THE STEIN–WEISS TYPE INEQUALITIES
FOR THE B–RIESZ POTENTIALS

A. D. GADJIYEV, V. S. GULIYEV, A. SERBETCI AND E. V. GULIYEV

(Communicated by J. Pečarić)

Abstract. We establish two inequalities of Stein-Weiss type for the Riesz potential operator $I_{\alpha,\gamma}$ ($B$–Riesz potential operator) generated by the Laplace-Bessel differential operator $\Delta_k$ in the weighted Lebesgue spaces $L_{p,|x|^{\beta,\gamma}}$. We obtain necessary and sufficient conditions on the parameters for the boundedness of $I_{\alpha,\gamma}$ from the spaces $L_{p,|x|^{\beta,\gamma}}$ to $L_{q,|x|^{-\lambda,\gamma}}$, and from the spaces $L_{1,|x|^{\beta,\gamma}}$ to the weak spaces $WL_{q,|x|^{-\lambda,\gamma}}$. In the limiting case $p = Q/\alpha$ we prove that the modified $B$–Riesz potential operator $\widetilde{I}_{\alpha,\gamma}$ is bounded from the spaces $L_{p,|x|^{\beta,\gamma}}$ to the weighted $B – BMO$ spaces $BMO_{|x|^{-\lambda,\gamma}}$.

As applications, we get the boundedness of $I_{\alpha,\gamma}$ from the weighted $B$-Besov spaces $B^\gamma_{p\theta,|x|^{\beta,\gamma}}$ to the spaces $B^\gamma_{q\theta,|x|^{-\lambda,\gamma}}$. Furthermore, we prove two Sobolev embedding theorems on weighted Lebesgue $L_{p,|x|^{\beta,\gamma}}$ and weighted $B$-Besov spaces $B^\gamma_{p\theta,|x|^{\beta,\gamma}}$ by using the fundamental solution of the $B$-elliptic equation $\Delta_B^{\alpha/2}$.

1. Introduction and main results

Let $\mathbb{R}^n_{k,+} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0, \ldots, x_k > 0\}$, $1 \leq k \leq n$. We denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all classes of measurable functions $f$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx\right)^{1/p}, \quad 1 \leq p < \infty,$$

where $x' = (x_1, \ldots, x_k)$, and $\gamma = (\gamma_1, \ldots, \gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + \ldots + \gamma_k$ and $(x')^\gamma = x_1^{\gamma_1} \ldots x_k^{\gamma_k}$. If $p = \infty$, we assume

$$L_{\infty,\gamma} \equiv L_{\infty} = \{f : \|f\|_{L_{\infty,\gamma}} = \text{ess sup}_{x \in \mathbb{R}^n_{k,+}} |f(x)| < \infty\}.$$

For any measurable set $E \subset \mathbb{R}^n_{k,+}$, let $|E|_\gamma = \int_E (x')^\gamma dx$. The weak $L_{p,\gamma}$ space $WL_{p,\gamma} \equiv WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leq p < \infty$, is defined as the set of locally integrable functions $f$, with


Keywords and phrases: Laplace-Bessel differential operator, $B$-Riesz potential, Stein-Weiss type inequalities, weighted Lebesgue space, weighted $B$-Besov space.

A. D. Gadjiev was partially supported by the grant of INTAS (project 06-1000017-8792).

V. S. Guliyev was partially supported by BGP II (project ANSF/AZM1-3110-BA-08).
finite norm
\[ \|f\|_{L^p_{\gamma}} = \sup_{r > 0} r \left\{ x \in \mathbb{R}^n_{k,+} : |f(x)| > r \right\}^{1/p}. \]

Let \( w \) be a weight function on \( \mathbb{R}^n_{k,+} \), i.e., \( w \) is a non-negative and measurable function on \( \mathbb{R}^n_{k,+} \), then for all measurable functions \( f \) on \( \mathbb{R}^n_{k,+} \) the weighted Lebesgue space \( L^p_{\gamma}(\mathbb{R}^n_{k,+}) \) and the weak weighted Lebesgue space \( \text{WL}^p_{\gamma}(\mathbb{R}^n_{k,+}) \) are defined by
\[ L^p_{\gamma} = \{ f : \|f\|_{L^p_{\gamma}} = \|wf\|_{L^p_{\gamma}} < \infty \} \]
and
\[ \text{WL}^p_{\gamma} = \{ f : \|f\|_{\text{WL}^p_{\gamma}} = \|wf\|_{\text{WL}^p_{\gamma}} < \infty \}, \]
respectively.

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The potential and related topics associated with the Laplace-Bessel differential operator
\[ \Delta_B = \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, k \]
have been the research interests of many mathematicians such as B. Muckenhoupt and E. Stein [26], K. Stempak [28], K. Trimèche [30], I. Kipriyanov [17], A. D. Gadjiev and I. A. Aliev [8], L. Lyakhov [23], I. A. Aliev and B. Rubin [1], V. S. Guliyev [12]-[14], and others.

In this paper we study the Riesz potential associated with \( \Delta_B \) (B-Riesz potential) defined by
\[ I_{\alpha,\gamma}f(x) = \int_{\mathbb{R}^n_{k,+}} T^\gamma|x|^\alpha - Q f(y)(y')^\gamma dy, \]
and the modified B-Riesz potential by
\[ \tilde{I}_{\alpha,\gamma}f(x) = \int_{\mathbb{R}^n_{k,+}} \left( T^\gamma|x|^\alpha - |y|^\alpha - Q \chi_{B_1}(y) \right) f(y)(y')^\gamma dy \]
in weighted Lebesgue spaces \( L^p_{\alpha,\beta}(\mathbb{R}^n_{k,+}) \), where \( T^\gamma \) is \( B \)-shift operators is defined below, \( B(x,r) = \{ y \in \mathbb{R}^n_{k,+} : |x - y| < r \} \) is the open ball centered at \( x \) with radius \( r \) in \( \mathbb{R}^n_{k,+} \) and \( B_r = B(0,r), \quad \mathbb{R}^n_{k,+} \setminus B_r, \) and \( 0 < \alpha < Q, \quad Q = n + |\gamma|. \)

V. Kokilashvili and A. Meskhi [21] proved the Stein-Weiss inequality for the fractional integral operator defined on nonhomogeneous spaces. In this paper we establish two inequalities of Stein-Weiss type (see [27]) for the \( B \)-Riesz potential \( I_{\alpha,\gamma}f \). We give the Stein-Weiss type inequality in Theorem 1, and a weak version of the Stein-Weiss inequality in Theorem 2 for \( I_{\alpha,\gamma}f \).
Theorem 1. Let $0 < \alpha < Q$, $1 < p \leq q < \infty$, $\beta < Q/p'$, $\lambda < Q/q$, $\beta + \lambda \geq 0$ ($\beta + \lambda > 0$, if $p = q$), $1/p - 1/q = (\alpha - \beta - \lambda)/Q$ and $f \in L_{p,|x|^\beta,Y}$. Then $I_{\alpha,Y}f \in L_{q,|x|^{-\lambda},Y}$ and the following inequality holds

$$
\left(\int_{\mathbb{R}^n_+} |x|^{-\lambda q} |I_{\alpha,Y}f(x)|^q (x')^\gamma dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n_+} |x|^\beta p |f(x)|^p (x')^\gamma dx\right)^{1/p},
$$

where $C$ is independent of $f$.

Theorem 2. Let $0 < \alpha < Q$, $1 < q < \infty$, $\beta \leq 0$, $\lambda < Q/q$, $\beta + \lambda \geq 0$, $1 - 1/q = (\alpha - \beta - \lambda)/Q$ and $f \in L_{1,|x|^\beta,Y}$. Then $I_{\alpha,Y}f \in WL_{q,|x|^{-\lambda},Y}$ and the following inequality holds

$$
\left(\int_{\{x \in \mathbb{R}^n_+: |x|^{-\lambda} |I_{\alpha,Y}f(x)| > \gamma\}} (x')^\gamma dx\right)^{1/q} \leq C \int_{\mathbb{R}^n_+} |x|^\beta |f(x)|(x')^\gamma dx,
$$

where $C$ is independent of $f$.

In the following, by using Stein-Weiss type Theorems 1 and 2, we obtain necessary and sufficient conditions on the parameters for the boundedness of the $B$-Riesz potential operator $I_{\alpha,Y}$ from the spaces $L_{p,|x|^\beta,Y}$ to $L_{q,|x|^{-\lambda},Y}$, and from the spaces $L_{1,|x|^\beta,Y}$ to the weak spaces $WL_{q,|x|^{-\lambda},Y}$. In the limiting case $p = Q/\alpha$ we prove that the modified $B$-Riesz potential operator $\tilde{I}_{\alpha}$ is bounded from the space $L_{p,|x|^\beta,Y}$ to the weighted $B$-BMO space $BMO_{|x|^{-\lambda},Y}$.

Theorem 3. Let $0 < \alpha < Q$, $1 \leq p \leq q < \infty$, $\beta < Q/p'$ ($\beta \leq 0$, if $p = 1$), $\lambda < Q/q$ ($\lambda \leq 0$, if $q = \infty$), $\alpha \geq \beta + \lambda \geq 0$ ($\beta + \lambda > 0$, if $p = q$).

1) If $1 < p < Q/(\alpha - \beta - \lambda)$, then the condition $1/p - 1/q = (\alpha - \beta - \lambda)/Q$ is necessary and sufficient for the boundedness of $I_{\alpha,Y}$ from $L_{p,|x|^\beta,Y}$ to $L_{q,|x|^{-\lambda},Y}$.

2) If $p = 1$, then the condition $1 - 1/q = (\alpha - \beta - \lambda)/Q$ is necessary and sufficient for the boundedness of $I_{\alpha,Y}$ from $L_{1,|x|^\beta,Y}$ to $WL_{q,|x|^{-\lambda},Y}$.

3) If $1 < p = Q/(\alpha - \beta - \lambda)$, then the operator $\tilde{I}_{\alpha,Y}$ is bounded from $L_{p,|x|^\beta,Y}$ to $BMO_{|x|^{-\lambda},Y}$.

Moreover, if the integral $I_{\alpha,Y}f$ exists almost everywhere for $f \in L_{p,|x|^\beta,Y}$, then $I_{\alpha,Y}f \in BMO_{|x|^{-\lambda},Y}$ and the following inequality holds

$$
\|I_{\alpha,Y}f\|_{BMO_{|x|^{-\lambda},Y}} \leq C \|f\|_{L_{p,|x|^\beta,Y}},
$$

where $C > 0$ is independent of $f$.

Remark 1. Note that in the case of $k = 1$ the statements 1) and 2) in Theorem 3 were proved in [9], and the statement 3) in [10].
Here the weighted \( B - BMO \) space \( BMO_{w, \gamma} \) is defined as the set of locally integrable functions \( f \) with finite norm

\[
\|f\|_{*, w, \gamma} = \sup_{x \in \mathbb{R}^n_+, r > 0} w(B_r)^{-1} \int_{B_r} |T_y f(x) - f_{B_r}(x)|(y')^\gamma dy < \infty,
\]

and \( B - BMO \) space (see [13]) \( BMO_\gamma(\mathbb{R}^n_{k,+}) \equiv BMO_{1, \gamma}(\mathbb{R}^n_{k,+}) \), where

\[
f_{B_r}(x) = \frac{1}{|B_r|} \int_{B_r} T_y f(x)(y')^\gamma dy,
\]

\[
|B_r| = \omega(n,k,\gamma)r^Q \quad \text{and} \quad \omega(n,k,\gamma) = \int_{B_1} (x')^\gamma dx = \pi^{(n-k)/2} 2^{-k} \prod_{i=1}^k \Gamma((\gamma_i + 1)/2)[\Gamma(\gamma_i/2)]^{-1}.
\]

Besov spaces in the setting of the Bessel differential operator on \((0, \infty)\) is studied by G. Altenburg [2], D. I. Cruz-Baez and J. Rodriguez, [5], M. Assal and H. Ben Abdallah [3], and the setting of the Laplace-Bessel differential operator on \( \mathbb{R}^n_{k,+} \), studied by V. Guliyev, A. Serbetci and Z. V. Safarov [16].

In Theorem 4 we prove the boundedness of \( I_{\alpha, \gamma} \) in the weighted Besov spaces associated with the Laplace-Bessel differential operator on \( \mathbb{R}^n_{k,+} \) (weighted \( B \)-Besov spaces)

\[
B^s_{p\theta,w,\gamma} = \left\{ f : \|f\|_{B^s_{p\theta,w,\gamma}} = \|f\|_{I_p,w,\gamma} + \left( \int_{\mathbb{R}^n_{k,+}} \frac{\|T^\gamma f(y') - f(y')\|_{L_{p,w,\gamma}(x')^\gamma dx}}{|x|^{Q+s\theta}} \right)^{1/\theta} < \infty \right\}
\]

for a power weight \( w \), \( 1 \leq p, \theta \leq \infty \) and \( 0 < s < 1 \).

**THEOREM 4.** Let \( 0 < \alpha < Q, 1 < p \leq q < \infty, \beta < Q/p', \lambda < Q/q, \alpha \geq \beta + \lambda \geq 0 \) (\( \beta + \lambda > 0 \), if \( p = q \)).

If \( 1 < p < Q/(\alpha - \beta - \lambda) \), \( 1/p - 1/q = (\alpha - \beta - \lambda)/Q \), \( 1 \leq \theta \leq \infty \) and \( 0 < s < 1 \),

\[
\text{then the operator } I_{\alpha, \gamma} \text{ is bounded from } B^s_{p\theta,|x|^\beta,\gamma} \text{ to } B^s_{q\theta,|x|^{-\lambda}, \gamma}. \text{ More precisely, there is a constant } C > 0 \text{ such that}
\]

\[
\|I_{\alpha, \gamma}f\|_{B^s_{q\theta,|x|^{-\lambda}, \gamma}} \leq C \|f\|_{B^s_{p\theta,|x|^\beta, \gamma}}
\]

holds for all \( f \in B^s_{p\theta,|x|^\beta, \gamma} \).

It is known that (see [18], [19]) there exists a positive constant \( C_0 \) such that \( G(x) = C_0|x|^{2-Q} \) is the fundamental solution of the Laplace-Bessel differential operator \( \Delta_B \).

**THEOREM 5.** [19] Let \( \alpha \) is an even positive integer such that \( 0 < \alpha < Q \). If the function \( f \) is finite, even with respect to the variables \( x_1, \ldots, x_k \) having \( \alpha \) continuous
derivatives by the variables \(x_1, \ldots, x_k\) and \(\alpha/2\) continuous derivatives by \(x_{k+1}, \ldots, x_n\), then the potential \(I_{\alpha, \gamma} f\) is a solution of the \(B\)-elliptic equation

\[
\Delta_B^{\alpha/2} u(x) = f(x).
\]

In the following we prove two Sobolev embedding theorems on weighted Lebesgue \(L_{p,|x|^{\beta, \gamma}}\) and weighted \(B\)-Besov spaces \(B^s_{p,\theta,|x|^{\beta, \gamma}}\) by using the fundamental solution of the \(B\)-elliptic equation \(\Delta_B^{\alpha/2}\). We expect that these results will be useful to investigate the regularity properties of \(B\)-elliptic differential equations.

From Theorems 3 and 5 we have

**Theorem 6.** Let \(f\) be defined as in Theorem 5 and \(\alpha\) be an even positive integer, \(0 < \alpha < Q\). \(1 \leq p < q < \infty\), \(\beta < Q/p\) (\(\beta \leq 0\), if \(p = 1\)), \(\lambda < Q/q\) (\(\lambda \leq 0\), if \(q = \infty\)), \(\alpha \geq \beta + \lambda \geq 0\) (\(\beta + \lambda > 0\), if \(p = q\)).

1) If \(f \in L_{p,|x|^{\beta, \gamma}}, \ 1 < p < Q/(\alpha - \beta - \lambda), \ 1/p - 1/q = (\alpha - \beta - \lambda)/Q\), then the following estimation holds:

\[
\|u\|_{L_{q,|x|^{-\lambda, \gamma}}} \leq C\|\Delta_B^{\alpha/2} u\|_{L_{p,|x|^{\beta, \gamma}}},
\]

where \(C > 0\) is independent of \(u\).

2) If \(f \in L_{1,|x|^{\beta, \gamma}}, \ 1/1 - 1/q = (\alpha - \beta - \lambda)/Q\), then the following estimation holds:

\[
\|u\|_{WL_{q,|x|^{-\lambda, \gamma}}} \leq C\|\Delta_B^{\alpha/2} u\|_{L_{1,|x|^{\beta, \gamma}}},
\]

where \(C > 0\) is independent of \(u\).

From Theorems 4 and 5 we have

**Theorem 7.** Let \(\alpha\) be an even positive integer, \(0 < \alpha < Q\), \(1 < p \leq q < \infty\), \(\beta < Q/p\), \(\lambda < Q/q\), \(\alpha \geq \beta + \lambda \geq 0\) (\(\beta + \lambda > 0\), if \(p = q\)).

If \(f \in B^s_{p,\theta,|x|^{\beta, \gamma}}, \ 1 < p < Q/(\alpha - \beta - \lambda), \ 1/p - 1/q = (\alpha - \beta - \lambda)/Q, \ 1 \leq \theta \leq \infty\) and \(0 < s < 1\), then the following estimation holds:

\[
\|u\|_{B^s_{q,\theta,|x|^{-\lambda, \gamma}}} \leq C\|\Delta_B^{\alpha/2} u\|_{B^s_{p,\theta,|x|^{\beta, \gamma}}},
\]

where \(C > 0\) is independent of \(u\).

2. Preliminaries

Denote the generalized shift operator (\(B\)-shift operator) by \(T^y\), acting according to the law

\[
T^y f(x) = C_{\gamma,k} \int_0^\pi \ldots \int_0^\pi f((x',y')_\beta,x'' - y'') \ d\nu(\beta),
\]
where \((x', y')_\beta = ((x_1, y_1)\beta_1, \ldots, (x_k, y_k)\beta_k)\), \((x_i, y_i)\beta_i = (x_i^2 - 2x_iy_i\cos\beta_i + y_i^2)^{1/2}\), \(1 \leq i \leq k\), \(d\nu(\beta) = \prod_{i=1}^{k} \sin^{\beta_i-1}\beta_i \, d\beta_1 \ldots d\beta_k\), \(1 \leq k \leq n\) and

\[
C_{\gamma, k} = \pi^{-k/2} \prod_{i=1}^{k} \Gamma((\gamma_i + 1)/2) [\Gamma(\gamma_i/2)]^{-1} = 2^k \pi^{-k} \omega(2k, k, \gamma).
\]

We remark that the generalized shift operator \(T^y\) is closely connected with the Laplace-Bessel differential operator \(\Delta_B\) (see [17, 22, 23] for details). Furthermore, \(T^y\) generates the corresponding \(B\)-convolution

\[
(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)[T^y g(x)](y') dy.
\]

**Lemma 1.** [9] Let \(0 < \alpha < Q\). Then for \(2|x| \leq |y|\), \(x, y \in \mathbb{R}^n_{k,+}\), the following inequality holds

\[
|T^y|x|^\alpha - Q - |y|^\alpha - Q| \leq 2^{Q-\alpha+1}|y|^\alpha - 1|x|. \tag{4}
\]

We will need the following Hardy-type transforms defined on \(\mathbb{R}^n_{k,+}\):

\[
H_\gamma f(x) = \int_{B_{|x|}} f(y)(y')^\gamma dy,
\]

and

\[
H'_\gamma f(x) = \int_{\mathbb{C}_{B_{|x|}}} f(y)(y')^\gamma dy.
\]

The following two theorems related to the boundedness of these transforms were proved in [6] (see also [7], Section 1.1).

**Theorem A.** Let \(1 < q < \infty\). Suppose that \(v\) and \(w\) are a.e. positive functions on \(\mathbb{R}^n_{k,+}\). Then

(a) The operator \(H_\gamma\) is bounded from \(L_{1,w,\gamma}\) to \(WL_{q,v,\gamma}\) if and only if

\[
A_1 \equiv \sup_{r > 0} \left( \int_{B_r} v^q(x)(x')^\gamma dx \right)^{1/q} \sup_{B_r} w^{-1}(x) < \infty;
\]

(b) The operator \(H'_\gamma\) is bounded from \(L_{1,w,\gamma}\) to \(WL_{q,v,\gamma}\) if and only if

\[
A_2 \equiv \sup_{r > 0} \left( \int_{B_r} v^q(x)(x')^\gamma dx \right)^{1/q} \sup_{\mathbb{C}_{B_r}} w^{-1}(x) < \infty.
\]

Moreover, there exist positive constants \(a_j\), \(j = 1, \ldots, 4\), depending only on \(q\) such that \(a_1A_1 \leq \|H\| \leq a_2A_1\) and \(a_3A_2 \leq \|H'\| \leq a_4A_2\).
THEOREM B. Let $1 < p \leq q < \infty$. Suppose that $v$ and $w$ are a.e. positive functions on $\mathbb{R}^n_{k,+}$. Then

(a) The operator $H_\gamma$ is bounded from $L_{p,w,\gamma}$ to $L_{q,v,\gamma}$ if and only if

$$A_3 \equiv \sup_{r>0} \left( \int_{B_r} v^q(x)(x')^\gamma dx \right)^{1/q} \left( \int_{B_r} w^{-p'}(x)(x')^\gamma dx \right)^{1/p'} < \infty,$$

$p' = p/(p-1)$;

(b) The operator $H'_\gamma$ is bounded from $L_{p,w,\gamma}$ to $L_{q,v,\gamma}$ if and only if

$$A_4 \equiv \sup_{r>0} \left( \int_{B_r} v^q(x)(x')^\gamma dx \right)^{1/q} \left( \int_{B_r} w^{-p'}(x)(x')^\gamma dx \right)^{1/p'} < \infty.$$

Moreover, there exist positive constants $b_j$, $j = 1, \ldots, 4$, depending only on $p$ and $q$ such that $b_1 A_3 \leq \|H\| \leq b_2 A_3$ and $b_3 A_4 \leq \|H'\| \leq b_4 A_4$.

We will need the case that we substitute $L_{p,v,\gamma}$ with the homogeneous space $(X, \rho, \mu)$ in Theorems A and B in which $X = \mathbb{R}^n_{k,+}$, $\rho(x,y) = |x-y|$ and $d\mu(x) = (x')^\gamma dx$.

DEFINITION 1. The weight function $w$ belongs to the class $A_{p,\gamma}$ for $1 < p, q < \infty$, if

$$\sup_{x,r} \left( \frac{\left| B(x,r) \right|^{-1}_\gamma}{\left| B(x,r) \right|^{-1}_\gamma} \int_{B(x,r)} w(y)(y')^\gamma dy \right)^{p-1} \left( \frac{\left| B(x,r) \right|^{-1}_\gamma}{\left| B(x,r) \right|^{-1}_\gamma} \int_{B(x,r)} w^{-\frac{1}{p-1}}(y)(y')^\gamma dy \right) < \infty$$

and $w$ belongs to $A_{1,\gamma}$, if there exists a positive constant $C$ such that for any $x \in \mathbb{R}^n_{k,+}$ and $r > 0$

$$\left| B(x,r) \right|^{-1}_\gamma \int_{B(x,r)} w(y)(y')^\gamma dy \leq C \operatorname{ess inf}_{y \in B(x,r)} w(y).$$

The properties of the class $A_{p,\gamma}$ are analogous to those of the Muckenhoupt classes. In particular, if $w \in A_{p,\gamma}$, then $w \in A_{p-\varepsilon,\gamma}$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1,\gamma}$ for any $p_1 > p$.

Note that, $|x|^\alpha \in A_{p,\gamma}$, $1 < p < \infty$, if and only if $-\frac{Q}{p} < \alpha < \frac{Q}{p}$; and $|x|^\alpha \in A_{1,\gamma}$, if and only if $-Q < \alpha \leq 0$.

For the $B$-maximal function (see [12, 13])

$$M_\gamma f(x) = \sup_{r>0} \left| B_r \right|^{-1}_\gamma \int_{B_r} T^y |f(x)|(y')^\gamma dy$$

the following analogue of Muckenhoupt theorem (see [25]) is valid.

THEOREM C. 1. If $f \in L_{1,w,\gamma}$ and $w \in A_{1,\gamma}$, then $M_\gamma f \in WL_{1,w,\gamma}$ and

$$\|M_\gamma f\|_{WL_{1,w,\gamma}} \leq C_{1,w,\gamma} \|f\|_{L_{1,w,\gamma}},$$

(5)
where \( C_{1,w,γ} \) depends only on \( γ, k \) and \( n \).

2. If \( f \in L_{p,w,γ} \) and \( w \in A_{p,γ} \), \( 1 < p < ∞ \), then \( M_{γ}f \in L_{p,w,γ} \) and
\[
\|M_{γ}f\|_{L_{p,w,γ}} \leq C_{p,w,γ}\|f\|_{L_{p,w,γ}},
\]
where \( C_{p,w,γ} \) depends only on \( w, p, γ, k \) and \( n \).

Proof. Following [13] and [29], we define a maximal function on a space of homogeneous type (see [4]). By this we mean a topological space \( X \) equipped with a continuous pseudometric \( ρ \) and a positive measure \( μ \) satisfying the doubling condition
\[
μ(E(x,2r)) \leq cμ(E(x,r)),
\]
where \( c \) does not depend on \( x \) and \( r > 0 \). Here \( E(x,r) = \{y \in X : ρ(x,y) < r\} \). Denote
\[
M_{μ}f(x) = \sup_{r > 0} μ(E(x,r))^{−1} \int_{E(x,r)} |f(y)|dμ(y).
\]

Let \((X, ρ, μ)\) be a homogeneous type space. It is known that the maximal function \( M_{μ} \) is weighted weak \((1,1)\) type, \( w \in A_{1,γ} \), that is
\[
\int_{\{x \in X : M_{μ}f(x) > τ\}} w(x) \, dμ(x) \leq \left( \frac{C_{1,w,γ}}{τ} \int_{X} |f(x)|w(x) \, dμ(x) \right),
\]
and is weighted \((p,p)\) type, \( 1 < p < ∞ \) and \( w \in A_{p,γ} \) (see [20], [24]), that is
\[
\int_{X} |M_{μ}f(x)|^{p} w(x)^{p} \, dμ(x) \leq C_{p,w,γ} \int_{X} |f(x)|^{p}w(x)^{p} \, dμ(x).
\]

In [13] and [29] it is proved that the following inequality
\[
M_{γ}f(x) \leq CM_{μ}f(x)
\]
holds, where constant \( C > 0 \) does not depend on \( f \) and \( x \).

In (8) and (9) if we take \( X = \mathbb{R}^{n}_{k,+}, \rho(x,y) = |x−y| \) and \( dμ(x) = (x')^{γ}dx \), then we have
\[
\|M_{γ}f\|_{p,w,γ} \leq C\|M_{μ}f\|_{p,w,γ} \leq C_{p,w,γ}\|f\|_{p,w,γ}, \quad 1 < p < ∞,
\]
and for \( p = 1 \)
\[
\int_{\{x \in \mathbb{R}^{n}_{k,+} : M_{γ}f(x) > τ\}} w(x) \, (x')^{γ}dx \leq \int_{\{x \in X : M_{μ}f(x) > τ\}} w(x) \, dμ(x)
\]
\[
\leq \frac{C_{1,w,γ}}{τ} \int_{\mathbb{R}^{n}_{k,+}} |f(x)|w(x) \, dμ(x).
\]

REMARK 2. Note that in the case \( k = 1 \) Theorem C was proved in [11].

We will need the following Hardy-Littlewood-Sobolev theorem for \( I_{α,γ} \).
Theorem D. Let $0 < \alpha < Q$ and $1 \leq p < Q/\alpha$. Then

1) If $1 < p < Q/\alpha$, then the condition $1/p - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $L_{p,\gamma}$ to $L_{q,\gamma}$.

2) If $p = 1$, then the condition $1 - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $L_{1,\gamma}$ to $WL_{q,\gamma}$.

3) If $1 < p = Q/\alpha$, then the operator $\widetilde{I}_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}$ to $BMO_{\gamma}$. Moreover, if the integral $I_{\alpha,\gamma} f$ exists almost everywhere for $f \in L_{p,\gamma}$, then $I_{\alpha,\gamma} f \in BMO_{\gamma}$ and the following inequality is valid

$$
\|I_{\alpha,\gamma} f\|_{BMO_{\gamma}} \leq C\|f\|_{L_{p,\gamma}},
$$

where $C > 0$ is independent of $f$.

Remark 3. Note that statements 1) and 2) in Theorem D was proved in [8] in the case $k = 1$ and [12, 13] in the case $k = n$ and [14, 23] in the case $1 \leq k \leq n$, and statement 3) in [13] in the case $k = 1$.

3. Proof of the theorems

Proof of Theorem 1. We write

$$
\left(\int_{\mathbb{R}^n_k} |x|^{-\lambda q} |I_{\alpha,\gamma} f(x)|^q (x')^\gamma dx\right)^{1/q} \leq I_1 + I_2 + I_3
$$

$$=
\left(\int_{\mathbb{R}^n_k} |x|^{-\lambda q} \left(\int_{B[a/2]} |f(y)| T^y|x|^\alpha - Q (y')\gamma dy\right)^q (x')^\gamma dx\right)^{1/q}
$$

$$+ \left(\int_{\mathbb{R}^n_k} |x|^{-\lambda q} \left(\int_{B[a]\setminus B[a/2]} |f(y)| T^y|x|^\alpha - Q (y')\gamma dy\right)^q (x')^\gamma dx\right)^{1/q}
$$

$$+ \left(\int_{\mathbb{R}^n_k} |x|^{-\lambda q} \left(\int_{\mathbb{R}^n_k} |f(y)| T^y|x|^\alpha - Q (y')\gamma dy\right)^q (x')^\gamma dx\right)^{1/q}.
$$

It is easy to check that if $|y| < |x|/2$, then $|x| \leq |y| + |x - y| < |x|/2 + |x - y|$. Hence $|x|/2 < |x - y|$ and $T^y|x|^\alpha - Q \leq (|x|/2)^{\alpha - Q}$. Indeed,

$$
T^y|x|^\alpha - Q = C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi |(x',y')_\beta, x'' - y''|^{\alpha - Q} d\nu(\beta)
$$

$$\geq C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi |(x' - y', x'' - y'')|^{\alpha - Q} d\nu(\beta)
$$

$$= |x - y|^{\alpha - Q} \geq (|x|/2)^{\alpha - Q}.
$$

Then we get

$$
I_1 \leq 2^{Q - \alpha} \left(\int_{\mathbb{R}^n_k} |x|^{(\alpha - Q - \lambda)q} (H_{\gamma} f(x))^q (x')^\gamma dx\right)^{1/q}.
$$
Further, taking into account the inequality \(-\lambda q < (Q - \alpha)q - Q\) (i.e., \(\alpha < Q/q' + \lambda\)) we obtain

\[
\left( \int_{B_t} |x|^{-(\lambda + \alpha - Q)q(x')^q} dx \right)^{1/q} = C_1 t^{\alpha - \lambda - Q/q'},
\]

where \(C_1 = \left( \frac{\omega(n,k,\gamma)}{q/q' + (\lambda - \alpha)q/Q} \right)^{1/q}\). Similarly, by virtue of the condition \(\beta p < Q(p - 1)\) (i.e., \(\beta < Q/p'\)) it follows that

\[
\left( \int_{B_t} |x|^{-(\beta + Q - \alpha)q(x')^q} dx \right)^{1/p'} = C_2 t^{Q/p' - \beta},
\]

where \(C_2 = \left( \frac{\omega(n,k,\gamma)}{1 - \beta p'/Q} \right)^{1/p'}\).

Summarizing these estimates we find that

\[
\sup_{t > 0} \left( \int_{B_t} |x|^{(-\lambda + \alpha - Q)q(x')^q} dx \right)^{1/q} \left( \int_{B_t} |x|^{-(\beta + p')q(x')^q} dx \right)^{1/p'} = C_1 C_2 \sup_{t > 0} t^{\alpha - \beta - \lambda + Q/q - Q/p} < \infty
\]

\[\Leftrightarrow \alpha - \beta - \lambda = Q/p - Q/q.\]

Now the first part of Theorem B gives us the inequality

\[
I_1 \leq b_2 C_1 C_2 2^{Q - \alpha} \left( \int_{\mathbb{R}^n} |x|^\beta |f(x)|^p(x')^q dx \right)^{1/p}.
\]

If \(|y| > 2|x|\), then \(|y| \leq |x| + |x - y| < |y|/2 + |x - y|\). Hence \(|y|/2 < |x - y|\) and the inequality \(T^y|x|^{\alpha - Q} \leq (|y|/2)^{\alpha - Q}\) can be shown immediately by similar method that of the inequality (10). Consequently, we get

\[
I_3 \leq 2^{Q - \alpha} \left( \int_{\mathbb{R}^n} |x|^{-\lambda q} \left( H_f^\alpha (|f(y)||y|^{\alpha - Q}) (x) \right)^q (x')^q dx \right)^{1/q}.
\]

Further, taking into account the inequality \(-\lambda q > -Q\) (i.e., \(\lambda < Q/q\)) we have

\[
\left( \int_{B_t} |x|^{-\lambda q(x')^q} dx \right)^{1/q} = C_3 t^{Q/q - \lambda},
\]

where \(C_3 = \left( \frac{\omega(n,k,\gamma)}{1 - \lambda q/Q} \right)^{1/q}\). By the condition \(\beta p > \alpha p - Q\) (i.e., \(\alpha < Q/p + \beta\)) it follows that

\[
\left( \int_{B_t} |x|^{-(\beta + Q - \alpha)p'}(x')^q dx \right)^{1/p'} = C_4 t^{Q/p' - (Q + \beta - \alpha)},
\]

where \(C_4 = \left( \frac{\omega(n,k,\gamma)}{(1 + (\beta - \alpha)/Q)p' - 1} \right)^{1/p'}\).
Thus we find
\[
\sup_{t > 0} \left( \int_{B_t} |x|^{-\lambda q} (x')^y \, dx \right)^{1/q} \left( \int_{B_t} |x|^{-(\beta + Q - \alpha)p'} (x')^y \, dx \right)^{1/p'} = C_3 C_4 \sup_{t > 0} t^{\alpha - \beta - \lambda + Q/q - Q/p < \infty} \iff \alpha - \beta - \lambda = Q/p - Q/q.
\]

Now the second part of Theorem B gives us the inequality
\[
I_3 \leq b_4 C_3 C_4 2^Q \alpha \left( \int_{\mathbb{R}^n_{k+}} |x|^{\beta} |f(x)|^p (x')^y \, dx \right)^{1/p}.
\]

To estimate \( I_2 \) we consider the cases \( \alpha < Q/p \) and \( \alpha > Q/p \), separately. If \( \alpha < Q/p \), then the condition
\[
\alpha = \beta + \lambda + Q/p - Q/q \geq Q/p - Q/q
\]
implies \( q \leq p^* \), where \( p^* = Qp/(Q - \alpha p) \). Assume that \( q < p^* \). In the sequel we use the notation
\[
D_k \equiv \{ x \in \mathbb{R}^n_{k+} : 2^k \leq |x| < 2^{k+1} \},
\]
and
\[
\bar{D}_k \equiv \{ x \in \mathbb{R}^n_{k+} : 2^{k-2} \leq |x| < 2^{k+2} \}.
\]

By Hölder’s inequality with respect to the exponent \( p^*/q \) and Theorem D we get
\[
I_2 = \left( \int_{\mathbb{R}^n_{k+}} |x|^{-\lambda q} \left( \int_{B_{2|\bar{x}|} \setminus B_{|\bar{x}|/2}} |f(y)| T^y |x|^{\alpha - Q} (y')^y \, dy \right)^{q} (x')^y \, dx \right)^{1/q}
\]
\[
= \left( \sum_{k \in \mathbb{Z}} \int_{D_k} |x|^{-\lambda q} \left( \int_{B_{2|\bar{x}|} \setminus B_{|\bar{x}|/2}} |f(y)| T^y |x|^{\alpha - Q} (y')^y \, dy \right)^{q} (x')^y \, dx \right)^{1/q}
\]
\[
\leq \left( \sum_{k \in \mathbb{Z}} \left( \int_{D_k} \left( \int_{B_{2|\bar{x}|} \setminus B_{|\bar{x}|/2}} |f(y)| T^y |x|^{\alpha - Q} (y')^y \, dy \right)^{p^*} (x')^y \, dx \right)^{q/p^*} \right)^{1/q}
\]
\[
\times \left( \int_{D_k} |x|^{-\lambda q p^*/p^* - q} (x')^y \, dx \right)^{p^*/q - q} \right)^{1/q}
\]
\[
\leq C_5 \left( \sum_{k \in \mathbb{Z}} 2^{k[-\lambda q + \frac{p^* - q}{p^*} Q]} \left( \int_{D_k} |I_{\alpha, Y} \left( f X_{D_k} \right) (x)|^{p^*} (x')^y \, dx \right)^{q/p^*} \right)^{1/q}
\]
\[
\leq C_6 \left( \sum_{k \in \mathbb{Z}} 2^{k[-\lambda q + \frac{p^* - q}{p^*} Q]} \left( \int_{D_k} |f(x)|^{p} (x')^y \, dx \right)^{q/p} \right)^{1/q}
\]
\[
\leq C_7 \left( \int_{\mathbb{R}^n_{k+}} |x|^{\beta} |f(x)|^{p} (x')^y \, dx \right)^{1/p}.
\]
If \( q = p^* \), then \( \beta + \lambda = 0 \). By using directly Theorem D we get

\[
I_2 \leq C_8 \left( \sum_{k \in \mathbb{Z}} 2^{k\beta} p^* \int_{D_k} |I_{\alpha,y} (f \chi_{D_k}) (x)|^{p^*} (x')^\gamma dx \right)^{1/p^*}
\]

\[
\leq C_9 \left( \sum_{k \in \mathbb{Z}} 2^{k\beta} p^* \left( \int_{D_k} |f(x)|^p (x')^\gamma dx \right)^{p^*/p} \right)^{1/p^*}
\]

\[
\leq C_{10} \left( \int_{\mathbb{R}^n_{k,+}} |x|^{\beta p}|f(x)|^p (x')^\gamma dx \right)^{1/p}.
\]

For \( \alpha > Q/p \) by Hölder’s inequality with respect to the exponent \( p \) we get the following inequality

\[
I_2 \leq \left( \int_{\mathbb{R}^n_{k,+}} |x|^{-\lambda q} \left( \int_{B_{2|x|}/2} |f(y)|^p (y')^\gamma dy \right)^{q/p} \right) \times \left( \int_{B_{2|x|}/2} (T^y |x|^{\alpha - Q})^{p'} (y')^\gamma dy \right)^{q/p'} (x')^\gamma dx \right)^{1/q}.
\]

On the other hand by using (2) and the inequality \( \alpha > Q/p \), we obtain

\[
\int_{B_{2|x|}/2} (T^y |x|^{\alpha - Q})^{p'} (y')^\gamma dy \leq \int_{B_{2|x|}/2} |x - y|^{(\alpha - Q)p'} (y')^\gamma dy
\]

\[
\leq \int_0^\infty |B_{2|x|} \cap B(x, \tau^{(\alpha - Q)p'})| d\tau
\]

\[
\leq \int_0^{|x|^{(\alpha - Q)p'}} |B_{2|x|}| \tau^{\gamma} d\tau + \int_{|x|^{(\alpha - Q)p'}}^\infty \frac{Q}{Q} |B(x, \tau^{(\alpha - Q)p'})| \tau^{\gamma} Q d\tau
\]

\[
= C_{11} |x|^{(\alpha - Q)p' + Q} + C_{12} \int_{|x|^{(\alpha - Q)p'}}^\infty \tau^{\gamma} Q d\tau
\]

where the positive constant \( C_{13} \) does not depend on \( x \). The latter estimate yields

\[
I_2 \leq C_{14} \left( \sum_{k \in \mathbb{Z}} \int_{D_k} |x|^{-\lambda q + [(\alpha - Q)p' + Q]/p'} \left( \int_{B_{2|x|}/2} |f(y)|^p (y')^\gamma dy \right)^{q/p} (x')^\gamma dx \right)^{1/q}
\]

\[
\leq C_{14} \left( \sum_{k \in \mathbb{Z}} \int_{D_k} |f(y)|^p (y')^\gamma dy \right)^{q/p} \left( \int_{D_k} |x|^{-\lambda q + [(\alpha - Q)p' + Q]/p'} (x')^\gamma dx \right)^{1/q}
\]

\[
\leq C_{14} \left( \sum_{k \in \mathbb{Z}} 2^{k(-\lambda + \alpha - Q + Q/p' + Q/q)q} \left( \int_{D_k} |f(y)|^p (y')^\gamma dy \right)^{q/p^*} \right)^{1/q}.
\]
Thus Theorem 1 is proved. □

Proof of Theorem 2. We write

$$\left( \int_{\{x \in \mathbb{R}^n_{k+} : |x|^{-\lambda} |I_{\alpha,\gamma} f(x)>\tau \}} (x')^\gamma dx \right)^{1/q} \leq J_1 + J_2 + J_3$$

\[= \left( \int_{\{x \in \mathbb{R}^n_{k+} : |x|^{-\lambda} \int_{B_{|x|/2}} |f(y)| \, T^\gamma |x|^{\alpha-\lambda} \gamma dy > \tau/3 \}} (x')^\gamma dx \right)^{1/q} \]

\[+ \left( \int_{\{x \in \mathbb{R}^n_{k+} : |x|^{-\lambda} \int_{B_2|x|} \int_{B_{|x|/2}} |f(y)| \, T^\gamma |x|^{\alpha-\lambda} \gamma dy > \tau/3 \}} (x')^\gamma dx \right)^{1/q} \]

\[+ \left( \int_{\{x \in \mathbb{R}^n_{k+} : |x|^{-\lambda} \int_{B_2|x|} \int_{B_2|x|} |f(y)| \, T^\gamma |x|^{\alpha-\lambda} \gamma dy > \tau/3 \}} (x')^\gamma dx \right)^{1/q}.\]

Then it is clear that

$$J_1 \leq \left( \int_{\{x \in \mathbb{R}^n_{k+} : 2^{Q-\alpha}|x|^{\alpha-\lambda} \gamma f(x)>\tau/3 \}} (x')^\gamma dx \right)^{1/q}.$$

Further, taking into account the inequality $-\lambda q < (Q-\alpha)q - Q$ (i.e., $\alpha < Q - Q/q + \lambda$) we have

$$\int_{B_t} |x|^{(-\lambda + \alpha - Q)q} (x')^\gamma dx = C_1 t^{(-\lambda + \alpha - Q)q+Q}. \]

By the condition $\beta \leq 0$ it follows that $\sup_{B_t} |x|^{-\beta} = t^{-\beta}$.

Summarizing these estimates we find that

$$\sup_{t>0} \left( \int_{B_t} |x|^{(-\lambda + \alpha - Q)q} (x')^\gamma dx \right)^{1/q} \sup_{B_t} |x|^{-\beta} = C_1 \sup_{t>0} t^{Q/q-\lambda + \alpha - Q - \beta} < \infty,$$

$$\Leftrightarrow \alpha - \beta - \lambda = Q - Q/q.$$

Now in the case $p = 1$ the first part of Theorem A gives us the inequality

$$J_1 \leq \frac{C_{16}}{\tau} \int_{\mathbb{R}^n_{k+}} |x|^\beta \gamma f(x)(x')^\gamma dx,$$
where the positive constant $C_{16}$ does not depend on $f$.

Further, we have

$$J_3 \leq \left( \int_{\{x \in \mathbb{R}_+^n \Delta 2Q^{-\alpha} |x|^-\lambda H_{2,\alpha}^+(\|f(y)\|y^\alpha -Q)(x) > \tau/3 \}} (x')^\gamma dx \right)^{1/q}.$$  

Taking into account the inequality $-\lambda q > -Q$ (i.e., $\lambda < Q/q$) we get

$$\int_{B_t} |x|^{-\lambda q} (x')^\gamma dx = C_{17} t^{-\lambda q + Q},$$

where the positive constant $C_{17}$ depends only on $\alpha$ and $\lambda$. Analogously, by virtue of the condition $\beta \geq \alpha - Q$ it follows that

$$\sup_{\mathcal{C}_{B_t}} |x|^{-\beta + \alpha - Q} = t^{-\beta + \alpha - Q}.$$  

Then we obtain

$$\sup_{t > 0} \left( \int_{B_t} |x|^{-\lambda q} (x')^\gamma dx \right)^{1/q} \sup_{\mathcal{C}_{B_t}} |x|^{-\beta + \alpha - Q} = C_{17} \sup_{t > 0} t^{Q/q - \lambda + \alpha - Q - \beta} < \infty$$

$$\Leftrightarrow \alpha - \beta - \lambda = Q - Q/q.$$  

Now in the case $p = 1$, from the second part of Theorem A we get the inequality

$$J_3 \leq \frac{C_{18}}{\tau} \int_{\mathbb{R}_+^n} |x|^\beta |f(x)|(x')^\gamma dx,$$

where the positive constant $C_{18}$ does not depend on $f$.

We now estimate $J_2$. From $\beta + \lambda \geq 0$ and Theorem D, we get

$$J_2 = \left( \sum_{k \in \mathbb{Z}} \int_{\{x \in D_k : |x|^-\lambda \int_{B_{2|x|/2}} |f(y)| T^\gamma |x|^\alpha - Q(y')^\gamma dy > \tau/3 \}} (x')^\gamma dx \right)^{1/q}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} \int_{\{x \in D_k : \int_{B_{2|x|/2}} |f(y)| |y|^\beta T^\gamma |x|^\alpha - \beta - Q(y')^\gamma dy > c\tau \}} (x')^\gamma dx \right)^{1/q}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} \int_{\{x \in D_k : |(f(\cdot)| |x|^\beta \chi_{D_k})(\cdot) > c\tau \}} (x')^\gamma dx \right)^{1/q}$$

$$\leq \left( \sum_{k \in \mathbb{Z}} \left( \frac{C_{19}}{\tau} \int_{D_k} |f(x)||x|^\beta (x')^\gamma dx \right)^q \right)^{1/q}$$

$$\leq \left( \frac{C_{20}}{\tau} \int_{\mathbb{R}_+^n} |x|^\beta |f(x)|(x')^\gamma dx \right)^{1/q}.$$  

Thus the proof of the theorem is completed. □

Proof of Theorem 3. Sufficiency part of Theorem 3 follows from Theorems 1 and 2.

Necessity. 1) Suppose that the operator $I_{\alpha,\gamma}$ is bounded from $L_{p,|x|^\beta,\gamma}$ to $L_{q,|x|^{-\lambda},\gamma}$ and $1 < p < Q/(\alpha - \beta - \lambda)$.

Define $f_t(x) =: f(tx)$ for $t > 0$. Then it can be easily shown that

$$\|f_t\|_{L_{p,|x|^\beta,\gamma}} = t^{-\frac{q}{p} - \beta} \|f\|_{L_{p,|x|^\beta,\gamma}},$$

$$(I_{\alpha,\gamma}f_t)(x) = t^{-\alpha}I_{\alpha,\gamma}f(tx),$$

and

$$\|I_{\alpha,\gamma}f_t\|_{L_{q,|x|^{-\lambda},\gamma}} = t^{-\alpha - \frac{q}{q} + \lambda} \|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}}.$$

From the boundedness of $I_{\alpha,\gamma}$, we have

$$\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} \leq C\|f\|_{L_{p,|x|^\beta,\gamma}},$$

where $C$ does not depend on $f$. Then we get

$$\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} = t^{\alpha + Q/q - \lambda} \|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} \leq Ct^{\alpha + Q/q - \lambda} \|f_t\|_{L_{p,|x|^\beta,\gamma}} \leq Ct^{\alpha + Q/q - \lambda - Q/p - \beta} \|f\|_{L_{p,|x|^\beta,\gamma}},$$

If $1/p - 1/q < (\alpha - \beta - \lambda)/Q$, then for all $f \in L_{p,|x|^\beta,\gamma}$ we have $\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} \rightarrow 0$ as $t \rightarrow 0$.

If $1/p - 1/q > (\alpha - \beta - \lambda)/Q$, then for all $f \in L_{p,|x|^\beta,\gamma}$ we have $\|I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda},\gamma}} \rightarrow 0$ as $t \rightarrow \infty$.

Therefore we obtain the equality $1/p - 1/q = (\alpha - \beta - \lambda)/Q$.

2) The proof of necessity for the case 2) is similar to that of the case 1); therefore we omit it.

3) Let $f \in L_{p,|x|^\beta,\gamma}$, $1 < p = Q/(\alpha - \beta - \lambda)$. For given $t > 0$ we denote

$$f_1(x) = f(x)\chi_{B_2}(x), \quad f_2(x) = f(x) - f_1(x),$$

where $\chi_{B_2}$ is the characteristic function of the set $B_2$. Then

$$\widetilde{I}_{\alpha,\gamma}f(x) = \widetilde{I}_{\alpha,\gamma}f_1(x) + \widetilde{I}_{\alpha,\gamma}f_2(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \int_{B_2} \left(T^y|x|^\alpha - |y|^\alpha \chi_{B_1}(y)\right)f(y)(y')^ydy,$$
and
\[
F_2(x) = \int_{B_{2t}} \left( T^y |x|^{\alpha - Q} - |y|^{\alpha - Q} \chi_{B_1}(y) \right) f(y)(y') \gamma dy.
\]

Note that the function \( f_1 \) has compact (bounded) support and thus
\[
a_1 = - \int_{B_{2t} \setminus B_{\text{min}(1,2t)}} |y|^{\alpha - Q} f(y)(y') \gamma dy
\]
is finite.

Note also that
\[
F_1(x) - a_1 = \int_{B_{2t}} T^y |x|^{\alpha - Q} f(y)(y') \gamma dy - \int_{B_{2t} \setminus B_{\text{min}(1,2t)}} |y|^{\alpha - Q} f(y)(y') \gamma dy \\
+ \int_{B_{2t} \setminus B_{\text{min}(1,2t)}} |y|^{\alpha - Q} f(y)(y') \gamma dy \\
= \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - Q} f_1(y)(y') \gamma dy = I_{\alpha,\gamma} f_1(x).
\]

Therefore
\[
|F_1(x) - a_1| \leq \int_{\mathbb{R}^n_{k,+}} |y|^{\alpha - Q} |T^y f_1(x)(y')| \gamma dy \\
= \int_{B(x,2t)} |y|^{\alpha - Q} |T^y f(x)(y')| \gamma dy.
\]

Further, for \( x \in B_t, y \in B(x,2t) \) we have
\[
|y| \leq |x| + |x - y| < 3t.
\]

Consequently, for all \( x \in B_t \) we have
\[
|F_1(x) - a_1| \leq \int_{B_{3t}} |y|^{\alpha - Q} |T^y f(x)(y')| \gamma dy. \quad (12)
\]

By Theorem C and inequality (12), for \( (\alpha - \beta - \lambda)p = Q \) we have
\[
t^{\alpha - \beta - \lambda} \int_{B_t} |T^z F_1(x) - a_1|(z') \gamma dz \\
\leq Ct^{\alpha - \beta - \lambda} \int_{B_t} T^z \left( \int_{B_{3t}} |y|^{\alpha - Q} T^y |f(x)(y')| \gamma dy \right) (z') \gamma dz \\
\leq Ct^{\alpha - \beta - \lambda} \cdot t^{Q/p'} \left( \int_{B_t} T^z (M_{\gamma}(f(x)))^p (z') \gamma dz \right)^{1/p} \\
\leq Ct^\beta \left( \int_{B_t} T^z (M_{\gamma}(f(x)))^p (z') \gamma dz \right)^{1/p}
\]
\[ \leq C \left( \int_{B_t} |z|^\beta p \ T^z (M_\gamma (f(x)))^p (z')^\gamma dz \right)^{1/p} \]
\[ = C \left( \int_{\mathbb{R}^n_{+k}} T^z \left( \chi_{B_t} |x|^\beta p \right) (M_\gamma (f(x)))^p (z')^\gamma dz \right)^{1/p} \]
\[ = C \left( \int_{\mathbb{R}^n_{+k}} |z|^\beta p (M_\gamma (f(x)))^p (z')^\gamma dz \right)^{1/p} \]
\[ \leq C \|f\|_{L^p,|x|^\beta,\gamma} . \] (13)

Denote
\[ a_2 = \int_{B_{max(1,2t)}} |y|^{\alpha - Q} f(y)(y')^\gamma dy \]
and estimate \(|F_2(x) - a_2|\) for \(x \in B_t: \)
\[ |F_2(x) - a_2| \leq \int_{B_{2t}} |f(y)| |T^y|x|^{\alpha - Q} - |y|^{\alpha - Q} y_n^\gamma dy. \]

Applying Lemma 1 and Hölder’s inequality we get
\[ |F_2(x) - a_2| \leq 2^{Q - \alpha + 1} |x| \int_{B_{2t}} |f(y)||y|^{\alpha - Q - 1} y_n^\gamma dy \]
\[ \leq 2^{Q - \alpha + 1} |x| \left( \int_{B_t} |y|^\beta |f(y)|^p y_n^\gamma dy \right)^{1/p} \left( \int_{B_t} |y|(-\beta + \alpha - Q - 1)p' y_n^\gamma dy \right)^{1/p'} \]
\[ \leq C|x|^\beta - 1 |f|_{L^p,|x|^\beta,\gamma} \]
\[ \leq C|x|^\beta \|f\|_{L^p,|x|^\beta,\gamma} \]
\[ \leq C|x|^\beta \|f\|_{L^p,|x|^\beta,\gamma} . \]

Note that if \(|x| \leq t\) and \(|z| \leq 2t\), then \(T^z|x| \leq |x| + |z| \leq 3t\). Thus for \((\alpha - \beta - \lambda)p = Q\) we obtain
\[ |T^zF_2(x) - a_2| \leq T^z|F_2(x) - a_2| \leq C|x|^\lambda \|f\|_{L^p,|x|^\beta,\gamma} . \] (14)

Denote
\[ a_f = a_1 + a_2 = \int_{B_{max(1,2t)}} |y|^{\alpha - Q} f(y)(y')^\gamma dy. \]

Finally, from (13) and (14) we have
\[ \sup_{x,f} T^{Q - \lambda} \int_{B_t} T^y \tilde{T} f(x) - a_f (y')^\gamma dy \leq C \|f\|_{L^p,|x|^\beta,\gamma} . \]
Thus
\[
\left\| \widetilde{I}_{\alpha,\gamma}f \right\|_{BM_{O,|x|^{-\lambda,\gamma}}} \leq 2C \sup_{x,f} t^{-Q-\lambda} \int_{B_t} \left| T^y \widetilde{I}_{\alpha,\gamma}f(x) - a_f \right| (y')^\gamma dy \leq C \|f\|_{L_p,|x|^\beta,\gamma}.
\]
Thus Theorem 3 is proved. \(\square\)

If we take \(p = q, \beta = 0\) or \(p = q, \lambda = 0\) in Theorem 3, then we get the following

**COROLLARY 1.**
1) Let \(0 < \alpha < Q/p, 1 < p < \infty\), then \(I_{\alpha,\gamma}\) is bounded from \(L_p,|x|^{-\alpha,\gamma}\) to \(L_p,|x|^{-\lambda,\gamma}\).
2) Let \(0 < \alpha < Q/p', 1 < p < \infty\), then \(I_{\alpha,\gamma}\) is bounded from \(L_p,|x|^{\alpha,\gamma}\) to \(L_p,\gamma\).

**Proof of Theorem 4.** By the definition of the weighted \(B\)-Besov spaces it suffices to show that
\[
\|T^y I_{\alpha,\gamma}f - I_{\alpha,\gamma}f\|_{L_{q,|x|^{-\lambda,\gamma}}} \leq C \|T^y f - f\|_{L_p,|x|^\beta,\gamma}.
\]
It is easy to see that \(T^y\) commutes with \(I_{\alpha,\gamma}\), i.e., \(T^y I_{\alpha,\gamma}f = I_{\alpha,\gamma}(T^y f)\). Hence we obtain
\[
|T^y I_{\alpha,\gamma}f - I_{\alpha,\gamma}f| = |I_{\alpha,\gamma}(T^y f) - I_{\alpha,\gamma}f| \leq I_{\alpha,\gamma}(|T^y f - f|).
\]
Taking \(L_{q,|x|^{-\lambda,\gamma}}\)-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of \(I_{\alpha,\gamma}\) from \(L_p,|x|^\beta,\gamma\) to \(L_{q,|x|^{-\lambda,\gamma}}\). \(\square\)

From Theorem 4 we get the following result on the boundedness of \(I_{\alpha,\gamma}\) on the \(B\)-Besov spaces \(B_{p^\theta,\gamma}^s \equiv B_{p^\theta,1,\gamma}^s\).

**COROLLARY 2.** Let \(0 < \alpha < Q, 1 < p < Q/\alpha, 1/p - 1/q = \alpha/Q, 1 \leq \theta \leq \infty\) and \(0 < s < 1\). Then the operator \(I_{\alpha,\gamma}\) is bounded from \(B_{p^\theta,\gamma}^s\) to \(B_{q^\theta,\gamma}^s\). More precisely, there is a constant \(C > 0\) such that
\[
\|I_{\alpha,\gamma}f\|_{B_{q^\theta,\gamma}^s} \leq C \|f\|_{B_{p^\theta,\gamma}^s}
\]
holds for all \(f \in B_{p^\theta,\gamma}^s\).

**Acknowledgements**

The authors are thankful to the referee for his/her valuable comments and suggestions.

**REFERENCES**


[2] G. ALTEMBURG, Eine Realisierung der Theorie der abstrakten Besov-Raume \(B_q^s(A)\) \(s > 0\), \(1 \leq q \leq \infty\) und der Lebesgue-Raume \(H_p^\gamma\) auf der Grundlage Besselscher Differential-operatoren, Z. Anal. Anwendungen, 3, 1 (1984), 43–63.


[22] B.M. LEVITAN, Bessel function expansions in series and Fourier integrals, Uspekhi Mat. Nauk 6, 2 (42) (1951), 102–143. (Russian)


(Received May 13, 2010)

A. D. Gadjiev
Institute of Mathematics and Mechanics
Baku
Azerbaijan
e-mail: akif.gadjiev@mail.az

V. S. Guliyev
Ahi Evran University
Department of Mathematics
Kirşehir
Turkey
e-mail: vagif@guliyev.com

A. Serbetci
Ankara University
Department of Mathematics, Ankara
Turkey
e-mail: serbetci@science.ankara.edu.tr

E. V. Guliyev
Institute of Mathematics and Mechanics
Baku
Azerbaijan
e-mail: emin@guliyev.com