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ON SOME INEQUALITIES FOR DERIVATIVES OF POLYNOMIALS AND RATIONAL FUNCTIONS

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Abstract. Using methods of geometric function theory, we get new inequalities for derivatives of polynomials and rational functions. These theorems refine and supplement some known results.

1. Introduction and auxiliary statements

In this paper algebraic polynomials

$$P_n(z) = c_n z^n + \ldots + c_0, \quad c_n \neq 0, \quad c_l \in \mathbb{C}, \quad l = 0, \ldots, n, \tag{1}$$

and rational functions

$$r(z) = \frac{c_m z^m + \ldots + c_0}{\prod\limits_{k=1}^{n} (z - a_k)}, \; c_j, a_k \in \mathbb{C}, \quad |a_k| > 1, \; k = 1, \ldots, n, \; j = 1, \ldots, m, \tag{2}$$

will be considered. We'll apply an approach which was proposed by V. N. Dubinin in his recent articles [1, 2]. This approach consists in constructing an analytic function associated with the given polynomial and applying some methods of geometric function theory to this function.

In the paper [2] V. N. Dubinin proved the following boundary analog of Schwarz's lemma.

THEOREM 1.1. *Let a function $z = F(w)$ be regular on an open set G, $0 \in G \subset \{w : |w| < 1\}$; $F(w) \neq 0$ for $w \in G \setminus \{0\}$; $|F(w)| < 1$ for $w \in G$, and let, in a neighborhood of the origin*

$$F(w) = d_n w^n + \ldots \;, \quad d_n \neq 0.$$

Assume that all limit boundary values of $|F(w)|$ in G equal one. Then

$$|F(w)| \geqslant |w|^n, \quad w \in G.$$

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If, in addition, the boundary of the domain G *contains an arc* λ *of the circle* $|w|=1$*, then, for an arbitrary interior point* w_0 *of* λ*, the inequality*

$$|F'(w_0)| \leqslant \frac{n^2(|d_n|+1)}{(n+1)|d_n|+n-1} \leqslant n \tag{3}$$

is valid. Equality is attained for the function $F(w)=d_nw^n, |d_n|=1$*, and the domain* $G=\{w:|w|<1\}$.

This theorem will be used to prove the first three theorems in the second section. Also we need some auxiliary results from univalent function theory.

We denote by $\mathscr{B}$ the class of functions $w=f(z)$ that are regular and univalent in the unit disk $U:=\{z:|z|<1\}$ and are normalized by the conditions $f(0)=0$ and $|f(z)|<1$ for $z\in U$. We also introduce the following notation:

$$\lambda_f(z)=\frac{1+|f(z)|}{1+|z|},\ \Lambda_f(z)=\frac{1-|f(z)|}{1-|z|},\ \ f\in\mathscr{B}.$$

For every function $w=f(z)$ of the class $\mathscr{B}$ and every point z with $|z|<1$ we have

$$\frac{\lambda_f(z)}{\Lambda_f(z)} \leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant \frac{\Lambda_f(z)}{\lambda_f(z)}. \tag{4}$$

Equality is attained only for the functions $f(z)\equiv \alpha z$ with $|\alpha|=1$ (See [1, 3]).

E. Landau proved that if a function $f(z)=az+...$ is regular in the closed unit disk $|z|\leqslant 1$, and $|f(z)|\leqslant M$ there then the function f is univalent in the disk with the center at the point 0 and the radius $M/|a|-\sqrt{(M/|a|)^2-1}$ (See for example [4]).

To study cases of equality in theorems 2.1–2.3 we will need the following well known representations of Chebyshev polynomials of the first, the second, the third, and the fourth kinds [5]:

$$\begin{aligned}
T_n(z) &= \frac{1}{2}((z+\sqrt{z^2-1})^n+(z-\sqrt{z^2-1})^n),\\
U_n(z) &= \frac{(z+\sqrt{z^2-1})^{n+1}-(z-\sqrt{z^2-1})^{n+1}}{2\sqrt{z^2-1}},\\
V_n(z) &= \frac{(z+\sqrt{z^2-1})^{n+\frac{1}{2}}+(z-\sqrt{z^2-1})^{n+\frac{1}{2}}}{(z+\sqrt{z^2-1})^{\frac{1}{2}}+(z-\sqrt{z^2-1})^{\frac{1}{2}}},\\
W_n(z) &= \frac{(z+\sqrt{z^2-1})^{n+\frac{1}{2}}-(z-\sqrt{z^2-1})^{n+\frac{1}{2}}}{(z+\sqrt{z^2-1})^{\frac{1}{2}}-(z-\sqrt{z^2-1})^{\frac{1}{2}}}
\end{aligned}$$

respectively.

2. Inequalities for polynomials and rational functions

The first three theorems of this section are devoted to Bernstein-type inequalities for polynomial with various curved majorants dependent on Chebyshev polynomials of the first kind. There are many papers related to polynomials having curved majorants, for instance [6, 7]. An interest in Bernstein-type inequalities for such polynomials arises, for example, because of their importance in inverse problems of approximation theory.

THEOREM 2.1. *If the polynomial $P(z)$ as in (1) has real coefficients and satisfies the condition*

$$|P(z)|\sqrt{1-T_k^2(z)} \leqslant 1,\ z\in[-1,1],$$

for some positive integer k then for all $x\in[-1,1]$ the inequality

$$\begin{aligned} &k|P(x)T_k(x)T_k'(x)-P'(x)(1-T_k^2(x))| \\ &\leqslant \frac{(n+k)^2(2^n+|c_n|)\sqrt{(1-(1-T_k^2(x))P^2(x))}}{2^n(n+k+1)+(n+k-1)|c_n|}|T_k'(x)| \end{aligned} \tag{5}$$

holds.

If n is divisible by k and $P(z)=U_{n/k}(T_k(z))$ then this inequality becomes equality for all $x\in[-1,1]$.

Proof. Consider the function

$$z=F(w)=\Phi\left[\frac{i}{2}\left(w^k-\frac{1}{w^k}\right)P\left(\frac{1}{2}\left(w+\frac{1}{w}\right)\right)\right],$$

where $\zeta=\Phi(\omega)=\omega-\sqrt{\omega^2-1}$ denotes the branch of the analytic function (the inverse of the Zhukovskii mapping) which maps the exterior of the interval $[-1,1]$ onto the unit disk $|\zeta|<1$ conformally and univalently.

$$\begin{aligned} \frac{1}{2}\left(F(w)+\frac{1}{F(w)}\right) &= \frac{1}{2}\left(d_{n+k}w^{n+k}\ldots+\frac{1}{d_{n+k}w^{n+k}+\ldots}\right) \\ &= \frac{i}{2}\left(w^k-\frac{1}{w^k}\right)\left(c_n\left(\frac{1}{2}\left(w+\frac{1}{w}\right)\right)^n+\ldots\right). \end{aligned}$$

Letting w approach 0

$$d_{n+k}=\frac{i2^n}{c_n}.$$

The function $z = F(w)$ satisfies the conditions of theorem 1.1 on the set

$$G := \left\{ w : |w| < 1, \frac{i}{2}\left(w^k - \frac{1}{w^k}\right) P\left(\frac{1}{2}\left(w + \frac{1}{w}\right)\right) \notin [-1,1] \right\}.$$

Indeed, the function $F(w)$ is regular on G; at boundary points of the set G we have $\frac{i}{2}\left(w^k - \frac{1}{w^k}\right) P\left(\frac{1}{2}\left(w + \frac{1}{w}\right)\right) \in [-1,1]$. Therefore, limit boundary values of $|F(w)|$ on G equal 1. Hence, almost at all points w_0 of the unit circle $|w| = 1$, the inequality (3) holds.

A direct computation shows that

$$\begin{aligned} |F'(w_0)| = {} & \frac{1}{\sqrt{1 + \frac{1}{4}\left(w_0^k - \frac{1}{w_0^k}\right)^2 P^2\left(\frac{1}{2}\left(w_0 + \frac{1}{w_0}\right)\right)}} \\ & \times \left| \frac{k}{2}\left(w_0^k + \frac{1}{w_0^k}\right) P\left(\frac{1}{2}\left(w_0 + \frac{1}{w_0}\right)\right) \right. \\ & \left. + \frac{1}{4} P'\left(\frac{1}{2}\left(w_0 + \frac{1}{w_0}\right)\right)\left(w_0 - \frac{1}{w_0}\right)\left(w_0^k - \frac{1}{w_0^k}\right) \right|. \end{aligned}$$

After substitution $x = \frac{1}{2}\left(w_0 + \frac{1}{w_0}\right)$, and using formulas

$$\frac{1}{2}\left(w_0^k + \frac{1}{w_0^k}\right) = T_k(x), \quad \frac{i}{2}\left(w_0^k - \frac{1}{w_0^k}\right) = \sqrt{1 - T_k^2(x)},$$

$$T_k'(x) = \frac{k\left(w_0^k - \frac{1}{w_0^k}\right)}{\left(w_0 - \frac{1}{w_0}\right)},$$

we obtain (5). If n is divisible by k and $P(z) = U_{n/k}(T_k(z))$ then $F(w) \equiv iw^{n+k}$. Therefore this inequality becomes the equality. The theorem is proved. □

By a remark in article [2] theorem 2.1 is stronger than theorem 6 in the paper [8], if $|c_n| > 2^n/(n+k-1)^2$.

To prove the following two theorems we should consider the functions

$$z = F(w) = \Phi\left[\frac{1}{2}\left(w^k + \frac{1}{w^k}\right) P\left(\frac{1}{2}\left(w^2 + \frac{1}{w^2}\right)\right)\right],$$

and

$$z = F(w) = \Phi\left[\frac{i}{2}\left(w^k - \frac{1}{w^k}\right) P\left(\frac{1}{2}\left(w^2 + \frac{1}{w^2}\right)\right)\right],$$

on the sets

$$G := \left\{ w : |w| < 1, \frac{1}{2}\left(w^k + \frac{1}{w^k}\right) P\left(\frac{1}{2}\left(w^2 + \frac{1}{w^2}\right)\right) \notin [-1,1] \right\},$$

and

$$G := \left\{ w : |w| < 1, \frac{i}{2}\left(w^k - \frac{1}{w^k}\right) P\left(\frac{1}{2}\left(w^2 + \frac{1}{w^2}\right)\right) \notin [-1,1] \right\},$$

respectively.

THEOREM 2.2. *If the polynomial $P(z)$ as in (1) has real coefficients and satisfies the condition*

$$|P(z)|\sqrt{\frac{1+T_k(z)}{2}} \leqslant 1,\ z \in [-1,1],$$

for some positive integer k then for all $x \in [-1,1]$ the inequality

$$|2P'(x)(1+T_k(x)) + T_k'(x)P(x)|$$

$$\leqslant \frac{(2n+k)^2(2^n+|c_n|)\sqrt{(1+T_k(x))(2-(1+T_k(x))P^2(x))}}{((2n+k+1)2^n+(2n+k-1)|c_n|)\sqrt{1-x^2}} \tag{6}$$

holds.

If n is divisible by k and $P(z) = V_{n/k}(T_k(z))$ then this inequality becomes equality for all $x \in [-1,1]$.

THEOREM 2.3. *If the polynomial $P(z)$ as in (1) has real coefficients and satisfies the condition*

$$|P(z)|\sqrt{\frac{1-T_k(z)}{2}} \leqslant 1,\ z \in [-1,1],$$

for some positive real k then for all $x \in [-1,1]$ the inequality

$$|2P'(x)(T_k(x)-1) + T_k'(x)P(x)|$$

$$\leqslant \frac{(2n+k)^2(2^n+|c_n|)\sqrt{(1+T_k(x))(2+(T_k(x)-1)P^2(x))}}{((2n+k+1)2^n+(2n+k-1)|c_n|)\sqrt{1-x^2}} \tag{7}$$

holds.

If n is divisible by k and $P(z) = W_{n/k}(T_k(z))$ then this inequality becomes equality for all $x \in [-1,1]$.

If $|c_n| > 2^n/(2n+k-1)^2$ then theorem 2.2 and theorem 2.3 imply theorem 7 and theorem 6 in the article [8]. In case $k = 1$, all mentioned theorems supplement some results in the papers [9, 10].

THEOREM 2.4. *If the rational function as in* (2) *satisfies the condition* $\max\{r(z): |z|=1\}=1$ *then the following inequalities hold*

$$\begin{aligned}
&|z|\frac{(|z^{1+m-n}B(z)|+|r(z)|)(|z|-R)}{(|z^{1+m-n}B(z)|-|r(z)|)(|z|+R)}\\
&\quad\leqslant\left|z(1+m-n)-z^2\frac{r'(z)}{r(z)}+\overline{B'\left(\frac{1}{\overline{z}}\right)}B(z)\right|\\
&\quad\leqslant|z|\frac{(|z^{1+m-n}B(z)|-|r(z)|)(|z|+R)}{(|z^{1+m-n}B(z)|+|r(z)|)(|z|-R)},\quad |z|>R\geqslant 1,
\end{aligned}$$

where

$$R=\frac{\prod\limits_{k=1}^{n}|a_k|}{|c_m|}+\sqrt{\frac{\prod\limits_{k=1}^{n}|a_k|^2}{|c_m|^2}-1}\geqslant 1,$$

and

$$B(z)=\prod_{k=1}^{n}\frac{1-\overline{a}_k z}{z-a_k}.$$

If $r(z)=z^kB(z), k\in\mathbb{N}_0$, *then* $R=1$ *and these inequalities become equalities for all points* z, $|z|>1$.

Proof. The function

$$f(\zeta)=\zeta^{1+m-n}\overline{r\left(\frac{1}{\overline{\zeta}}\right)}B(\zeta)=\frac{\overline{c}_m}{(-1)^n\prod\limits_{k=1}^{n}a_k}\zeta+\ldots,$$

is regular in the closed unit disk $\overline{U}$ and its modulus bounded by 1. Hence, the function $f(\zeta)$ is univalent in the disk with center at the origin and the radius

$$R_1=\prod_{k=1}^{n}|a_k|/|c_m|-\sqrt{\prod_{k=1}^{n}|a_k|^2/|c_m|^2-1},\ \ R_1\leqslant 1.$$

Therefore, the function $f_1(\zeta)=f(R_1\zeta)\in\mathscr{B}$ and we can apply the inequality (4)

$$\frac{(1+|f_1(\zeta)|)(1-|\zeta|)}{(1-|f_1(\zeta)|)(1+|\zeta|)}\leqslant\left|\frac{\zeta f_1'(\zeta)}{f_1(\zeta)}\right|\leqslant\frac{(1-|f_1(\zeta)|)(1+|\zeta|)}{(1+|f_1(\zeta)|)(1-|\zeta|)},\ \ |\zeta|<1,$$

Hence

$$
\begin{aligned}
&\frac{(1+|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1-|\zeta|)}{(1-|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1+|\zeta|)}\\
&\quad\leqslant\left|\frac{\zeta((R_1\zeta)^{1+m-n}\overline{r\left(\frac{1}{R_1\overline{\zeta}}\right)}B(R_1\zeta))'}{(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)}\right|\\
&\quad\leqslant\frac{(1-|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1+|\zeta|)}{(1+|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1-|\zeta|)}
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{(1+|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1-|\zeta|)}{(1-|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1+|\zeta|)}\\
&\quad\leqslant|R_1\zeta|\left|\frac{(1+m-n)}{R_1\zeta}-\frac{\overline{r'(\frac{1}{R_1\overline{\zeta}})}}{(R_1\zeta)^2\overline{r(\frac{1}{R_1\overline{\zeta}})}}+\frac{B'(R_1\zeta)}{B(R_1\zeta)}\right|\\
&\quad\leqslant\frac{(1-|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1+|\zeta|)}{(1+|(R_1\zeta)^{1+m-n}\overline{r(\frac{1}{R_1\overline{\zeta}})}B(R_1\zeta)|)(1-|\zeta|)},\ |\zeta|<1.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
&|z|\frac{(|z^{1+m-n}B(z)|+|r(z)|)(|z|-R)}{(|z^{1+m-n}B(z)|-|r(z)|)(|z|+R)}\\
&\quad\leqslant\left|z(1+m-n)-z^2\frac{r'(z)}{r(z)}+\overline{B'\left(\frac{1}{\overline{z}}\right)}B(z)\right|\\
&\quad\leqslant|z|\frac{(|z^{1+m-n}B(z)|-|r(z)|)(|z|+R)}{(|z^{1+m-n}B(z)|+|r(z)|)(|z|-R)},\ \ |z|>R\geqslant 1,
\end{aligned}
$$

where, $R=1/R_1$.

If $r(z)=z^kB(z), k\in\mathbb{N}_0$, then we have $f(\zeta)\equiv\zeta$, therefore, $R_1=R=1$, and inequalities become equalities. The theorem is proved. $\square$

THEOREM 2.5. *If the polynomial as in* (1) *has all its zeros in the closed unit disk* $|z| \leqslant 1$ *then the following inequalities hold*

$$
\begin{aligned}
&|z| \frac{(|z^{1-n}P(z)| + |P(1/\overline{z})|)(|z| - R)}{(|z^{1-n}P(z)| - |P(1/\overline{z})|)(|z| + R)} \\
&\quad \leqslant \left| z(1-n) + \overline{(\log P)'}(1/\overline{z}) + z^2 (\log P)'(z) \right| \\
&\quad \leqslant |z| \frac{(|z^{1-n}P(z)| - |P(1/\overline{z})|)(|z| + R)}{(|z^{1-n}P(z)| + |P(1/\overline{z})|)(|z| - R)}, \quad |z| > R \geqslant 1,
\end{aligned}
$$

where

$$
R = \left| \frac{c_n}{c_0} \right| + \sqrt{\left| \frac{c_n}{c_0} \right|^2 - 1} \geqslant 1.
$$

If the polynomial has all its zeros on the unit circle $|z| = 1$ *then* $R = 1$, *and inequalities become equalities.*

THEOREM 2.6. *If the polynomial as in* (1) *has no zeros in the unit disk* $|z| < 1$ *then the following inequalities hold*

$$
\begin{aligned}
&|z| \frac{(|z^{1+n}P(1/\overline{z})| + |P(z)|)(|z| - R)}{(|z^{1+n}P(1/\overline{z})| - |P(z)|)(|z| + R)} \\
&\quad \leqslant \left| z(1+n) - \overline{(\log P)'}(1/\overline{z}) - z^2 (\log P)'(z) \right| \\
&\quad \leqslant |z| \frac{(|z^{1+n}P(1/\overline{z})| - |P(z)|)(|z| + R)}{(|z^{1+n}P(1/\overline{z})| + |P(z)|)(|z| - R)}, \quad |z| > R \geqslant 1,
\end{aligned}
$$

where

$$
R = \left| \frac{c_0}{c_n} \right| + \sqrt{\left| \frac{c_0}{c_n} \right|^2 - 1} \geqslant 1.
$$

If the polynomial has all its zeros on the unit circle $|z| = 1$ *then* $R = 1$, *and inequalities become equalities.*

To prove these theorems it's sufficient to consider the functions

$$
f_2(\zeta) = \frac{\zeta^{1-n} P(\zeta)}{\overline{P(1/\overline{\zeta})}}
$$

and

$$
f_3(\zeta) = \frac{\zeta^{1+n} \overline{P(1/\overline{\zeta})}}{P(\zeta)}
$$

correspondingly, and to use the same way like in the proof of theorem 2.4. Theorems 2.4–2.6 continue a topic of the paper [11] where sharp inequalities for modulus of rational functions were obtained, and supplement in part some results in the article of M.A. Qazi and Q.I. Rahman [12].

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