GENERALIZATIONS OF CONVERSE JENSEN’S INEQUALITY AND RELATED RESULTS

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Abstract. In this paper we prove generalizations of Converse Jensen’s inequality for convex functions defined on convex hulls. As consequences we get generalizations of the Hermite-Hadamard inequality for convex functions defined on \( k \)-simplices in \( \mathbb{R}^k \). We also present some related results which generalize results in [8].

1. Introduction

Let \( U \) be a convex subset of \( \mathbb{R}^k \) and \( n \in \mathbb{N} \). If \( f : U \to \mathbb{R} \) is a convex function, \( x_1, \ldots, x_n \in U \) and \( p_1, \ldots, p_n \) nonnegative real numbers with \( P_n = \sum_{i=1}^{n} p_i \), then the well known Jensen’s inequality

\[
f \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i)
\]

holds.

If the following conditions are satisfied

\[
p_1 > 0, \quad p_i \leq 0 \quad (i = 2, \ldots, n) \quad P_n > 0,
\]

then Reversed Jensen’s inequality

\[
f \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i)
\]

holds (see [14]).

The convex hull of vectors \( x_1, \ldots, x_n \in \mathbb{R}^k \) is represented by \( K = \text{conv}(\{x_1, \ldots, x_n\}) \).

Barycentric coordinates over \( K \) are continuous functions \( \lambda_1, \lambda_2, \ldots, \lambda_n \) on \( K \) with following properties:

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\end{itemize}
\begin{enumerate}
    \item $\lambda_i(x) \geq 0$, \quad $i = 1, \ldots, n$,
    \item $\sum_{i=1}^{n} \lambda_i(x) = 1$,
    \item $x = \sum_{i=1}^{n} \lambda_i(x)x_i$.
\end{enumerate}

If $x_2 - x_1, \ldots, x_n - x_1$ are linearly independent vectors, then each $x \in K$ can be written in unique way as convex combination of $x_1, \ldots, x_n$ in the form (3).

We also consider $k$-simplex $S = [v_1, \ldots, v_{k+1}]$ in $\mathbb{R}^k$ which is convex hull of its vertices $v_1, v_2, \ldots, v_{k+1} \in \mathbb{R}^k$. Barycentric coordinates $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ over $S$ are nonnegative linear polynomials on $S$ and have special form (see the third section).

The next variant of Jensen’s inequality was proved by A. Matković and J. Pečarić [8].

**THEOREM A.** Let $U$ be a convex subset in $\mathbb{R}^k$, $x_1, \ldots, x_n \in U$ and $y_1, \ldots, y_m \in \text{conv}(\{x_1, \ldots, x_n\})$. If $f$ is a convex function on $U$, then the inequality

$$f \left( \frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j y_j}{P_n - W_m} \right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(y_j)}{P_n - W_m}$$  (1.3)

holds for all positive real numbers $p_1, \ldots, p_n$ and $w_1, \ldots, w_m$ satisfying the condition

$$p_i \geq W_m \quad \text{for all} \quad i = 1, \ldots, n,$$

where $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$.

In the following, let $E$ be a nonempty set and $L$ be a linear class of functions $f : E \to \mathbb{R}$ having the properties:

(L1) if $f, g \in L$ then $(af + bg) \in L$ for all $a, b \in \mathbb{R}$

(L2) $1 \in L$ where $1(t) = 1$ for all $t \in E$.

We consider positive linear functionals $A : L \to \mathbb{R}$. That is, we assume:

(A1) $A(af + bg) = aA(f) + bA(g)$ for all $f, g \in L$, $a, b \in \mathbb{R}$ (linearity)

(A2) if $f \in L$, $f(t) \geq 0$ for all $t \in E$ then $A(f) \geq 0$ (positivity).

From (A1) we obtain

(A1’) $A \left( \sum_{i=1}^{k} a_i g_i \right) = \sum_{i=1}^{k} a_i A(g_i)$ for $g_1, \ldots, g_k \in L$, $a_1, \ldots, a_k \in \mathbb{R}$ (linearity).
If in addition $A(1) = 1$ is satisfied, we say that $A$ is a positive normalized linear functional.

With $L^k$ we denote a linear class of functions $g : E \to \mathbb{R}^k$ defined by

$$g(t) = (g_1(t), \ldots, g_k(t)), g_i \in L \ (i = 1, \ldots, k).$$

We also consider linear operators $\tilde{A} : L^k \to \mathbb{R}^k$ defined by

$$\tilde{A}(g) = (A(g_1), \ldots, A(g_k)).$$

If $A(1) = 1$ is satisfied, then using (A1) we also have

(A3) $A(f(g)) = f(\tilde{A}(g))$ for every linear function $f$ on $\mathbb{R}^k$.

Next we introduce the functional versions of Jensen’s inequality and some related results which we generalize in sequel.

B. Jessen [14, p. 47] gave the following generalization of Jensen’s inequality for positive linear functionals.

**Theorem B.** (Jessen’s inequality) Let $L$ satisfy properties L1, L2 on nonempty set $E$ and $A$ be a positive normalized linear functional on $L$. Let $f$ be a continuous convex function on an interval $I \subset \mathbb{R}$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have $A(g) \in I$ and

$$f(A(g)) \leq A(f(g)). \quad (1.4)$$

The next theorem, proved by J. Pečarić and P. R. Beesack, presents generalization of Theorem Lah-Ribarić (see [10, p. 98], [14, p. 98]).

**Theorem C.** (Converse Jessen’s inequality) Let $L$ satisfy properties L1, L2 and $A$ be a positive normalized linear functional on $L$. Let $f$ be a convex function on an interval $I = [m, M] \subset \mathbb{R}$ ($-\infty < m < M < \infty$). Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have

$$A(f(g)) \leq \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M). \quad (1.5)$$

Using Theorem C, Beesack and Pečarić also proved the next result [14, p. 101].

**Theorem D.** Let $L$, $A$ and $f$ be as in Theorem C. Let $J$ be an interval in $\mathbb{R}$ such that $f(J) \subset J$. If $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have

$$F(A(f(g)), f(A(g))) \leq \max_{x \in [m, M]} F \left( \frac{M - x}{M - m} f(m) + \frac{x - m}{M - m} f(M), f(x) \right) = \max_{\theta \in [0,1]} F \left( \theta f(m) + (1 - \theta) f(M), \theta m + (1 - \theta) M \right). \quad (1.6)$$
REMARK 1. If we choose $F(x, y) = x - y$, as a simple consequence of Theorem D it follows

$$A(f(g)) - f(A(g)) \leq \max_{\theta \in [0, 1]} \left[ \theta f(m) + (1 - \theta)f(M) - f(\theta m + (1 - \theta)M) \right].$$  \hspace{1cm} (1.7)

Choosing $F(x, y) = \frac{x}{y}$, it follows

$$\frac{A(f(g))}{f(A(g))} \leq \max_{\theta \in [0, 1]} \left[ \frac{\theta f(m) + (1 - \theta)f(M)}{f(\theta m + (1 - \theta)M)} \right].$$ \hspace{1cm} (1.8)

It is obviously that the main results in [15], [16] and [17] can be obtained as direct consequences of Theorem D published many years earlier.

Additional generalization of Jessen’s inequality (1.4) is proved by E. J. McShane (see [9], [14, p. 48]).

**THEOREM E.** (McShane’s inequality) Let $L$ satisfy properties L1, L2, $A$ be a positive normalized linear functional on $L$ and $\tilde{A} = (A, ..., A) : L^k \to \mathbb{R}^k$ a linear operator. Let $f$ be a continuous convex function on a closed convex set $U \subset \mathbb{R}^k$. Then for all $g \in L^k$ such that $g(E) \subset U$ and $f(g) \in L$, we have that $\tilde{A}(g) \in U$ and

$$f(\tilde{A}(g)) \leq A(f(g)).$$ \hspace{1cm} (1.9)

It is known that for a convex function $f : [a, b] \to \mathbb{R}$ the Hermite-Hadamard inequality

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$ \hspace{1cm} (1.10)

holds.

In this paper, as our main results we present generalizations of Theorem C and Theorem D for convex functions defined on convex hulls. As consequences, we obtain generalizations of the Hermite-Hadamard inequality (1.10) for convex functions defined on $k$-simplices in $\mathbb{R}^k$. Some related results can be found in [5], [6], [7]. We also present related results which generalize results in [8].

2. Main results

For $n \in \mathbb{N}$ we denote

$$\Delta^n = \left\{ (\Lambda_1, ..., \Lambda_n) : \Lambda_i \geq 0, \forall i \in \{1, ..., n\}, \sum_{i=1}^{n} \Lambda_i = 1 \right\}.$$

The next theorem presents generalization of Theorem C.
THEOREM 1. Let \( L \) satisfy properties \( L1, L2 \) on nonempty set \( E \) and \( A \) be a positive normalized linear functional on \( L \). Let \( f \) be a convex function on \( K \) and \( \lambda_1, \ldots, \lambda_n \) barycentric coordinates over \( K \). Then for all \( g \in L^k \) such that \( g(E) \subset K \) and \( f(g), \lambda_i(g) \in L \) \((i = 1, \ldots, n)\) we have

\[
A(f(g)) \leq \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i).
\]

(2.1)

Proof. For each \( t \in E \) we have \( g(t) \in K \). Then there exist barycentric coordinates \( \lambda_i(g(t)) \geq 0 \) \((i = 1, \ldots, n)\) such that \( \sum_{i=1}^{n} \lambda_i(g(t)) = 1 \) and

\[
g(t) = \sum_{i=1}^{n} \lambda_i(g(t)) x_i.
\]

Since \( f \) is convex on \( K \), then

\[
f(g(t)) = f \left( \sum_{i=1}^{n} \lambda_i(g(t)) x_i \right) \leq \sum_{i=1}^{n} \lambda_i(g(t)) f(x_i).
\]

Now, applying a functional \( A \) on the last inequality we get

\[
A(f(g)) \leq A \left( \sum_{i=1}^{n} \lambda_i(g) f(x_i) \right) = \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i). \quad \square
\]

REMARK 2. If all the assumptions of Theorem 1 are satisfied and in addition \( f \) is continuous, then

\[
f(\tilde{A}(g)) \leq A(f(g)) \leq \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i)
\]

The first inequality is consequence of Theorem E and the second of Theorem 1.

Using Theorem 1 we prove generalization of Theorem E and the second of Theorem 1.

THEOREM 2. Let \( L \) satisfy properties \( L1, L2 \) on nonempty set \( E \), \( A \) be a positive normalized linear functional on \( L \) and \( \Lambda = (A_1, \ldots, A) : L^k \to \mathbb{R}^k \) a linear operator. Let \( x_1, \ldots, x_n \in \mathbb{R}^k \) and \( K = \text{conv}(\{x_1, \ldots, x_n\}) \). Let \( f \) be a convex function on \( K \) and \( \lambda_1, \ldots, \lambda_n \) barycentric coordinates over \( K \). If \( J \) is an interval in \( \mathbb{R} \) such that \( f(K) \subset J \) and \( F : J \times J \to \mathbb{R} \) is an increasing function in the first variable, then for all \( g \in L^k \) such that \( g(E) \subset K \) and \( f(g), \lambda_i(g) \in L \) \((i = 1, \ldots, n)\) we have

\[
F \left( A(f(g)), f(\tilde{A}(g)) \right) \leq F \left( \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i), f(\tilde{A}(g)) \right) \leq \max_{(\Lambda_1, \ldots, \Lambda_n) \in \Lambda^n} F \left( \sum_{i=1}^{n} \Lambda_i f(x_i), f \left( \sum_{i=1}^{n} \Lambda_i x_i \right) \right).
\]

(2.2)
Proof. For each $t \in E$ we have $g(t) \in K$. Then there exist barycentric coordinates $\lambda_i(g(t)) \geq 0$ ($i = 1, ..., n$) such that $\sum_{i=1}^{n} \lambda_i(g(t)) = 1$ and

$$g(t) = \sum_{i=1}^{n} \lambda_i(g(t)) x_i.$$ 

Since $A$ is a positive normalized linear functional on $L$ and $\tilde{A} = (A, ..., A)$ a linear operator on $L^k$, we have

$$\tilde{A}(g) = (A(g_1), ..., A(g_k)) = \sum_{i=1}^{n} A(\lambda_i(g)) x_i,$$

where

$$A(\lambda_i(g)) \geq 0, \quad i = 1, ..., n$$

and

$$\sum_{i=1}^{n} A(\lambda_i(g)) = A \left( \sum_{i=1}^{n} \lambda_i(g) \right) = A(1) = 1.$$ 

Therefore, $\tilde{A}(g) \in K$.

Since $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, using (2.1) we have

$$F \left( A(f(g)), f(\tilde{A}(g)) \right) \leq F \left( \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i), f(\tilde{A}(g)) \right). \quad (2.3)$$

By substitutions

$$A(\lambda_i(g)) = \Lambda_i \quad (i = 1, ..., n),$$

it follows

$$\tilde{A}(g) = \sum_{i=1}^{n} \Lambda_i x_i.$$ 

Now we have

$$F \left( \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i), f(\tilde{A}(g)) \right) = F \left( \sum_{i=1}^{n} \Lambda_i f(x_i), f \left( \sum_{i=1}^{n} \Lambda_i x_i \right) \right) \leq \max_{(\Lambda_1, ..., \Lambda_n) \in \Delta^n} F \left( \sum_{i=1}^{n} \Lambda_i f(x_i), f \left( \sum_{i=1}^{n} \Lambda_i x_i \right) \right). \quad (2.4)$$

By combining (2.3) and (2.4) we get (2.2). \hfill \Box

Remark 3. If we choose $F(x, y) = x - y$, as a simple consequence of Theorem 2 it follows

$$A(f(g)) - f(\tilde{A}(g)) \leq \max_{(\Lambda_1, ..., \Lambda_n) \in \Delta^n} \left( \sum_{i=1}^{n} \Lambda_i f(x_i) - f \left( \sum_{i=1}^{n} \Lambda_i x_i \right) \right). \quad (2.5)$$
Choosing $F(x,y) = \frac{x}{y}$, it follows
\[
\frac{A(f(g))}{f(A(g))} \leq \max_{(\Lambda_1, \ldots, \Lambda_n) \in \Delta^n} \left( \frac{\sum_{i=1}^{n} \Lambda_i f(x_i)}{f\left(\sum_{i=1}^{n} \Lambda_i x_i\right)}\right).
\tag{2.6}
\]

The inequalities (2.5) and (2.6) present generalizations of (1.7) and (1.8).

Replacing $F$ by $-F$ in Theorem 2 we get the next theorem.

**Theorem 3.** Let $L$ satisfy properties $L_1, L_2$ on nonempty set $E$, $A$ be a positive normalized linear functional on $L$ and $\tilde{A} = (A_1, \ldots, A_n) : L^k \to \mathbb{R}^k$ a linear operator. Let $x_1, \ldots, x_n \in \mathbb{R}^k$ and $K = \text{conv} \{x_1, \ldots, x_n\}$. Let $f$ be a convex function on $K$ and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over $K$. If $J$ is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F : J \times J \to \mathbb{R}$ is an decreasing function in the first variable, then for all $g \in L^k$ such that $g(E) \subset K$ and $f(g), \lambda_i(g) \in L$ ($i = 1, \ldots, n$) we have
\[
F \left( A(f(g)), f(\tilde{A}(g)) \right) \geq F \left( \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i), f(\tilde{A}(g)) \right) \\
\geq \min_{(\Lambda_1, \ldots, \Lambda_n) \in \Delta^n} F \left( \sum_{i=1}^{n} \Lambda_i f(x_i), f\left(\sum_{i=1}^{n} \Lambda_i x_i\right)\right).
\]

**3. Convex functions on $k$-simplices in $\mathbb{R}^k$**

In this section we give analogs to Theorem 1 and Theorem 2 for convex functions defined on $k$-simplices in $\mathbb{R}^k$. As a consequence we obtain generalizations of the Hermite-Hadamard inequality (1.10).

Let $S = [v_1, v_2, \ldots, v_{k+1}]$ be $k$-simplex in $\mathbb{R}^k$ with vertices $v_1, v_2, \ldots, v_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1, \ldots, \lambda_{k+1}$ over $S$ are nonnegative linear polynomials that satisfy Lagrange's property:

\[
\lambda_i(v_j) = \delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j 
\end{cases}.
\]

Therefore, it is known that for each $x \in S$ the barycentric coordinates $\lambda_1(x), \ldots,
\[ \lambda_{k+1}(\mathbf{x}) \] have the form

\[
\begin{align*}
\lambda_1(\mathbf{x}) &= \frac{\text{Vol}_k([\mathbf{x}, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)}, \\
\lambda_2(\mathbf{x}) &= \frac{\text{Vol}_k([\mathbf{v}_1, \mathbf{x}, \mathbf{v}_3, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)}, \\
&\vdots \\
\lambda_{k+1}(\mathbf{x}) &= \frac{\text{Vol}_k([\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x}])}{\text{Vol}_k(S)},
\end{align*}
\] (3.1)

where \( \text{Vol}_k \) denotes \( k \)-dimensional Lebesgue measure on \( S \).

Here, for example, \([\mathbf{v}_1, \mathbf{x}, \ldots, \mathbf{v}_{k+1}] \) denotes the subsimplex obtained by replacing \( \mathbf{v}_2 \) by \( \mathbf{x} \), i.e. the subsimplex opposite to \( \mathbf{v}_2 \), when adding \( \mathbf{x} \) as a new vertex.

In other words, we see that the barycentric coordinates \( \lambda_1, \ldots, \lambda_{k+1} \) for each \( \mathbf{x} \in S \) can be presented as the ratios of the volume of subsimplex with one vertex in \( \mathbf{x} \) and the volume of \( S \) (see Picture 1).

\begin{center}
\begin{tikzpicture}
  \filldraw[blue!20] (0,0) -- (1,0) -- (0.5,1) -- cycle;
  \filldraw[green!20] (0,0) -- (0.5,1) -- (1,0) -- cycle;
  \filldraw[red!20] (0,0) -- (0.5,1) -- (0.5,0) -- cycle;
  \draw (0,0) -- (1,0) -- (0.5,1) -- cycle;
  \draw (0.5,0) -- (0.5,1);
  \node at (0.25,0.5) {\( \mathbf{x} \)};
  \node at (0,0) {\( \mathbf{v}_1 \)};
  \node at (1,0) {\( \mathbf{v}_2 \)};
  \node at (0.5,1) {\( \mathbf{v}_3 \)};
\end{tikzpicture}
\end{center}

\textit{Picture 1.} 2-simplex \( S = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \) in \( \mathbb{R}^2 \) divided into 3 subsimplices.

The signed volume \( \text{Vol}_k(S) \) is given by \((k+1) \times (k+1)\) determinant

\[
\text{Vol}_k(S) = \frac{1}{k!} \begin{vmatrix}
1 & 1 & \cdots & 1 \\
\mathbf{v}_{11} & \mathbf{v}_{21} & \cdots & \mathbf{v}_{k+11} \\
\mathbf{v}_{12} & \mathbf{v}_{22} & \cdots & \mathbf{v}_{k+12} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{v}_{1k} & \mathbf{v}_{2k} & \cdots & \mathbf{v}_{k+1k}
\end{vmatrix},
\]

where \( \mathbf{v}_1 = (v_{11}, v_{12}, \ldots, v_{1k}), \ldots, \mathbf{v}_{k+1} = (v_{k+11}, v_{k+12}, \ldots, v_{k+1k}) \) (see [18]).

Since vectors \( \mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_{k+1} - \mathbf{v}_1 \) are linearly independent, then each \( \mathbf{x} \in S \) can be written in unique way as convex combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_{k+1} \) in the form

\[
\mathbf{x} = \frac{\text{Vol}_k([\mathbf{x}, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)} \mathbf{v}_1 + \ldots + \frac{\text{Vol}_k([\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x}])}{\text{Vol}_k(S)} \mathbf{v}_{k+1}.
\] (3.2)
Now we present an analog of Theorem 1 for convex functions defined on $k$-simplices in $\mathbb{R}^k$.

**Theorem 4.** Let $L$ satisfy properties $L1, L2$ on nonempty set $E$, $A$ be a positive normalized linear functional on $L$ and $\tilde{A} = (A, ..., A) : L^k \rightarrow \mathbb{R}^k$ a linear operator. Let $f$ be a convex function on $k$-simplex $S = [v_1, v_2, ..., v_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, ..., \lambda_{k+1}$ barycentric coordinates over $S$. Then for all $g \in L^k$ such that $g(E) \subset S$ and $f(g) \in L$ we have

$$A(f(g)) \leq \sum_{i=1}^{k+1} A(\lambda_i(g)) f(v_i)$$

(3.3)

$$= \frac{\text{Vol}_k\left(\tilde{A}(g), v_2, ..., v_{k+1}\right)}{\text{Vol}_k(S)} f(v_1) + ... + \frac{\text{Vol}_k\left(v_1, v_2, ..., \tilde{A}(g)\right)}{\text{Vol}_k(S)} f(v_{k+1}).$$

**Proof.** For each $t \in E$ we have $g(t) \in S$. Then there exist the barycentric coordinates

$$\lambda_1(g(t)) = \frac{\text{Vol}_k\left(g(t), v_2, ..., v_{k+1}\right)}{\text{Vol}_k(S)} = \frac{1}{k!} \begin{vmatrix} 1 & 1 & \cdots & 1 \\
1 & g_1(t) & v_2 & v_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
g_k(t) & v_{2k} & \cdots & v_{k+1k}\end{vmatrix},$$

$$\vdots$$

$$\lambda_{k+1}(g(t)) = \frac{\text{Vol}_k\left(v_1, ..., v_k, g(t)\right)}{\text{Vol}_k(S)} = \frac{1}{k!} \begin{vmatrix} 1 & 1 & \cdots & 1 \\
1 & v_{11} & v_{k1} & g_1(t) \\
\vdots & \vdots & \ddots & \vdots \\
v_{1k} & \cdots & v_{kk} & g_k(t) \\
v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix},$$

such that $\sum_{i=1}^{k+1} \lambda_i(g(t)) = 1$ and $g(t) = \sum_{i=1}^{k+1} \lambda_i(g(t)) v_i$.

Since $f$ is convex on $S$, then

$$f(g(t)) \leq \sum_{i=1}^{k+1} \lambda_i(g(t)) f(v_i).$$
Using the Laplace expansion of the determinant we can easily check that $\lambda_i(g) \in L$ for all $i = 1, \ldots, k + 1$.

Now, applying $A$ on the last inequality we have

$$A(f(g)) \leq A \left( \sum_{i=1}^{k+1} \lambda_i(g) f(v_i) \right) = \sum_{i=1}^{k+1} A(\lambda_i(g)) f(v_i), \quad (3.4)$$

where

$$A(\lambda_1(g)) = \frac{1}{k!} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & \cdots & v_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1} \end{vmatrix} = \frac{\text{Vol}_k \left( [\tilde{A}(g), v_2, \ldots, v_{k+1}] \right)}{\text{Vol}_k(S)}, \quad \text{and}$$

$$A(\lambda_{k+1}(g)) = \frac{1}{k!} \begin{vmatrix} 1 & \cdots & 1 & 1 \\ v_{11} & v_{k1} & A(g_1) & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1} \end{vmatrix} = \frac{\text{Vol}_k \left( [v_1, \ldots, v_k, \tilde{A}(g)] \right)}{\text{Vol}_k(S)},$$

$$\vdots$$

By combining (3.4) and (3.5) we obtain (3.3). \(\square\)

Using Theorem 4 we prove an analog of Theorem 2.

**Theorem 5.** Let $L$ satisfy properties L1, L2 on nonempty set $E$, $A$ be a positive normalized linear functional on $L$ and $\tilde{A} = (A_1, \ldots, A) : L^k \to \mathbb{R}^k$ a linear operator. Let $f$ be a convex function on $k$-simplex $S = [v_1, v_2, \ldots, v_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ barycentric coordinates over $S$. If $J$ is an interval in $\mathbb{R}$ such that $f(S) \subset J$ and $F : J \times J \to \mathbb{R}$ an increasing function in the first variable, then for all $g \in L^k$ such that $g(E) \subset S$ and $f(g) \in L$ we have

$$F \left( A(f(g)), f(\tilde{A}(g)) \right) \leq \max_{x \in S} F \left( \frac{\text{Vol}_k([x, v_2, \ldots, v_{k+1}])}{\text{Vol}_k(S)} f(v_1) + \ldots + \frac{\text{Vol}_k([v_1, \ldots, v_k, x])}{\text{Vol}_k(S)} f(v_{k+1}), f(x) \right)$$

$$= \max_{(\Lambda_1, \ldots, \Lambda_{k+1}) \in \Delta^{k+1}} \left( \sum_{i=1}^{k+1} \Lambda_i f(v_i) \right) \left( \sum_{i=1}^{k+1} \Lambda_i f(v_i) \right), \quad (3.6)$$
Proof. Since for each \( t \in E \) we have \( \mathbf{g}(t) \in S \), then it follows \( \tilde{A}(\mathbf{g}) \in S \) (see the first part of proof of Theorem 2).

Since \( F : J \times J \to \mathbb{R} \) is an increasing function in the first variable, by Theorem 4 we have

\[
F \left( A(f(\mathbf{g})), f(\tilde{A}(\mathbf{g})) \right) \\
\leq F \left( \frac{\text{Vol}_k([\tilde{A}(\mathbf{g}), \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)} f(\mathbf{v}_1) + \ldots + \frac{\text{Vol}_k([\mathbf{v}_1, \ldots, \mathbf{v}_k, \tilde{A}(\mathbf{g})])}{\text{Vol}_k(S)} f(\mathbf{v}_{k+1}), f(\tilde{A}(\mathbf{g})) \right) \\
\leq \max_{x \in S} F \left( \frac{\text{Vol}_k([x, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)} f(\mathbf{v}_1) + \ldots + \frac{\text{Vol}_k([\mathbf{v}_1, \ldots, \mathbf{v}_k, x])}{\text{Vol}_k(S)} f(\mathbf{v}_{k+1}), f(x) \right).
\]

The equality in (3.6) is simple consequence of substitutions

\[
\Lambda_1 = \frac{\text{Vol}_k([\tilde{A}(\mathbf{g}), \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)}, \ldots, \Lambda_{k+1} = \frac{\text{Vol}_k([\mathbf{v}_1, \ldots, \mathbf{v}_k, x])}{\text{Vol}_k(S)},
\]

and

\[
x = \sum_{i=1}^{k+1} \Lambda_i \mathbf{v}_i. \quad \square
\]

**Remark 4.** Replacing \( F \) by \(-F\) in Theorem 5 we can get an analog of Theorem 3 for convex functions defined on \( k \)-simplices in \( \mathbb{R}^k \).

**Remark 5.** If all the assumptions of Theorem 4 are satisfied and in addition \( f \) is continuous, then

\[
f(\tilde{A}(\mathbf{g})) \leq A(f(\mathbf{g}))
\]

\[
\leq \sum_{i=1}^{k+1} A(\lambda_i(\mathbf{g})) f(\mathbf{v}_i)
\]

\[
= \frac{\text{Vol}_k([\tilde{A}(\mathbf{g}), \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)} f(\mathbf{v}_1) + \ldots + \frac{\text{Vol}_k([\mathbf{v}_1, \ldots, \mathbf{v}_k, \tilde{A}(\mathbf{g})])}{\text{Vol}_k(S)} f(\mathbf{v}_{k+1}).
\]

The first inequality is consequence of Theorem E and the second of Theorem 4.

**Example 1.** Let \( p_1, \ldots, p_{k+1} \geq 0 \) such that \( \sum_{i=1}^{k+1} p_i = 1 \). We define the functional \( A : L \to \mathbb{R} \) by

\[
A(\mathbf{g}) = \sum_{i=1}^{k+1} p_i g(t_i).
\]

It is obviously that \( A \) is positive normalized linear functional on \( L \). Then the linear operator \( \tilde{A} = (A, \ldots, A) : L^k \to \mathbb{R}^k \) is defined by

\[
\tilde{A}(\mathbf{g}) = \sum_{i=1}^{k+1} p_i \mathbf{g}(t_i).
\]
We set $g(t_i) = v_i$ for all $i = 1, \ldots, k + 1$. Let $S = [v_1, v_2, \ldots, v_{k+1}]$ be $k$-simplex in $\mathbb{R}^k$ and $f$ be a continuous convex function on $S$ such that $f(g) \in L$. Then as a simple consequence of (3.7) it follows

$$f \left( \sum_{i=1}^{k+1} p_i v_i \right) \leq A(f(g)) \leq \sum_{i=1}^{k+1} p_i f(v_i).$$

Setting $p_1 = \ldots = p_{k+1} = \frac{1}{k+1}$ we get

$$f \left( \frac{1}{k+1} \sum_{i=1}^{k+1} v_i \right) \leq A(f(g)) \leq \frac{1}{k+1} \sum_{i=1}^{k+1} f(v_i).$$

Related results are obtained in [1], [20].

**Example 2.** Let $S = [v_1, v_2, \ldots, v_{k+1}]$ be $k$-simplex in $\mathbb{R}^k$ and $f$ a continuous convex function on $S$. Let $L = (E, \mathcal{A}, \lambda)$ be a measure space with positive measure $\lambda$. We define the functional $A : L \to \mathbb{R}$ by

$$A(g) = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t).$$

It is obviously that $A$ is positive normalized linear functional on $L$. Then the linear operator $\tilde{A} = (A_1, \ldots, A_k) : L^k \to \mathbb{R}^k$ is defined by

$$\tilde{A}(g) = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t).$$

We denote $\overline{g} = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t).$ If $g(E) \subset S$ and $f(g) \in L$, then from (3.7) it follows

$$f(\overline{g}) \leq A(f(g)) \leq \frac{\text{Vol}_k([\overline{g}, v_2, \ldots, v_{k+1}])}{\text{Vol}_k(S)} f(v_1) + \ldots + \frac{\text{Vol}_k([v_1, \ldots, v_k, \overline{g}])}{\text{Vol}_k(S)} f(v_{k+1}),$$

Related results are obtained as consequences of Choquet’s theory (see [4], [11], [12], [13], [19]).

4. Related results

In this section we present generalizations of results in [8].

The next theorem generalizes Theorem A.

**Theorem 6.** Let $L$ satisfy properties $L1, L2$ on nonempty set $E$, $A$ be a positive linear functional on $L$ and $\tilde{A} = (A_1, \ldots, A_k) : L^k \to \mathbb{R}^k$ a linear operator. Let $x_1, \ldots, x_n \in \mathbb{R}^k$ and $K = \text{conv}(\{x_1, \ldots, x_n\})$. Let $f$ be a convex function on $K$ and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over $K$. Then for all $g \in L^k$ such that $g(E) \subset K$ and $f(g), \lambda_i(g) \in L$
(i = 1, ..., n) and positive real numbers \( p_1, \ldots, p_n \), with \( P_n = \sum_{i=1}^{n} p_i \), satisfying the condition

\[ p_i \geq A(1) \quad \text{for all} \quad i = 1, \ldots, n, \quad (4.1) \]

we have

\[
f\left( \frac{\sum_{i=1}^{n} p_i x_i - \tilde{A}(g)}{P_n - A(1)} \right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i)}{P_n - A(1)} \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - A(f(g))}{P_n - A(1)}. \quad (4.2)
\]

**Proof.** For each \( t \in E \) we have \( g(t) \in K \). Then there exist barycentric coordinates \( \lambda_i(g(t)) \geq 0 \) \( (i = 1, \ldots, n) \) such that \( \sum_{i=1}^{n} \lambda_i(g(t)) = 1 \) and \( g(t) = \sum_{i=1}^{n} \lambda_i(g(t)) x_i \).

Since \( f \) is convex on \( K \), then

\[
f(g(t)) \leq \sum_{i=1}^{n} \lambda_i(g(t)) f(x_i). \quad (4.3)
\]

Applying a positive linear functional \( A \) on (4.3) we get

\[
A(f(g)) \leq \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i),
\]

where

\[
\sum_{i=1}^{n} A(\lambda_i(g)) = A\left( \sum_{i=1}^{n} \lambda_i(g) \right) = A(1)
\]

and

\[
A(1) \geq A(\lambda_i(g)) \geq 0 \quad \text{for all} \quad i = 1, \ldots, n.
\]

Also we have

\[
\tilde{A}(g) = \sum_{i=1}^{n} A(\lambda_i(g)) x_i.
\]

Now we can write

\[
\frac{\sum_{i=1}^{n} p_i x_i - \tilde{A}(g)}{P_n - A(1)} = \frac{1}{P_n - A(1)} \left( \sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} A(\lambda_i(g)) x_i \right)
\]

\[
= \frac{1}{P_n - A(1)} \sum_{i=1}^{n} (p_i - A(\lambda_i(g))) x_i.
\]

We have

\[
\frac{1}{P_n - A(1)} \sum_{i=1}^{n} (p_i - A(\lambda_i(g))) = 1
\]
and
\[ \frac{1}{P_n - A(1)} (p_i - A(\lambda_i(g))) \geq 0 \quad \text{for all } i = 1, \ldots, n, \]
since
\[ p_i \geq A(1) \geq A(\lambda_i(g)) \quad \text{for all } i = 1, \ldots, n. \]
Therefore, expression \( \sum_{i=1}^{n} \frac{p_i x_i - \lambda_i(g)}{P_n - A(1)} \) is convex combination of vectors \( x_1, \ldots, x_n \) and belongs to \( K \).

Since \( f \) is convex on \( K \), we have
\[
\begin{align*}
&f \left( \frac{\sum_{i=1}^{n} p_i x_i - \lambda_i(g)}{P_n - A(1)} \right) = f \left( \frac{1}{P_n - A(1)} \sum_{i=1}^{n} (p_i - A(\lambda_i(g))) x_i \right) \\
&\leq \frac{1}{P_n - A(1)} \sum_{i=1}^{n} (p_i - A(\lambda_i(g))) f(x_i) \\
&= \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i)}{P_n - A(1)} \\
&\leq \frac{\sum_{i=1}^{n} p_i f(x_i) - A(f(g))}{P_n - A(1)}. \quad \square
\end{align*}
\]

**Corollary 1.** Let \( L \) satisfy properties L1, L2 on nonempty set \( E \) and \( A \) be a positive normalized linear functional on \( L \). Let \( f \) be a convex function on an interval \( I = [m, M] \subset \mathbb{R} \) \((-\infty < m < M < \infty)\). Then for all \( g \in L \) such that \( g(E) \subset I \) and \( f(g) \in L \), we have
\[
\begin{align*}
f(m + M - A(g)) &\leq \frac{A(g) - m}{M - m} f(m) + \frac{M - A(g)}{M - m} f(M) \\
&\leq f(m) + f(M) - A(f(g)). \quad (4.4)
\end{align*}
\]

**Proof.** For each \( t \in E \) we have \( g(t) \in I = [m, M] \).

Since interval \( I = [m, M] \) is 1-simplex with vertices \( m \) and \( M \), then the barycentric coordinates have the special form:
\[
\lambda_1(g(t)) = \frac{M - g(t)}{M - m} \quad \text{and} \quad \lambda_2(g(t)) = \frac{g(t) - m}{M - m}
\]

Then applying a functional \( A \) we have
\[
\begin{align*}
A(\lambda_1(g)) &= \frac{M - A(g)}{M - m} \quad \text{and} \quad A(\lambda_2(g)) = \frac{A(g) - m}{M - m}. \quad (4.5)
\end{align*}
\]
Choosing \( n = 2, \ p_1 = p_2 = 1, \ x_1 = m, \ x_2 = M \) from (4.2) it follows

\[
f(m + M - A(g)) \leq f(m) + f(M) - \left[ \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M) \right]
\]

\[
= \frac{A(g) - m}{M - m} f(m) + \frac{M - A(g)}{M - m} f(M)
\]

\[
\leq f(m) + f(M) - A(f(g)). \quad \Box
\]

**Remark 6.** The inequalities in (4.4) are also obtained in [3]. Some related results are obtained in [2].

**Theorem 7.** Let \( L \) satisfy properties L1, L2 on nonempty set \( E \), \( A = (A,...,A): L^k \to \mathbb{R}^k \) a linear operator. Let \( \mathbf{x}_1,...,\mathbf{x}_n \in \mathbb{R}^k \) and \( K = \text{conv}\{\mathbf{x}_1,...,\mathbf{x}_n\} \). Let \( f \) be a convex function on \( K \) and \( \lambda_1,...,\lambda_n \) barycentric coordinates over \( K \). Then for all \( \mathbf{g} \in L^k \) such that \( \mathbf{g}(E) \subset K \) and \( f(\mathbf{g}), \lambda_i(\mathbf{g}) \in L \) \((i = 1,...,n)\) and positive real numbers \( p_1,...,p_n \) satisfying the conditions \( P_n - A(1) > 0 \), where \( P_n = \sum_{i=1}^n p_i \), and

\[
\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(1)} \in K,
\]

we have

\[
f \left( \frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(1)} \right) \geq \frac{P_nf \left( \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - A(1) f \left( \frac{1}{A(1)} \tilde{A}(\mathbf{g}) \right)}{P_n - A(1)}
\]

\[
\geq \frac{P_n f \left( \frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) f(\mathbf{x}_i)}{P_n - A(1)}. \quad (4.7)
\]

**Proof.** For each \( t \in E \) we have \( \mathbf{g}(t) \in K \). Then there exist barycentric coordinates \( \lambda_i(\mathbf{g}(t)) \geq 0 \) \((i = 1,...,n)\) such that \( \sum_{i=1}^n \lambda_i(\mathbf{g}(t)) = 1 \) and

\[
\mathbf{g}(t) = \sum_{i=1}^n \lambda_i(\mathbf{g}(t)) \mathbf{x}_i.
\]

Also we have

\[
\tilde{A}(\mathbf{g}) = \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \mathbf{x}_i.
\]

We can easily see that

\[
\frac{1}{A(1)} \tilde{A}(\mathbf{g}) = \frac{1}{A(1)} \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \mathbf{x}_i \in K,
\]
Since
\[
\frac{1}{A(1)} \sum_{i=1}^{n} A(\lambda_i(g)) = 1 \quad \text{and} \quad \frac{1}{A(1)} A(\lambda_i(g)) \geq 0, \quad i = 1, \ldots, n.
\]
Since \( f \) is convex on \( K \), then
\[
f \left( \frac{1}{A(1)} \tilde{A}(g) \right) \leq \frac{1}{A(1)} \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i). \tag{4.8}
\]
Using first (1.2) and then (4.8) we have
\[
\frac{P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right)}{P_n - A(1)} \geq \frac{P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right)}{P_n - A(1)}
\]
and then
\[
P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right) \geq \frac{P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right)}{P_n - A(1)}. \tag{4.9}
\]

**Remark 7.** If positive real numbers \( p_1, \ldots, p_n \) satisfy the condition (4.1), then the condition (4.6) is also satisfied since \( K \) is convex set. Then (4.2) can be extended as follows
\[
P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right) \leq \frac{P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right)}{P_n - A(1)}
\]
and
\[
P_n f \left( \frac{1}{n} \sum_{i=1}^{n} p_i x_i \right) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - A(1) \left( \frac{1}{A(1)} \tilde{A}(g) \right)}{P_n - A(1)}.
\]

**Corollary 2.** Let \( L \) satisfy properties L1, L2 on nonempty set \( E \) and \( A \) be a positive normalized linear functional on \( L \). Let \( f \) be a convex function on an interval \( I = [m, M] \subset \mathbb{R} \ (\infty < m < M < \infty) \). Then for all \( g \in L \) such that \( g(E) \subset I \) and \( f(g) \in L \), we have
\[
f(m+M-A(g)) \geq 2f \left( \frac{m+M}{2} \right) - f(A(g)) \]
\[
\geq 2f \left( \frac{m+M}{2} \right) - \left[ \frac{M-A(g)}{M-m} f(m) + \frac{A(g)-m}{M-m} f(M) \right]. \tag{4.10}
\]
Proof. Choosing $n = 2$, $x_1 = m$, $x_2 = M$, $p_1 = p_2 = 1$ and using (4.5), the inequalities in (4.10) easily follows from (4.7). □

Next we give generalizations of Corollary 1 and Corollary 2 for convex functions defined on $k$-simplices in $\mathbb{R}^k$.

**COROLLARY 3.** Let $L$ satisfy properties L1, L2 on nonempty set $E$, $A$ be a positive normalized linear functional on $L$ and $\tilde{A} = (A_1, ... , A_k) : L^k \to \mathbb{R}^k$ a linear operator. Let $f$ be a convex function on $k$-simplex $S = [v_1, v_2, ..., v_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, ..., \lambda_{k+1}$ barycentric coordinates over $S$. Then for all $g \in L^k$ such that $g(E) \subset S$ and $f(g) \in L$ we have

\[
(k + 1)f \left( \frac{1}{k+1} \sum_{i=1}^{k+1} v_i \right) - \frac{1}{k} \sum_{i=1}^{k+1} \tilde{A}(g) f(v_i) \leq \frac{(k + 1)f \left( \frac{1}{k+1} \sum_{i=1}^{k+1} v_i \right) - f(\tilde{A}(g))}{k} \leq f \left( \frac{\sum_{i=1}^{k+1} v_i - \tilde{A}(g)}{k} \right) \leq \frac{\sum_{i=1}^{k+1} f(v_i) - \frac{1}{k} \sum_{i=1}^{k+1} \lambda_i(\tilde{A}(g)) f(v_i)}{k} \leq \frac{\sum_{i=1}^{k+1} f(v_i) - A(f(g))}{k}. \tag{4.11}
\]

Proof. Since barycentric coordinates $\lambda_1, ..., \lambda_{k+1}$ over $k$-simplex $S$ in $\mathbb{R}^k$ are nonnegative linear polynomials, then $A(\lambda_i(g)) = \lambda_i(A(g))$ for all $i = 1, ..., k+1$.

Choosing $x_i = v_i$ for all $i = 1, ..., k+1$ and $p_1 = p_2 = ... = p_{k+1} = 1$, the inequalities in (4.11) easily follow from (4.2) and (4.7). □

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