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ON NECESSARY AND SUFFICIENT CONDITIONS FOR VALIDITY OF SOME CHEBYSHEV–TYPE INEQUALITIES

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Abstract. We obtain necessary and sufficient conditions for validity of some Chebyshev-Type inequalities.

1. Introduction

Let f, g: $[a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Further, let $p: [a,b] \to \mathbb{R}_0^+$ be an integrable function. Then (see, for example, [1, Chap. IX])

$$\int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \Big(\int_a^b p(x)dx \Big)^{-1}. \qquad (1.1)$$

If one of the functions f or g is nonincreasing and the other nondecreasing the reversed inequality is true, i.e.,

$$\int_a^b p(x)f(x)g(x)dx \leqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \Big(\int_a^b p(x)dx \Big)^{-1} \qquad (1.2)$$

Inequalities (1.1) and (1.2) are known as Chebyshev's inequalities. Inequality (1.1) was formulated in 1882 by P. L. Chebyshev in the paper [2]. In [2] P. L. Chebyshev gives without a proof some properties of remainder terms of certain infinite fractions. One of these properties implies that if p, f, g are integrable functions and $p(x) > 0$ on $[a,b]$ and if

$$\operatorname{sgn} \frac{df(x)}{dx} = \operatorname{sgn} \frac{dg(x)}{dx}$$

then inequality (1.1) is valid. In 1883 in [3] P. L. Chebyshev published the proofs of this result.

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Note that inequalities (1.1) and (1.2) attracted great interest of researchers. So, there exists a number of proofs of these inequalities, given by other authors (see, for example, [4–6]). A lot of analogues and generalizations of inequalities (1.1) and (1.2) is also known. In particular, these results can be found in Chapter IX of the book [1] by D.S. Mitrinović, J. E. Pečarić and A. M. Fink which trace completely the historical and chronological developments of Chebyshev's and related inequalities (see also [7, 8]).

In the paper, we study a question of what minimal conditions on functions $p: [a,b] \to \mathbb{R}^+$ and $g\colon [a,b] \to \mathbb{R}_0^+$ one has to impose for the inequality

$$\int_a^b p(x)f(x)g(x)dx \geqslant \left(\int_a^b p^r(x)f^r(x)dx\right)^{1/r} \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx\right)^{-1/r} \tag{1.3}$$

or the inequality

$$\int_a^b p(x)f(x)g(x)dx \leqslant \left(\int_a^b p^r(x)f^r(x)dx\right)^{1/r} \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx\right)^{-1/r} \tag{1.4}$$

to be satisfied for any nonincreasing on $[a,b]$ function $f\colon [a,b] \to \mathbb{R}_0^+$ with r being an arbitrary positive number.

Our investigations of this question were begun in the team-work [9] of A. I. Stepanets and the author (see also [10]). In [9], the authors studied the quantities of the best approximations of integrals of functions by integrals of finite rank. In fact, inequalities of the kind (1.4) were needed to get exact values of the upper bonds for theses quantities on a certain class of functions. In [9], the function f was nonincreasing and the functions p was nondecreasing but the function g was of special type and it was not monotone as it demands above for validity of (1.2). The authors got weaker conditions (than the monotonicity of the function g) on functions p and g that guarantee validity of inequality (1.4) (and, in particular, inequality (1.2)) for any nonincreasing on $[a,b]$ function f.

In this paper, we obtain necessary and sufficient conditions on arbitrary functions $p\colon [a,b] \to \mathbb{R}^+$ and $g\colon [a,b] \to \mathbb{R}_0^+$ such that inequality (1.4) or inequality (1.3) holds for any nonincreasing on $[a,b]$ function f.

2. Main results

We formulate here main theorems.

THEOREM 2.1. *Let $r \in (0,1]$, and let $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ be integrable functions. Then for any nonincreasing function $f\colon [a,b] \to \mathbb{R}_0^+$, inequality (1.3) is valid if and only if for all $s \in (a,b)$,*

$$\int_a^s p(x)g(x)dx \left(\int_a^s p^r(x)dx\right)^{-1} \geqslant \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx\right)^{-1}. \tag{2.1}$$

In the case where $r \in (1,\infty)$, the following statement is true:

THEOREM 2.2. *Let $r \in (1,\infty)$, and let $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ be integrable functions. Then for any nonincreasing function $f\colon [a,b] \to \mathbb{R}_0^+$, inequality (1.3) is valid if and only if for all $s \in (a,b)$,*

$$\int_a^s p(x)g(x)dx\left(\int_a^s p^r(x)dx\right)^{-1/r} \geqslant \int_a^b p(x)g(x)dx\left(\int_a^b p^r(x)dx\right)^{-1/r}. \tag{2.2}$$

Note that, in the case where $r = 1$ and where f is an increasing function on $[a,b]$, the validity of inequality (1.1) for similar conditions follows from J.F. Steffensen's result (see [11], [1, Chap. IX]):

if F is an increasing function on $[a,b]$, and if F, G and H are integrable functions on $[a,b]$ such that for all $s \in [a,b]$

$$\int_a^s G(x)dx\left(\int_a^b G(x)dx\right)^{-1} \leqslant \int_a^s H(x)dx\left(\int_a^b H(x)dx\right)^{-1}.$$

then

$$\int_a^b F(x)G(x)dx\left(\int_a^b G(x)dx\right)^{-1} \leqslant \int_a^b F(x)H(x)dx\left(\int_a^b H(x)dx\right)^{-1}.$$

Putting $F(x) = f(x)$, $H(x) = p(x) > 0$, $G(x) = p(x)g(x)$ in this statement, we conclude that inequality (1.1) holds if $p(x) > 0$, if f is an increasing function on $[a,b]$ and if for all $s \in [a,b]$,

$$\int_a^s p(x)g(x)dx\left(\int_a^s p(x)dx\right)^{-1} \leqslant \int_a^b p(x)g(x)dx\left(\int_a^b p(x)dx\right)^{-1}. \tag{2.3}$$

We also mention the paper of M. Biernacki [12]. It follows from results of M. Biernacki that inequality (1.1) also holds in the case where both functions

$$f(x) \quad \text{and} \quad \int_a^x p(t)g(t)dt\left(\int_a^x p(t)dt\right)^{-1}$$

are increasing or decreasing. These conditions are similar to conditions (2.1)–(2.3) but they are less general.

In the case where the product $p^{1-r}(x)g(x)$ is nonincreasing, for a given $r > 0$, the derivative of the function

$$\Phi_r(s) = \int_a^s p(x)g(x)dx\left(\int_a^s p^r(x)dx\right)^{-1} \tag{2.4}$$

is nonpositive on $(a,b]$. Thus, the function $\Phi_r(s)$ is also nonincreasing and condition (2.1) holds. Therefore, the following statement is true:

COROLLARY 2.1. *Let f, g: $[a,b] \to \mathbb{R}_0^+$ and p: $[a,b] \to \mathbb{R}^+$ be integrable functions. If for a given $r \in (0,1]$, the product $p^{1-r}(x)g(x)$ is nonincreasing on $[a,b]$, and if the function f is also nonincreasing on $[a,b]$, then inequality (1.5) is valid.*

Now we consider inequality (1.4).

THEOREM 2.3. *Let $r \in (0,1)$, and let g: $[a,b] \to \mathbb{R}_0^+$ and p: $[a,b] \to \mathbb{R}^+$ be integrable functions. Then for any nonincreasing function f: $[a,b] \to \mathbb{R}_0^+$, inequality (1.4) is valid if and only if for all $s \in (a,b)$,*

$$\int_a^s p(x)g(x)dx \left(\int_a^s p^r(x)dx \right)^{-1/r} \leqslant \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx \right)^{-1/r}. \tag{2.5}$$

In the case where $r \in [1,\infty)$, the following theorem is true:

THEOREM 2.4. *Let $r \in [1,\infty)$, and let g: $[a,b] \to \mathbb{R}_0^+$ and p: $[a,b] \to \mathbb{R}^+$ be integrable functions. Then for any nonincreasing function f: $[a,b] \to \mathbb{R}_0^+$, inequality (1.4) is valid if and only if for all $s \in (a,b)$,*

$$\int_a^s p(x)g(x)dx \left(\int_a^s p^r(x)dx \right)^{-1} \leqslant \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx \right)^{-1}. \tag{2.6}$$

Note that in the case where f: $[a,b] \to \mathbb{R}_0^+$ and p: $[a,b] \to \mathbb{R}^+$ are nonincreasing functions and where for $x \in [a,\sigma)$, the function $g(x) = 0$ and for $x \in [\sigma,b]$, the function $g(x) = 1/p(x)$, $a < \sigma < b$, the sufficiency of conditions (2.5) and (2.6) (for corresponding $r \in (0,\infty)$) for validity of (1.4) follows from the proofs of Lemma 2 and of Lemma 3 of the paper [9] (see also [10, Chapt. 4 and Chapt. 7]).

In the case where for a given $r > 0$, the product $p^{1-r}(x)g(x)$ is nondecreasing, the derivative of the function $\Phi_r(s)$ defined by equality (2.4) is nonnegative on $(a,b]$. Thus, the function $\Phi_r(s)$ is also nondecreasing and condition (2.6) holds. Therefore, the following statement is true:

COROLLARY 2.2. *Let f, g: $[a,b] \to \mathbb{R}_0^+$ and p: $[a,b] \to \mathbb{R}^+$ be integrable functions. If for a given $r \in [1,\infty)$, the product $p^{1-r}(x)g(x)$ is nondecreasing on $[a,b]$, and if the function f is nonincreasing on $[a,b]$, then inequality (1.4) is valid.*

REMARK. Statements on necessary and sufficient conditions for inequalities (1.3) and (1.4) to be valid for any *nondecreasing* function f are the same as in Theorems 1–4 but all integrals of the kind $\int_a^s (\cdot)dx$ in conditions (2.1), (2.2), (2.5) and (2.6) should be replaced by integrals of the kind $\int_s^b (\cdot)dx$.

3. Proofs of the theorems

We mainly use the following discrete analogues of the theorems.

3.1. Discrete analogues of Theorems 2.1–2.4.

LEMMA 3.1. *Let $r \in (0,1]$, and let $b = \{b_k\}_{k=1}^n$ and $p = \{p_k\}_{k=1}^n$ be nonnegative number sequences, $n \in \mathbb{N}$, $p_k > 0$. Then for any nonnegative nonincreasing sequence $a = \{a_k\}_{k=1}^n$, the inequality*

$$\sum_{k=1}^n p_k a_k b_k \geqslant \left(\sum_{i=1}^n p_i^r a_i^r\right)^{1/r} \sum_{k=1}^n p_k b_k \left(\sum_{k=1}^n p_k^r\right)^{-1/r} \tag{3.1}$$

is valid if and only if the following condition is satisfied:

$$\min_{s\in[1,n]} \sum_{k=1}^s p_k b_k \left(\sum_{k=1}^s p_k^r\right)^{-1} = \sum_{k=1}^n p_k b_k \left(\sum_{k=1}^n p_k^r\right)^{-1}. \tag{3.2}$$

Proof. Sufficiency. First, let us verify that condition (3.2) is sufficient for validity of inequality (3.1) in the case where $n = 2$. For this purpose, set

$$c = (p_1 a_1)^r + (p_2 a_2)^r, \quad x_1 = (p_1 a_1)^r, \quad \alpha_k = p_k b_k, \quad \beta_k = p_k^{-1}, \ k = 1,2, \tag{3.3}$$

and consider on the interval $[0,c]$ the function

$$h(x) = \alpha_1 \beta_1 x^{1/r} + \alpha_2 \beta_2 (c-x)^{1/r}. \tag{3.4}$$

If $r \neq 1$, $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then the unique critical point of this function, namely

$$x_* = c(\alpha_1\beta_1)^{\frac{r}{r-1}} \left((\alpha_1\beta_1)^{\frac{r}{r-1}} + (\alpha_2\beta_2)^{\frac{r}{r-1}}\right)^{-1} = c(\alpha_2\beta_2)^{\frac{r}{1-r}} \left((\alpha_1\beta_1)^{\frac{r}{1-r}} + (\alpha_2\beta_2)^{\frac{r}{1-r}}\right)^{-1}, \tag{3.5}$$

is the minimum point. Consequently, for $x \in [0,x_*]$, the function h does not increase and, for $x \in [x_*,c]$, it does not decrease. Hence, this function attains its minimum value on any interval $[x_0,c] \subset [x_*,c]$ at the point x_0. Thus, $\forall x \in [x_0,c] \subset [x_*,c]$

$$h(x) \geqslant h(x_0). \tag{3.6}$$

Setting

$$x_0 = c\beta_1^{-r}(\beta_1^{-r} + \beta_2^{-r})^{-1}, \tag{3.7}$$

we see that $[x_0,c] = \{x \in [0,c] \ : \ \beta_1 x^{\frac{1}{r}} \geqslant \beta_2 (c-x)^{\frac{1}{r}}\}$. Hence, if the sequence a does not increase then, by virtue of (3.3), we get

$$\beta_1 x_1^{\frac{1}{r}} = a_1 \geqslant a_2 = \beta_2 (c-x_1)^{\frac{1}{r}},$$

and, hence, $x_1 \in [x_0, c]$. Therefore, if we have $x_0 \geqslant x_*$ then, according to (3.6), we get

$$h(x_1) \geqslant h(x_0), \tag{3.8}$$

whence, taking into account notations in (3.3), (3.4) and (3.7), we obtain (3.1):

$$\sum_{k=1}^{2} p_k a_k b_k = h(x_1) \geqslant h(x_0) = \left(\sum_{i=1}^{2} p_i^r a_i^r\right)^{1/r} \sum_{k=1}^{2} p_k b_k \left(\sum_{k=1}^{2} p_k^r\right)^{-1/r}. \tag{3.9}$$

So, let us show that $x_0 \geqslant x_*$. By (3.2), taking into account (3.3), we get

$$\min_{s\in[1,2]} \sum_{k=1}^{s} \alpha_k \left(\sum_{k=1}^{s} \beta_k^{-r}\right)^{-1} = \sum_{k=1}^{2} \alpha_k \left(\sum_{k=1}^{2} \beta_k^{-r}\right)^{-1}.$$

Then, by the relation

$$\min_{k=1,2} \delta_k \gamma_k^r \leqslant \frac{\delta_1 + \delta_2}{\gamma_1^{-r} + \gamma_2^{-r}} \leqslant \max_{k=1,2} \delta_k \gamma_k^r \tag{3.10}$$

which holds for any numbers $\delta_k \geqslant 0$, $\gamma_k > 0$ and $r > 0$, and which is equality if and only if $\delta_1 \gamma_1^r = \delta_2 \gamma_2^r$, we have

$$\alpha_1 \beta_1^r \geqslant \alpha_2 \beta_2^r. \tag{3.11}$$

By virtue of (3.11), (3.5) and (3.7), we get

$$x_0 - x_* = \frac{c\left((\alpha_1\beta_1^r)^{\frac{r}{1-r}} - (\alpha_2\beta_2^r)^{\frac{r}{1-r}}\right)}{\left(\beta_1^{-r} + \beta_2^{-r}\right)\left((\alpha_1\beta_1)^{\frac{r}{r-1}} + (\alpha_2\beta_2)^{\frac{r}{r-1}}\right)} \geqslant 0.$$

Therefore, we see that indeed, $x_0 \geqslant x_*$ and hence, relation (3.9) is true.

If $r = 1$ then by virue of (3.11) the function h is nondecreasing on any interval $[x_0, c] \subset [0, c]$. If $\alpha_2 = 0$ and $\alpha_1 \neq 0$ then the function h also does not decrease on any interval $[x_0, c] \subset [0, c]$. Therefore, in these cases, inequality (3.6) holds, and, hence, relation (3.9) holds too.

By virtue of (3.11), only in the trivial case where $\alpha_1 = \alpha_2 = 0$, equality $\alpha_1 = 0$ holds.

Thus, for $n = 2$, the sufficiency of condition (3.2) for validity of inequality (3.1) is proved.

In general case, we prove by induction on n the proposition about sufficiency of condition (3.2) for validity of inequality (3.1).

The case $n = 1$ is obvious.

In the case where $n = 2$, it is proved above.

Assume that for $n = m - 1 \geqslant 1$, this proposition is true.

Let us show that for $n = m$, it is also true.

First, let us verify that for a certain number s, $s < m - 1$, the following equality is true:

$$\min_{j=s,s+1} \frac{\sum_{k=s}^{j} p_k b_k}{\sum_{k=s}^{j} p_k^r} = \frac{p_s b_s + p_{s+1} b_{s+1}}{p_s^r + p_{s+1}^r}. \tag{3.12}$$

Indeed, if for all $s < m-1$,

$$\min_{j=s,s+1} \sum_{k=s}^{j} p_k b_k \left(\sum_{k=s}^{j} p_k^r \right)^{-1} = b_s p_s^{1-r} < \sum_{k=s}^{s+1} p_k b_k \left(\sum_{k=s}^{s+1} p_k^r \right)^{-1}, \tag{3.13}$$

then using (3.10) from (3.13), we get

$$b_1 p_1^{1-r} < b_2 p_2^{1-r} < \ldots < b_m p_m^{1-r}.$$

By virtue of (3.10), it follows that

$$\sum_{k=1}^{m} p_k b_k \left(\sum_{k=1}^{m} p_k^r \right)^{-1} > \min_{k\in[1,m]} b_k p_k^{1-r} = b_1 p_1^{1-r}.$$

Hence, we obtain contradiction with (3.2). Therefore, there exist at least one number $s < m-1$ such that relation (3.12) is satisfied.

Let, for example, relation (3.12) holds for $s = 1$. Then we apply the proposition proved above for $n = 2$ to estimate the sum $\sum\limits_{k=1}^{2} p_k a_k b_k$. We get

$$\sum_{k=1}^{m} p_k a_k b_k \geqslant \left(p_1^r a_1^r + p_2^r a_2^r \right)^{1/r} (p_1 b_1 + p_2 b_2) \left(p_1^r + p_2^r \right)^{-1/r} + \sum_{k=3}^{m} p_k a_k b_k = \sum_{k=1}^{m-1} p'_k a'_k b'_k, \tag{3.14}$$

where

$$p'_k = \begin{cases} (p_1^r + p_2^r)^{1/r}, & k = 1, \\ p_{k+1}, & k = \overline{2, m-1}; \end{cases} \quad b'_k = \begin{cases} (p_1 b_1 + p_2 b_2) \left(p_1^r + p_2^r \right)^{-1/r}, & k = 1, \\ b_{k+1}, & k = \overline{2, m-1}; \end{cases} \tag{3.15}$$

$$a'_k = \begin{cases} \left(p_1^r a_1^r + p_2^r a_2^r \right)^{1/r} \left(p_1^r + p_2^r \right)^{-1/r}, & k = 1, \\ a_{k+1}, & k = \overline{2, m-1}. \end{cases} \tag{3.16}$$

The sum $\sum\limits_{k=1}^{m-1} p'_k a'_k b'_k$ contains $m-1$ items. For any nonincreasing sequence $a = \{a_k\}_{k=1}^n$, the sequence $a' = \{a'_k\}_{k=1}^n$ of the form as in (3.16) does not increase too. Indeed, according to (3.16) for any $k = 2, 3, \ldots, m-2$, we have

$$a'_k = a_{k+1} \geqslant a_{k+2} = a'_{k+1}.$$

The inequality $a'_1 \geqslant a'_2$ is equivalent to the inequality

$$\left(\frac{p_1^r a_1^r + p_2^r a_2^r}{p_1^r + p_2^r} \right)^{1/r} \geqslant a_3$$

which holds for any nonincreasing sequence a.

By virtue of (3.2) and (3.15), we conclude

$$\min_{s\in[1,m-1]}\frac{\sum\limits_{k=1}^{s}p'_kb'_k}{\sum\limits_{k=1}^{s}p'^{\,r}_k}\geqslant\min_{s\in[1,m]}\frac{\sum\limits_{k=1}^{s}p_kb_k}{\sum\limits_{k=1}^{s}p_k^r}=\frac{\sum\limits_{k=1}^{m}p_kb_k}{\sum\limits_{k=1}^{m}p_k^r}=\frac{\sum\limits_{k=1}^{m-1}p'_kb'_k}{\sum\limits_{k=1}^{m-1}p'^{\,r}_k}.$$

Consequently, for numbers a'_k, b'_k and p'_k, the induction assumption is satisfied. Hence, according to (3.14), (3.15) and (3.16), we obtain

$$\sum_{k=1}^{m}p_ka_kb_k\geqslant\sum_{k=1}^{m-1}p'_ka'_kb'_k\geqslant\left(\sum_{i=1}^{m-1}p'^{\,r}_i\,a'^{\,r}_i\right)^{1/r}\sum_{k=1}^{m-1}p'_kb'_k\left(\sum_{j=1}^{m-1}p'^{\,r}_j\right)^{-1/r}$$
$$=\left(\sum_{i=1}^{m}p_i^ra_i^r\right)^{1/r}\sum_{k=1}^{m}p_kb_k\left(\sum_{j=1}^{m}p_j^r\right)^{-1/r}$$

Therefore, in the case where condition (3.12) is satisfied for $s=1$, inequality (3.1) is true. In the same manner, one can prove that inequality (3.1) is true in the case where condition (3.12) is satisfied for any $1<s<m-1$.

Necessity. Assume that

$$\min_{s\in[1,n]}\sum_{k=1}^{s}p_kb_k\left(\sum_{k=1}^{s}p_k^r\right)^{-1}=\sum_{k=1}^{s^*}p_kb_k\left(\sum_{k=1}^{s^*}p_k^r\right)^{-1}<\sum_{k=1}^{n}p_kb_k\left(\sum_{k=1}^{n}p_k^r\right)^{-1}. \qquad (3.17)$$

Set

$$\alpha_1=\sum_{i=1}^{s^*}p_ib_i,\quad \alpha_2=\sum_{i=s^*+1}^{n}p_ib_i,\quad \beta_1=(\sum_{i=1}^{s^*}p_i^r)^{-1/r},\quad \beta_2=(\sum_{i=s^*+1}^{n}p_i^r)^{-1/r},$$

and for any $c>0$, consider on the interval $[0,c]$ the function h of the form as in (3.4).

Further, let us consider the sequence $a'=\{a'_i\}_{i=1}^n$ such that

$$a'_i=\begin{cases}\beta_1x_*^{\frac{1}{r}}, & i=1,2,\ldots,s^*,\\ \beta_2(c-x_*)^{\frac{1}{r}}, & i=s^*+1,\ldots,n,\end{cases}$$

where x_* is the point of the form as in (3.5).

According to the definition of a', we get

$$\sum_{k=1}^{n}p_ka'_kb_k=h(x_*) \qquad (3.18)$$

and

$$\left(\sum_{i=1}^{n}p_i^ra'^{\,r}_i\right)^{1/r}\sum_{k=1}^{n}p_kb_k\left(\sum_{k=1}^{n}p_k^r\right)^{-1/r}=h(x_0), \qquad (3.19)$$

where x_0 is a point of the form as in (3.7).

According to (3.10) from relation (3.17), we conclude that

$$\alpha_1\beta_1^r=\sum_{k=1}^{s^*}p_kb_k\left(\sum_{k=1}^{s^*}p_k^r\right)^{-1}<\sum_{k=s^*+1}^{n}p_kb_k\left(\sum_{k=s^*+1}^{n}p_k^r\right)^{-1}=\alpha_2\beta_2^r. \tag{3.20}$$

Consequently,

$$x_0-x_*=\frac{c\left((\alpha_1\beta_1^r)^{\frac{r}{1-r}}-(\alpha_2\beta_2^r)^{\frac{r}{1-r}}\right)}{\left(\beta_1^{-r}+\beta_2^{-r}\right)\left((\alpha_1\beta_1)^{\frac{r}{r-1}}+(\alpha_2\beta_2)^{\frac{r}{r-1}}\right)}<0,$$

and $x_0<x_*$. In view of the relation $[x_0,c]=\{x\in[0,c]\ :\ \beta_1x^{\frac{1}{r}}\geqslant\beta_2(c-x)^{\frac{1}{r}}\}$, we see that $\beta_1x_*^{\frac{1}{r}}\geqslant\beta_2(c-x_*)^{\frac{1}{r}}$ and for any $i=1,2,\dots,n-1$, the inequality $a_i'\geqslant a_{i+1}'$ holds.

Furthermore, as stated above, for $r\neq1$, $\alpha_1\neq0$ and $\alpha_2\neq0$, the point x_* of the form as in (3.5) is the minimum point of the function h. Consequently, the following inequality holds:

$$h(x_*)<h(x_0). \tag{3.21}$$

If $r=1$ (by virtue of (3.20)) or if $\alpha_1=0$ and $\alpha_2\neq0$, then the function h is nonincreasing on the interval $[0,c]$. Hence, in these cases, inequality (3.21) is also true. By virtue of (3.20), the equality $\alpha_2=0$ is impossible.

Combining relations (3.18), (3.19) and (3.21), we conclude that for the sequence a', inequality (3.1) does not hold. Thus, necessity of condition (3.2) is proved. □

LEMMA 3.2. *Let $r\in(1,\infty)$, and let $b=\{b_k\}_{k=1}^n$ and $p=\{p_k\}_{k=1}^n$ be nonnegative number sequences, $n\in\mathbb{N}$, $p_k>0$. Then for any nonnegative nonincreasing sequence $a=\{a_k\}_{k=1}^n$, inequality (3.1) is valid if and only if the following condition is satisfied:*

$$\min_{s\in[1,n]}\sum_{k=1}^{s}p_kb_k\left(\sum_{k=1}^{s}p_k^r\right)^{-1/r}=\sum_{k=1}^{n}p_kb_k\left(\sum_{k=1}^{n}p_k^r\right)^{-1/r}. \tag{3.22}$$

Proof. Necessity. Assume that

$$\min_{s\in[1,n]}\sum_{k=1}^{s}p_kb_k\left(\sum_{k=1}^{s}p_k^r\right)^{-1/r}=\sum_{k=1}^{s^*}p_kb_k\left(\sum_{k=1}^{s^*}p_k^r\right)^{-1/r}<\sum_{k=1}^{n}p_kb_k\left(\sum_{k=1}^{n}p_k^r\right)^{-1/r}. \tag{3.23}$$

Consider nonincreasing sequence $a'=\{a_i'\}_{i=1}^n$ of the form

$$a_i'=\begin{cases}(\sum_{k=1}^{s^*}p_k^r)^{-1/r}, & k=1,2,\dots,s^*,\\ 0, & k=s^*+1,\dots,n.\end{cases}$$

For this sequence, we have

$$\sum_{k=1}^{n}p_ka_k'b_k=\sum_{k=1}^{s^*}p_kb_k\left(\sum_{k=1}^{s^*}p_k^r\right)^{-1/r}$$

and

$$\left(\sum_{i=1}^{n} p_i^r a_i'^r\right)^{1/r} \sum_{k=1}^{n} p_k b_k \left(\sum_{k=1}^{n} p_k^r\right)^{-1/r} = \sum_{k=1}^{n} p_k b_k \left(\sum_{k=1}^{n} p_k^r\right)^{-1/r}.$$

Combining these relations and taking into account (3.23), we conclude that for the sequence a', inequality (3.1) does not hold. Thus, necessity of condition (3.22) is proved.

Sufficiency. We prove by induction on n the proposition about sufficiency of condition (3.22) for inequality (3.1) to be valid.

The case $n=1$ is obvious.

Assume that for $n=m-1\geqslant 1$, this proposition is true.

Let us show that for $n=m$, it is also true.

For this, we use notation (3.3) and consider on the interval $[0,c]$ the function h defined by relation (3.4) where $r\in(1,\infty)$.

Setting

$$x_0 = c\beta_1^{-r}(\beta_1^{-r}+\beta_2^{-r})^{-1}$$

and following the proof of Lemma 3.1, we conclude that $[x_0,c]=\{x\in[0,c] : \beta_1 x^{\frac{1}{r}} \geqslant \beta_2(c-x)^{\frac{1}{r}}\}$. By virtue of (3.3) and monotonicity of the sequence a, we have $x_1\in[x_0,c]$.

Further, we consider the following two cases:

$$1)\ h(x_1)\geqslant h(x_0) \tag{3.24}$$

and

$$2)\ h(x_1) < h(x_0). \tag{3.25}$$

In the first case, we use notation (3.15) and (3.16). Then by virtue of (3.24), we get

$$\sum_{k=1}^{m} p_k a_k b_k = h(x_1) + \sum_{k=3}^{m} p_k a_k b_k \geqslant h(x_0) + \sum_{k=3}^{m} p_k a_k b_k = \sum_{k=1}^{m-1} p_k' a_k' b_k'. \tag{3.26}$$

The sum $\sum\limits_{k=1}^{m-1} p_k' a_k' b_k'$ contains $m-1$ items. For any nonincreasing sequence $a=\{a_k\}_{k=1}^n$, the sequence $a'=\{a_k'\}_{k=1}^n$ of the form as in (3.16) does not increase too.

Furthermore, in view of relations (3.22) and (3.15), we conclude that

$$\min_{s\in[1,m-1]} \frac{\sum\limits_{k=1}^{s} p_k' b_k'}{\left(\sum\limits_{k=1}^{s} p_k'^r\right)^{1/r}} \geqslant \min_{s\in[1,m]} \frac{\sum\limits_{k=1}^{s} p_k b_k}{\left(\sum\limits_{k=1}^{s} p_k^r\right)^{1/r}} = \frac{\sum\limits_{k=1}^{m} p_k b_k}{\left(\sum\limits_{k=1}^{m} p_k^r\right)^{1/r}} = \frac{\sum\limits_{k=1}^{m-1} p_k' b_k'}{\left(\sum\limits_{k=1}^{m-1} p_k'^r\right)^{1/r}}. \tag{3.27}$$

Consequently, for the numbers a'_k, b'_k and p'_k, the induction assumption is satisfied. Hence, according to (3.26), (3.15) and (3.16), we get

$$\begin{aligned}\sum_{k=1}^{m} p_k a_k b_k &\geqslant \sum_{k=1}^{m-1} p'_k a'_k b'_k \geqslant \left(\sum_{i=1}^{m-1} p'^r_i a'^r_i\right)^{1/r} \sum_{k=1}^{m-1} p'_k b'_k \left(\sum_{k=1}^{m-1} p'^r_k\right)^{-1/r} \\ &= \left(\sum_{i=1}^{m} p^r_i a^r_i\right)^{1/r} \sum_{k=1}^{m} p_k b_k \left(\sum_{k=1}^{m} p^r_k\right)^{-1/r}.\end{aligned} \tag{3.28}$$

Therefore, in the case where condition (3.24) is satisfied, inequality (3.1) is true.

Let us show that in the case where condition (3.25) is satisfied, this inequality is also true. It is clear that by virtue of (3.25), the number $\alpha_2 \neq 0$. For $\alpha_1 \neq 0$, on [0, c] the function h has the single critical point

$$x_* = \frac{c(\alpha_2\beta_2)^{\frac{r}{1-r}}}{(\alpha_1\beta_1)^{\frac{r}{1-r}} + (\alpha_2\beta_2)^{\frac{r}{1-r}}},$$

which is a maximum point of the function. Consequently, for $x \in [0, x_*]$, the function h does not decrease and, for $x \in [x_*, c]$, it does not increase. By virtue of (3.25) and the inequality $x_1 > x_0$, we conclude that, for $x \in [x_1, c]$, the function h does not increase. Hence, for any $\widetilde{x} \in [x_1, c]$ the following inequality is true:

$$h(\widetilde{x}) < h(x_1). \tag{3.29}$$

If $\alpha_1 = 0$ then the function h does not increase on any interval $[x_0, c]$. Thus, relation (3.29) is also true.

Let $\widetilde{x}$ is a number such that $\beta_2(c-\widetilde{x})^{\frac{1}{r}} = a_3$. Then, taking into account accepted notations and monotonicity of the sequence a, we conclude that $\widetilde{x} \in [x_1, c]$ and, hence, inequality (3.29) is true.

Setting

$$p'_k = \begin{cases} p_1, & k = 1, \\ (p_2^r + p_3^r)^{1/r}, & k = 2, \\ p_{k+1}, & k = \overline{3, m-1}; \end{cases} \quad b'_k = \begin{cases} b_1, & k = 1, \\ (p_2 b_2 + p_3 b_3)\left(p_2^r + p_3^r\right)^{-1/r}, & k = 2, \\ b_{k+1}, & k = \overline{3, m-1}; \end{cases} \tag{3.30}$$

$$a'_k = \begin{cases} a_1, & k = 1, \\ \left(p_2^r a_2^r + p_3^r a_3^r\right)^{1/r} \left(p_2^r + p_3^r\right)^{-1/r}, & k = 2, \\ a_{k+1}, & k = \overline{3, m-1}, \end{cases} \tag{3.31}$$

we obtain

$$\sum_{k=1}^{m} p_k a_k b_k = h(x_1) + \sum_{k=3}^{m} p_k a_k b_k \geqslant h(\widetilde{x}) + \sum_{k=3}^{m} p_k a_k b_k = \sum_{k=1}^{m-1} p'_k a'_k b'_k.$$

The sum $\sum\limits_{k=1}^{m-1} p'_k a'_k b'_k$ contains $m-1$ items. For any nonincreasing sequence $a = \{a_k\}_{k=1}^n$, the sequence $a' = \{a'_k\}_{k=1}^n$ of the form as in (3.31) does not increase too. Furthermore, by virtue of relations (3.22) and (3.30), relation (3.27) is true.

Consequently, for the numbers a'_k, b'_k and p'_k, the induction assumption is satisfied. Hence, in this case, relation (3.28) also holds. Therefore, inequality (3.1) holds too. Lemma is completed. □

LEMMA 3.3. *Let $r \in (0,1)$, and let $b = \{b_k\}_{k=1}^n$ and $p = \{p_k\}_{k=1}^n$ be nonnegative number sequences, $n \in \mathbb{N}$, $p_k > 0$. Then for any nonnegative nonincreasing sequence $a = \{a_k\}_{k=1}^n$, the inequality*

$$\sum_{k=1}^n p_k a_k b_k \leqslant \left(\sum_{i=1}^n p_i^r a_i^r\right)^{1/r} \sum_{k=1}^n p_k b_k \left(\sum_{k=1}^n p_k^r\right)^{-1/r} \tag{3.32}$$

is valid, if and only if the following condition is satisfied:

$$\max_{s\in[1,n]} \sum_{k=1}^s p_k b_k \left(\sum_{k=1}^s p_k^r\right)^{-1/r} = \sum_{k=1}^n p_k b_k \left(\sum_{k=1}^n p_k^r\right)^{-1/r}. \tag{3.33}$$

LEMMA 3.4. *Let $r \in [1,\infty)$, and let $b = \{b_k\}_{k=1}^n$ and $p = \{p_k\}_{k=1}^n$ be nonnegative number sequences, $n \in \mathbb{N}$, $p_k > 0$. Then, for any nonnegative nonincreasing sequence $a = \{a_k\}_{k=1}^n$, inequality (3.32) is valid, if and only if the following condition is satisfied:*

$$\max_{s\in[1,n]} \sum_{k=1}^s p_k b_k \left(\sum_{k=1}^s p_k^r\right)^{-1} = \sum_{k=1}^n p_k b_k \left(\sum_{k=1}^n p_k^r\right)^{-1}. \tag{3.34}$$

The proof of Lemma 3.3 is similar to the proof of Lemma 3.2 and the proof of Lemma 3.4 is similar to the proof of Lemma 3.1.

It follows from Proposition 3 of the paper [13] that condition (3.34) is sufficient for inequality (3.32) to be true for any nonincreasing sequence $a = \{a_k\}_{k=1}^n$ and for any $r \in [1,\infty)$.

In the proof of Lemma 1 of the paper [14], in fact, it was shown that for any nonincreasing sequence $a = \{a_k\}_{k=1}^n$ and for any $r \in (0,1)$, inequality (3.32) is valid if condition (3.33) is satisfied.

3.2. Proofs of theorems

Necessity in Theorem 2.1 and Theorem 2.2 is proved by analogy with the proof of necessity in Lemma 3.1 and Lemma 3.2 correspondingly.

Sufficiency. Let $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ be integrable functions such that for corresponding number $r \in (0,\infty)$, condition (2.1) or (2.2) is satisfied.

First, let us prove the proposition that inequality (1.3) holds for any function f such that for a certain $n \in \mathbb{N}$, the following representation is true:

$$f(t) = a_k, \quad t \in (s_{k-1}, s_k),\ k = 1,2,\ldots,n, \tag{3.35}$$

where $a_1 > a_2 > \ldots > a_n \geqslant 0$ and $a = s_0 < s_1 < \ldots < s_n = b$.

For any $k = 1,2,\ldots,n$, we set

$$p_k = \left(\int\limits_{s_{k-1}}^{s_k} p^r(x)dx\right)^{1/r}, \quad b_k = \int\limits_{s_{k-1}}^{s_k} p(x)g(x)dx\left(\int\limits_{s_{k-1}}^{s_k} p^r(x)dx\right)^{-1/r}. \tag{3.36}$$

If condition (2.1) or (2.2) is satisfied, then correspondingly, the following relation is true:

$$\min_{s\in[1,n]} \sum_{k=1}^{s} p_k b_k \left(\sum_{k=1}^{s} p_k^r\right)^{-1} = \min_{k=\overline{1,n}} \int\limits_a^{s_k} p(x)g(x)dx\left(\int\limits_a^{s_k} p^r(x)dx\right)^{-1}$$
$$\geqslant \int\limits_a^b p(x)g(x)dx\left(\int\limits_a^b p^r(x)dx\right)^{-1} = \sum_{k=1}^{n} p_k b_k \left(\sum_{k=1}^{n} p_k^r\right)^{-1}$$

or

$$\min_{s\in[1,n]} \sum_{k=1}^{s} p_k b_k \left(\sum_{k=1}^{s} p_k^r\right)^{-r} = \min_{k=\overline{1,n}} \int\limits_a^{s_k} p(x)g(x)dx\left(\int\limits_a^{s_k} p^r(x)dx\right)^{-r}$$
$$\geqslant \int\limits_a^b p(x)g(x)dx\left(\int\limits_a^b p^r(x)dx\right)^{-r}$$
$$= \sum_{k=1}^{n} p_k b_k \left(\sum_{k=1}^{n} p_k^r\right)^{-r}$$

Consequently, for sequences $a = \{a_k\}_{k=1}^n$, $b = \{b_k\}_{k=1}^n$ and $p = \{p_k\}_{k=1}^n$, conditions of Lemma 3.1 or Lemma 3.2 hold. Hence, inequality (3.1) is true. By virtue of (3.35) and (3.36), we obtain necessary relation

$$\int\limits_a^b p(x)f(x)g(x)dx = \sum_{k=1}^{n} \int\limits_{s_{k-1}}^{s_k} p(x)f(x)g(x)dx = \sum_{k=1}^{n} p_k a_k b_k$$
$$\geqslant \left(\sum_{i=1}^{n} p_i^r a_i^r\right)^{\frac{1}{r}} \sum_{k=1}^{n} p_k b_k \left(\sum_{k=1}^{n} p_k^r\right)^{-\frac{1}{r}}$$
$$= \left(\int\limits_a^b p^r(x)f^r(x)dx\right)^{\frac{1}{r}} \int\limits_a^b p(x)g(x)dx\left(\int\limits_a^b p^r(x)dx\right)^{-\frac{1}{r}}.$$

To prove the *sufficiency* in general case let us consider the functions $f_n(t)$, $n \in \mathbb{N}$, such that

$$f_n(t) = \frac{kf(a)}{n}, \quad t:\ \frac{(k-1)f(a)}{n} < f(t) \leqslant \frac{kf(a)}{n},\ k = 1,2,\ldots,n. \tag{3.37}$$

We see that the inequality $f(t) \leqslant f_n(t)$ holds for all $n \in \mathbb{N}$ and $t \in [a,b]$. By virtue of the summability on $[a,b]$ of the product $p(t)f(t)g(t)$, the values

$$\int_a^b p(t)g(t)(f_n(t) - f(t))dt$$

converge to zero as $n \to \infty$. Furthermore, for any $n \in \mathbb{N}$, the function $f_n(t)$ is non-increasing and it takes finitely many values on $[a,b]$. Hence, this function satisfies conditions of the proposition proved above. Thus, in view of (3.37), we conclude that for any $\varepsilon > 0$ and for all sufficiently great n $(n > n_0(\varepsilon))$

$$\begin{aligned}\int_a^b p(t)f(t)g(t)dt &= \int_a^b p(t)g(t)f_n(t)dt - \int_a^b p(t)g(t)(f_n(t) - f(t))dt \\ &\geqslant \left(\int_a^b p^r(t)f_n^r(t)dt\right)^{1/r} \frac{\int_a^b p(t)g(t)dt}{\left(\int_a^s p^r(t)dt\right)^{1/r}} - \varepsilon \\ &\geqslant \left(\int_a^b p^r(t)f^r(t)dt\right)^{1/r} \frac{\int_a^b p(t)g(t)dt}{\left(\int_a^b p^r(t)dt\right)^{1/r}} - \varepsilon.\end{aligned}$$

Therefore, inequality (1.3) is true. Theorems 2.1 and 2.2 are completed.

Theorems 2.3 and 2.4 are proved by analogy. □

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