$
$$\leqslant \frac{\sum_{i=1}^n p_i f(\boldsymbol{x}_i)-A(f(\boldsymbol{g}))}{P_n-A(1)}. \tag{4.2}$$

Proof. For each $t\in E$ we have $\boldsymbol{g}(t)\in K$. Then there exist barycentric coordinates $\lambda_i(\boldsymbol{g}(t))\geqslant 0$ $(i=1,\ldots,n)$ such that $\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))=1$ and $\boldsymbol{g}(t)=\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i$.

Since f is convex on K, then

$$f(\boldsymbol{g}(t)) \leqslant \sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))f(\boldsymbol{x}_i). \tag{4.3}$$

Applying a positive linear functional A on (4.3) we get

$$A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))f(\boldsymbol{x}_i),$$

where

$$\sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) = A\left(\sum_{i=1}^n \lambda_i(\boldsymbol{g})\right) = A(1)$$

and

$$A(1)\geqslant A(\lambda_i(\boldsymbol{g}))\geqslant 0 \text{ for all } i=1,\ldots,n.$$

Also we have

$$\widetilde{A}(\boldsymbol{g}) = \sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i.$$

Now we can write

$$\frac{\sum_{i=1}^n p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(1)} = \frac{1}{P_n-A(1)}\left(\sum_{i=1}^n p_i\boldsymbol{x}_i-\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i\right)$$
$$= \frac{1}{P_n-A(1)}\sum_{i=1}^n (p_i-A(\lambda_i(\boldsymbol{g})))\boldsymbol{x}_i.$$

We have

$$\frac{1}{P_n-A(1)}\sum_{i=1}^n (p_i-A(\lambda_i(\boldsymbol{g})))=1$$

and

$$\frac{1}{P_n - A(1)} \left(p_i - A\left(\lambda_i(\boldsymbol{g})\right)\right) \geqslant 0 \text{ for all } i = 1, \ldots, n,$$

since

$$p_i \geqslant A(1) \geqslant A\left(\lambda_i(\boldsymbol{g})\right) \text{ for all } i = 1, \ldots, n.$$

Therefore, expression $\frac{\sum_{i=1}^{n} p_i \boldsymbol{x}_i - \widetilde{A}(\boldsymbol{g})}{P_n - A(1)}$ is convex combination of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and belongs to K.

Since f is convex on K, we have

$$\begin{aligned} f\left(\frac{\sum_{i=1}^{n} p_i \boldsymbol{x}_i - \widetilde{A}(\boldsymbol{g})}{P_n - A(1)}\right) &= f\left(\frac{1}{P_n - A(1)} \sum_{i=1}^{n} \left(p_i - A\left(\lambda_i(\boldsymbol{g})\right)\right) \boldsymbol{x}_i\right) \\ &\leqslant \frac{1}{P_n - A(1)} \sum_{i=1}^{n} \left(p_i - A\left(\lambda_i(\boldsymbol{g})\right)\right) f(\boldsymbol{x}_i) \\ &= \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f(\boldsymbol{x}_i)}{P_n - A(1)} \\ &\leqslant \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - A\left(f(\boldsymbol{g})\right)}{P_n - A(1)}. \qquad \square \end{aligned}$$

COROLLARY 1. *Let L satisfy properties L1, L2 on nonempty set E and A be a positive normalized linear functional on L. Let f be a convex function on an interval $I = [m, M] \subset \mathbb{R}$ $(-\infty < m < M < \infty)$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have*

$$\begin{aligned} f\left(m + M - A(g)\right) &\leqslant \frac{A(g) - m}{M - m} f(m) + \frac{M - A(g)}{M - m} f(M) \\ &\leqslant f(m) + f(M) - A(f(g)). \end{aligned} \tag{4.4}$$

Proof. For each $t \in E$ we have $\boldsymbol{g}(t) \in I = [m, M]$.

Since interval $I = [m, M]$ is 1-simplex with vertices m and M, then the barycentric coordinates have the special form:

$$\lambda_1(g(t)) = \frac{M - g(t)}{M - m} \quad \text{and} \quad \lambda_2(g(t)) = \frac{g(t) - m}{M - m}$$

Then applying a functional A we have

$$A(\lambda_1(g)) = \frac{M - A(g)}{M - m} \quad \text{and} \quad A(\lambda_2(g)) = \frac{A(g) - m}{M - m}. \tag{4.5}$$

Choosing $n=2,\ p_1=p_2=1,\ x_1=m,\ x_2=M$ from (4.2) it follows

$$\begin{aligned} f(m+M-A(g)) &\leqslant f(m)+f(M)-\left[\frac{M-A(g)}{M-m}f(m)+\frac{A(g)-m}{M-m}f(M)\right] \\ &= \frac{A(g)-m}{M-m}f(m)+\frac{M-A(g)}{M-m}f(M) \\ &\leqslant f(m)+f(M)-A(f(g)). \qquad \square \end{aligned}$$

REMARK 6. The inequalities in (4.4) are also obtained in [3]. Some related results are obtained in [2].

THEOREM 7. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive linear functional on L and $\widetilde{A}=(A,\dots,A): L^k \to \mathbb{R}^k$ a linear operator. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in \mathbb{R}^k$ and $K=conv(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset K$ and $f(\boldsymbol{g}),\lambda_i(\boldsymbol{g})\in L$ $(i=1,\dots,n)$ and positive real numbers $p_1,\dots,p_n$ satisfying the conditions $P_n-A(1)>0$, where $P_n=\sum_{i=1}^n p_i$, and*

$$\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(1)} \in K, \tag{4.6}$$

we have

$$\begin{aligned} f\left(\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(1)}\right) &\geqslant \frac{P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\boldsymbol{x}_i\right)-A(1)f\left(\frac{1}{A(1)}\widetilde{A}(\boldsymbol{g})\right)}{P_n-A(1)} \\ &\geqslant \frac{P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\boldsymbol{x}_i\right)-\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))f(\boldsymbol{x}_i)}{P_n-A(1)}. \end{aligned} \tag{4.7}$$

Proof. For each $t\in E$ we have $\boldsymbol{g}(t)\in K$. Then there exist barycentric coordinates $\lambda_i(\boldsymbol{g}(t))\geqslant 0$ $(i=1,\dots,n)$ such that $\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))=1$ and

$$\boldsymbol{g}(t)=\sum_{i=1}^{n}\lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Also we have

$$\widetilde{A}(\boldsymbol{g})=\sum_{i=1}^{n}A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i.$$

We can easily see that

$$\frac{1}{A(1)}\widetilde{A}(\boldsymbol{g})=\frac{1}{A(1)}\sum_{i=1}^{n}A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i\in K,$$

since

$$\frac{1}{A(1)}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)=1 \quad \text{and} \quad \frac{1}{A(1)}A\left(\lambda_i(\boldsymbol{g})\right)\geqslant 0, \quad i=1,\ldots,n.$$

Since f is convex on K, then

$$f\left(\frac{1}{A(1)}\widetilde{A}\left(\boldsymbol{g}\right)\right)\leqslant \frac{1}{A(1)}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)f(\boldsymbol{x}_i). \tag{4.8}$$

Using first (1.2) and then (4.8) we have

$$f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right)-A(1)\left(\frac{1}{A(1)}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{P_n-A\left(1\right)}\right)\geqslant \frac{P_nf\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right)-A(1)f\left(\frac{1}{A(1)}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{P_n-A\left(1\right)}$$

$$\geqslant \frac{P_nf\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right)-A(1)\frac{1}{A(1)}\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)f\left(\boldsymbol{x}_i\right)}{P_n-A\left(1\right)}. \qquad \square$$

REMARK 7. If positive real numbers $p_1,...,p_n$ satisfy the condition (4.1), then the condition (4.6) is also satisfied since K is convex set. Then (4.2) can be extended as follows

$$\begin{aligned}
\frac{P_nf\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right)-\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)f\left(\boldsymbol{x}_i\right)}{P_n-A\left(1\right)} &\leqslant \frac{P_nf\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right)-A(1)f\left(\frac{1}{A(1)}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{P_n-A\left(1\right)}\\
&\leqslant f\left(\frac{\sum_{i=1}^{n}p_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{P_n-A\left(1\right)}\right)\\
&\leqslant \frac{\sum_{i=1}^{n}p_if\left(\boldsymbol{x}_i\right)-\sum_{i=1}^{n}A\left(\lambda_i(\boldsymbol{g})\right)f\left(\boldsymbol{x}_i\right)}{P_n-A\left(1\right)}\\
&\leqslant \frac{\sum_{i=1}^{n}p_if\left(\boldsymbol{x}_i\right)-A(f(\boldsymbol{g}))}{P_n-A\left(1\right)}.
\end{aligned} \tag{4.9}$$

COROLLARY 2. *Let L satisfy properties L1, L2 on nonempty set E and A be a positive normalized linear functional on L. Let f be a convex function on an interval $I=[m,M]\subset\mathbb{R}$ $(-\infty<m<M<\infty)$. Then for all $g\in L$ such that $g(E)\subset I$ and $f(g)\in L$, we have*

$$\begin{aligned}
f\left(m+M-A(g)\right)&\geqslant 2f\left(\frac{m+M}{2}\right)-f\left(A(g)\right)\\
&\geqslant 2f\left(\frac{m+M}{2}\right)-\left[\frac{M-A(g)}{M-m}f(m)+\frac{A(g)-m}{M-m}f(M)\right].
\end{aligned} \tag{4.10}$$

Proof. Choosing $n=2,\ x_1=m,\ x_2=M,\ p_1=p_2=1$ and using (4.5), the inequalities in (4.10) easily follows from (4.7). $\square$

Next we give generalizations of Corollary 1 and Corollary 2 for convex functions defined on k-simplices in $\mathbb{R}^k$.

COROLLARY 3. *Let L satisfy properties L1, L2 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}=(A,...,A):L^k\to\mathbb{R}^k$ a linear operator. Let f be a convex function on k-simplex $S=[\boldsymbol{v}_1,\boldsymbol{v}_2,...,\boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1,...,\lambda_{k+1}$ barycentric coordinates over S. Then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset S$ and $f(\boldsymbol{g})\in L$ we have*

$$\begin{aligned}
&\frac{(k+1)f\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\boldsymbol{v}_i\right)-\sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(\boldsymbol{g}))f(\boldsymbol{v}_i)}{k}\\
&\leqslant\frac{(k+1)f\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\boldsymbol{v}_i\right)-f(\widetilde{A}(\boldsymbol{g}))}{k}\\
&\leqslant f\left(\frac{\sum_{i=1}^{k+1}\boldsymbol{v}_i-\widetilde{A}(\boldsymbol{g})}{k}\right)\\
&\leqslant\frac{\sum_{i=1}^{k+1}f(\boldsymbol{v}_i)-\sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(\boldsymbol{g}))f(\boldsymbol{v}_i)}{k}\\
&\leqslant\frac{\sum_{i=1}^{k+1}f(\boldsymbol{v}_i)-A(f(\boldsymbol{g}))}{k}.
\end{aligned}\tag{4.11}$$

Proof. Since barycentric coordinates $\lambda_1,...,\lambda_{k+1}$ over k-simplex S in $\mathbb{R}^k$ are nonnegative linear polynomials, then $A(\lambda_i(\boldsymbol{g}))=\lambda_i(\widetilde{A}(\boldsymbol{g}))$ for all $i=1,...,k+1$.

Choosing $\boldsymbol{x}_i=\boldsymbol{v}_i$ for all $i=1,...,k+1$ and $p_1=p_2=...=p_{k+1}=1$, the inequalities in (4.11) easily follow from (4.2) and (4.7). $\square$

REFERENCES

[1] M. BESSENYEI, *The Hermite-Hadamard inequality on Simplices,* Amer. Math. Monthly, **115**, 4 (2008), 339–345.

[2] B. GAVREA, J. JAKŠETIĆ AND J. PEČARIĆ, *On a global upper bound for Jessen' s inequality,* The Australian & New Zeland Industrial and Applied Mathematics Journal, **50** (2009), 246–257.

[3] W. S. CHEUNG, A. MATKOVIĆ AND J. PEČARIĆ, *A variant of Jessen's inequality and generalized means*, J. Ineq. Pure and Appl. Math., **7**, 1 (2006).

[4] A. FLOREA AND C. P. NICULESCU, *A Hermite-Hadamard inequality for convex-concave symmetric functions,* Bull. Math. Soc. Sci. Math. Roumanie Tome, **50**, 98 (2007), 149–156.

[5] A. GUESSAB, G. SCHMEISSER, *A Definiteness Theory for Cubature Formulae of Order Two*, Constructive Approximation, **24**, 4 (2006), 263–288.

[6] A. GUESSAB, G. SCHMEISSER, *Construction of positive definite cubature formulae and approximation of functions via Voronoi tessellations*, Advances in Computational Mathematics, **32**, 4 (2010), 25–41.

[7] A. GUESSAB, G. SCHMEISSER, *Negative Definite Cubature Formulae, Extremality and Delaunay Triangulation*, Constructive Approximation, **31**, 4 (2010), 95–113.

[8] A. MATKOVIĆ, J. PEČARIĆ, *A variant of Jensen's inequality for convex functions of several variables*, J. Math. Ineq., **1**, 1 (2007), 45–51.

[9] E. J. MCSHANE, *Jensen's inequality*, Bull. Amer. Math. Soc., **43** (1937), 521–527.

[10] D. S. MITRINOVIĆ, J. PEČARIĆ AND A. M. FINK, *Classical and new inequalities in analysis,* Kluwer Academic Publishers, The Netherlands, 1993.

[11] C. P. NICULESCU, *The Hermite-Hadamard inequality for convex functions of a vector variable*, Math. Ineq. Appl., **5** (2002).

[12] C. P. NICULESCU, *The Hermite-Hadamard inequality for convex functions on a global NPC space,* J. Math. Anal. Appl., **356** (2009).

[13] C. P. NICULESCU AND L.-E. PERSSON, *Old and new on the Hermite-Hadamard inequality,* Real Analysis Exchange, **29**, 2 (2003/2004), 663–685.

[14] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, 1992.

[15] S. SIMIĆ, *On a converse of Jensen's discrete inequality,* J. Inequal. and Appl., **2009** (2009), Article ID 153080, 6 pages.

[16] S. SIMIĆ, *On a new converse of Jensen's inequaity,* Pub. De L'Instit. Mathem., **85**, 99 (2009), 107–110.

[17] S. SIMIĆ, *On an upper bound for Jensen's inequality,* J. Inequal. Pure Appl. Math., **10**, 2 (2009), Art. 60.

[18] D. M. Y. SOMMERVILLE, *An Introduction to the Geometry of N Dimensions,* Reprinted by Dover Press (1958).

[19] T. TRIF, *Characterizations of convex functions of a vector variable via Hermite-Hadamard's inequality,* J. Math. Ineq., **2**, 1 (2008), 37–44.

[20] S. WĄSOWICZ, *Hermite-Hadamard-type inequalities in the aproximate integration,* Math. Ineq. Appl., **11** (2008), 693–700.

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