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ON CERTAIN SEQUENCES DERIVED FROM GENERALIZED EULER–MASCHERONI CONSTANTS

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Abstract. Let $0<\alpha<1$, and let

$$C_\alpha := \lim_{n\to\infty}\left(1+\frac{1}{2^\alpha}+\cdots+\frac{1}{n^\alpha}-\frac{n^{1-\alpha}}{1-\alpha}\right).$$

It is proved that there exists a unique sequence (ω_n) such that

$$1+\frac{1}{2^\alpha}+\cdots+\frac{1}{n^\alpha}=C_\alpha+\frac{(n+\omega_n)^{1-\alpha}}{1-\alpha}.$$

Moreover, the sequence (ω_n) is decreasing and satisfies $\frac{1}{2}\leqslant\omega_n\leqslant\frac{1}{4}\left[1+\left(1+\frac{1}{n}\right)^\alpha\right]$, whence $\lim_{n\to\infty}\omega_n=\frac{1}{2}$. This is only a special case of the more general results established in this paper. These results concern some sequences derived from generalized Euler–Mascheroni constants involving convex functions and complement similar ones obtained by V. Timofte [Integral estimates for convergent positive series. *J. Math. Anal. Appl.* **303** (2005), 90–102].

1. Introduction

Let $f:[1,\infty)\to(0,\infty)$ be a continuous function such that

$$S:=\sum_{n=1}^{\infty} f(n)<\infty. \tag{1}$$

V. Timofte [7, Proposition 1] proved that if the restriction of f to $[3/2,\infty)$ is convex, then for every $n\in\mathbb{N}$ (the set of all positive integers) there exists a unique real number θ_n such that

$$f(1)+f(2)+\cdots+f(n)+\int_{n+\theta_n}^{\infty} f(t)dt=S \tag{2}$$

and

$$\frac{1}{2}\leqslant\theta_n\leqslant\frac{1}{4}\left[1+\frac{f(n)}{f(n+1)}\right]. \tag{3}$$

In particular, we have $\lim_{n\to\infty}\theta_n=\frac{1}{2}$ if $\lim_{n\to\infty}\frac{f(n)}{f(n+1)}=1$.

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In the present paper we are concerned with the case when the series $\sum_{n=1}^{\infty} f(n)$ diverges, i.e., (1) is not satisfied. In this case, for the asymptotic behavior of the sum $f(1)+f(2)+\cdots+f(n)$ the reader is referred to the paper by J. Sándor [6]. Let be given a continuous decreasing function $f:[1,\infty)\to(0,\infty)$, and let (a_n) and (b_n) be the sequences defined by

$$a_n := f(1)+f(2)+\cdots+f(n)-\int_1^{n+1} f(t)dt,$$
$$b_n := f(1)+f(2)+\cdots+f(n)-\int_1^{n} f(t)dt.$$

Then the chain of inequalities

$$a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n \tag{4}$$

holds for every positive integer n, whence the sequences (a_n) and (b_n) are both convergent. Under the additional assumption that

$$\lim_{x\to\infty} f(x)=0, \tag{5}$$

the two sequences have the same limit, say γ_f (see [6, Theorem 1]). Moreover, due to (4) one has

$$a_n \leqslant \gamma_f \leqslant b_n \quad \text{for every } n\in\mathbb{N}.$$

Under the above assumptions (f is a continuous positive decreasing function defined on $[1,\infty)$ which satisfies (5)) let n be any positive integer, and let $F_n:[0,\infty)\to\mathbb{R}$ be the function defined by

$$F_n(x) := f(1)+f(2)+\cdots+f(n)-\int_1^{n+x} f(t)dt-\gamma_f. \tag{6}$$

Note that F_n is continuous and strictly decreasing on $[0,\infty)$. Since

$$F_n(0)=b_n-\gamma_f \geqslant 0 \quad \text{and} \quad F_n(1)=a_n-\gamma_f \leqslant 0,$$

it follows that there exists a unique real number $\omega_n\in[0,1]$ such that $F_n(\omega_n)=0$, i.e.,

$$f(1)+f(2)+\cdots+f(n)-\int_1^{n+\omega_n} f(t)dt=\gamma_f. \tag{7}$$

The main purpose of the present paper is to investigate the sequence (ω_n), defined by (7). In section 2 we prove that although the equations (2) and (7) defining the sequences (θ_n) and (ω_n), respectively, are of completely different nature, in the presence of the convexity of f the estimates provided for θ_n by (3) are valid for ω_n, too. In section 3 we prove that also the monotonicity of the two sequences is the same under the additional assumption that f is twice differentiable and f''/f' is monotone.

2. Convergence of the sequence (ω_n)

In order to prove that the estimates (3) are valid also for (ω_n), we are lead to consider (besides the sequences (a_n) and (b_n) introduced in section 1) the sequence (c_n), defined by

$$c_n := f(1) + f(2) + \cdots + f(n) - \int_1^{n+\frac{1}{2}} f(t)dt. \tag{8}$$

In the special case when $f(x) := \frac{1}{x}$ it is known that (c_n) converges faster than (a_n) and (b_n). In this case we have

$$b_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n,$$
$$c_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right),$$

and $\gamma_f = \gamma$, the classical Euler–Mascheroni constant. It is known that

$$\frac{1}{2n+\frac{2}{5}} < b_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \text{for all } n \in \mathbb{N} \quad \text{(see [8, 9])}$$

and that

$$\frac{1}{24(n+1)^2} < c_n - \gamma < \frac{1}{24n^2} \quad \text{for all } n \in \mathbb{N} \quad \text{(see [2])}. \tag{9}$$

On the other hand, in the general setting from section 1, J. Sándor [6, Theorem 2] proved that if $f : [1,\infty) \to (0,\infty)$ is a continuous decreasing convex function satisfying (5) and such that the function defined by $g(x) := xf(x)$ is concave, then

$$\frac{n}{2n+1} f(n) \leqslant b_n - \gamma_f \leqslant \frac{f(n)}{2} \quad \text{for all } n \in \mathbb{N},$$

whence

$$\lim_{n\to\infty} \frac{1}{f(n)} (b_n - \gamma_f) = \frac{1}{2}.$$

In what follows we prove that the sequence (c_n), defined for an arbitrary function f by (8), possesses similar properties with the particular sequence (c_n) obtained by specializing $f(x) := \frac{1}{x}$. More precisely, we prove that (c_n) is decreasing and converges to γ_f whenever f is convex and satisfies (5). Moreover, (c_n) converges faster than (a_n) and (b_n) if, in addition, f satisfies

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1. \tag{10}$$

THEOREM 1. *Let $f : [1,\infty) \to (0,\infty)$ be a continuous convex function. Then the inequalities*

$$c_{n+1} \leqslant c_n \tag{11}$$

and

$$a_n < c_n < b_n \tag{12}$$

hold for all $n \in \mathbb{N}$. *If, in addition,* f *is decreasing and satisfies* (5)*, then* (c_n) *converges to* γ_f*, the common limit of* (a_n) *and* (b_n).

Proof. Let n be any positive integer. Since f takes only positive values, (12) is obvious. The inequality (11) is equivalent to

$$f(n+1) \leqslant \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} f(t)dt,$$

which holds by virtue of the famous Hermite–Hadamard inequality (see, for instance, [3, pp. 150–152], [5, p. 15], [4, Section 1.9] or [1, Section 3.7])

$$(b-a)f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(t)dt \leqslant (b-a)\frac{f(a)+f(b)}{2},$$

valid for every convex function $f : [a,b] \to \mathbb{R}$.

If, in addition, f is decreasing and satisfies (5), then (a_n) and (b_n) are both convergent and have the same limit γ_f. By (12) we deduce that (c_n) converges to γ_f, too. □

THEOREM 2. *Let* $f : [1,\infty) \to (0,\infty)$ *be a continuous decreasing convex function satisfying* (5)*, and let* ω_n *be the unique real number in* $[0,1]$ *defined by* (7)*. Then*

$$\frac{1}{2} \leqslant \omega_n \leqslant \frac{1}{4}\left[1+\frac{f(n)}{f(n+1)}\right] \quad \textit{for all } n \in \mathbb{N}. \tag{13}$$

In particular, we have $\lim_{n\to\infty} \omega_n = \frac{1}{2}$ *if* f *satisfies* (10).

Proof. Let (c_n) be the sequence defined by (8). By Theorem 1 it follows that

$$\gamma_f \leqslant c_n \quad \text{for every } n \in \mathbb{N}.$$

Further, let $F_n : [0,\infty) \to \mathbb{R}$ be the strictly decreasing function defined by (6). Since $F_n(\omega_n) = 0$ and

$$F_n\left(\frac{1}{2}\right) = c_n - \gamma_f \geqslant 0,$$

it follows that $\omega_n \geqslant \frac{1}{2}$.

In order to derive the upper estimate for ω_n in (13), note first that

$$F_n\left(\frac{1}{2}\right) = F_n\left(\frac{1}{2}\right) - F_n(\omega_n) = \int_{n+\frac{1}{2}}^{n+\omega_n} f(t)dt.$$

Since f is convex and decreasing, by the Hermite–Hadamard inequality we deduce that

$$F_n\left(\frac{1}{2}\right) \geqslant \left(\omega_n - \frac{1}{2}\right) f\left(n+\frac{2\omega_n+1}{4}\right) \geqslant \left(\omega_n - \frac{1}{2}\right) f(n+1). \tag{14}$$

Next we claim that

$$\frac{f(n)-f(n+1)}{4} \geqslant F_n\left(\frac{1}{2}\right). \tag{15}$$

Indeed, taking into account that $F_n\left(\frac{1}{2}\right) = c_n - \gamma_f$, inequality (15) is equivalent to

$$c_n - \frac{f(n)-f(n+1)}{4} \leqslant \gamma_f. \tag{16}$$

Let (c_n') be the sequence defined by

$$c_n' := c_n - \frac{f(n)-f(n+1)}{4}.$$

Since f satisfies (5) and (c_n) converges to γ_f, it follows that (c_n') converges to γ_f, too. So, in order to establish (16) it suffices to prove that (c_n') is increasing. Note that

$$\begin{aligned} c_{n+1}' - c_n' &= \frac{f(n+2)+2f(n+1)+f(n)}{4} - \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} f(t)dt \\ &\geqslant \frac{f(n+2)+2f(n+1)+f(n)}{4} - \frac{f\left(n+\frac{3}{2}\right)+f\left(n+\frac{1}{2}\right)}{2} \end{aligned}$$

by virtue of the Hermite–Hadamard inequality. Thus, in order to complete the proof of (15) it remains to show that

$$f(n+2)+2f(n+1)+f(n) \geqslant 2f\left(n+\frac{3}{2}\right)+2f\left(n+\frac{1}{2}\right). \tag{17}$$

But inequality (17) is an immediate consequence of the celebrated Hardy–Littlewood-Pólya majorization inequality (see, for instance, [3, pp. 89–91], [5, p. 259, Theorem B], [4, Theorem 1.5.4] or [1, Section 3.4]): given a nonempty interval $I \subseteq \mathbb{R}$, a convex function $f : I \to \mathbb{R}$, and a positive integer m, let $x_1, \ldots, x_m, y_1, \ldots, y_m \in I$ be such that

(i) $x_1 \geqslant \cdots \geqslant x_m$ and $y_1 \geqslant \cdots \geqslant y_m$;

(ii) $x_1 + \cdots + x_k \geqslant y_1 + \cdots + y_k$ for $1 \leqslant k \leqslant m-1$;

(iii) $x_1 + \cdots + x_m = y_1 + \cdots + y_m$.

Then the following inequality holds:

$$f(x_1) + \cdots + f(x_m) \geqslant f(y_1) + \cdots + f(y_m). \tag{18}$$

Let $m = 4$ and consider the numbers

$$x_1 := n+2, \qquad x_2 := x_3 := n+1, \qquad x_4 := n;$$
$$y_1 := y_2 := n+\frac{3}{2}, \qquad y_3 := y_4 := n+\frac{1}{2}.$$

A simple computation shows that

- $x_1 \geqslant \cdots \geqslant x_4$ and $y_1 \geqslant \cdots \geqslant y_4$;
- $x_1 + \cdots + x_k \geqslant y_1 + \cdots + y_k \quad$ for all $k \in \{1,2,3\}$;
- $x_1 + \cdots + x_4 = y_1 + \cdots + y_4$.

Thus (18) ensures the validity of (17). Therefore (15) holds, as claimed.

By (14) and (15) it follows that

$$\left(\omega_n - \frac{1}{2}\right) f(n+1) \leqslant \frac{f(n) - f(n+1)}{4},$$

and this inequality implies the upper estimate in (13). $\square$

REMARK 1. Let $0 < \alpha < 1$, and let $f(x) := \frac{1}{x^\alpha}$ for all $x \in [1, \infty)$. Then we have

$$b_n = 1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}}{1-\alpha} + \frac{1}{1-\alpha}.$$

Let γ_f be the limit of (b_n), and let

$$C_\alpha := \gamma_f - \frac{1}{1-\alpha} = \lim_{n\to\infty} \left(1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}}{1-\alpha}\right).$$

Further, let $\omega_n \in [0,1]$ be the unique real number satisfying

$$1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{(n+\omega_n)^{1-\alpha}}{1-\alpha} + \frac{1}{1-\alpha} = \gamma_f,$$

i.e.,

$$1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} = C_\alpha + \frac{(n+\omega_n)^{1-\alpha}}{1-\alpha}.$$

By Theorem 2 it follows that $\frac{1}{2} \leqslant \omega_n \leqslant \frac{1}{4}\left[1 + \left(1 + \frac{1}{n}\right)^\alpha\right]$, for every $n \in \mathbb{N}$, hence $\lim_{n\to\infty} \omega_n = \frac{1}{2}$.

REMARK 2. In the special case when $f(x) := \frac{1}{x}$, the number ω_n satisfying (7) is given by

$$\omega_n := \exp\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\right) - n. \tag{19}$$

Theorem 2 provides the estimate

$$\frac{1}{2} \leqslant \exp\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\right) - n \leqslant \frac{1}{2} + \frac{1}{4n},$$

i.e.,

$$\begin{aligned}0 \leqslant c_n - \gamma \leqslant \ln\left(n+\frac{1}{2}+\frac{1}{4n}\right) - \ln\left(n+\frac{1}{2}\right) \\ = \ln\left(1+\frac{1}{4n\left(n+\frac{1}{2}\right)}\right) < \frac{1}{4n\left(n+\frac{1}{2}\right)} < \frac{1}{4n^2}.\end{aligned}$$

Although this estimate is not so accurate as (9), it has the advantage that it does not appear as an isolated fact, but was derived as a special case of a more general result.

REMARK 3. By (15) it follows that under the assumptions of Theorem 2 one has

$$0 \leqslant c_n - \gamma_f \leqslant F_n\left(\frac{1}{2}\right) \leqslant \frac{f(n)-f(n+1)}{4},$$

whence

$$0 \leqslant \frac{1}{f(n)}(c_n-\gamma_f) \leqslant \frac{1}{4}\left[1-\frac{f(n+1)}{f(n)}\right].$$

If, in addition, f satisfies (10), then

$$\lim_{n\to\infty}\frac{1}{f(n)}(c_n-\gamma_f)=0,$$

i.e., (c_n) converges faster to γ_f than (b_n).

If f does not satisfy (10), then the limit of the sequence (ω_n) is no longer $\frac{1}{2}$. In this case we have the following result concerning the convergence of (ω_n).

THEOREM 3. *Let $f : [1,\infty) \to (0,\infty)$ be a continuous decreasing convex function satisfying* (5)*, and let $\omega_n \in \left[\frac{1}{2},1\right]$ be the unique real number defined by* (7)*. If $\lim_{x\to\infty}\frac{f(x+t)}{f(x)}$ exists in $\mathbb{R}$ for every $t \in [0,1]$, then*

$$\lim_{n\to\infty}\omega_n = L(a), \tag{20}$$

where $a := \lim_{x\to\infty}\frac{f(x+1)}{f(x)}$ and $L : [0,1] \to \left[\frac{1}{2},1\right]$ is the function defined by

$$L(x) := \begin{cases} 1 & \text{if} \quad x=0 \\ \ln\left(\frac{x\ln x}{x-1}\right)/\ln x & \text{if} \quad 0<x<1 \\ 1/2 & \text{if} \quad x=1. \end{cases}$$

Proof. If $a=1$, then the conclusion follows by Theorem 2.

Next consider the case $0<a<1$. It is easily seen (see also [7, proof of Theorem 3]) that

$$\lim_{x\to\infty}\frac{f(x+t)}{f(x)} = a^t \quad \text{for all } t \in [0,\infty). \tag{21}$$

Following V. Timofte [7] in his proof of Theorem 3, let $\omega := L(a)$, and let (u_n) be the sequence defined by

$$u_n := f(1) + f(2) + \cdots + f(n) - \int_1^{n+\omega} f(t)dt - \gamma_f.$$

Then we have $\omega \in \left[\frac{1}{2}, 1\right]$. Taking into account (7) we deduce that

$$u_n = \int_{n+\omega}^{n+\omega_n} f(t)dt,$$

whence $u_n = (\omega_n - \omega) f(n + \lambda_n)$ with $\lambda_n \in \left[\frac{1}{2}, 1\right]$, by virtue of the mean value theorem for integrals. Therefore we have

$$|\omega_n - \omega| = \frac{|u_n|}{f(n+\lambda_n)} \leqslant \frac{|u_n|}{f(n+1)}.$$

By using (21) and the Cesáro-Stolz theorem it can be proved that

$$\lim_{n\to\infty} \frac{u_n}{f(n+1)} = 0,$$

whence $\lim_{n\to\infty} \omega_n = \omega = L(a)$ (we omit the details because they are the same as in [7, pp. 94–95]).

Finally, suppose that $a = 0$. In order to prove that $\lim_{n\to\infty} \omega_n = 1$, let $\varepsilon \in (0,1)$ be arbitrarily chosen, and let (d_n) be the sequence defined by

$$d_n := f(1) + f(2) + \cdots + f(n) - \int_1^{n+1-\varepsilon} f(t)dt.$$

By virtue of (21) we have

$$\lim_{x\to\infty} \frac{f(x+t)}{f(x)} = 0 \quad \text{for all } t \in [0,\infty),$$

whence

$$\lim_{n\to\infty} \frac{f(n+1)}{f\left(n+1-\frac{\varepsilon}{2}\right)} = 0.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\frac{f(n+1)}{f\left(n+1-\frac{\varepsilon}{2}\right)} < \frac{\varepsilon}{2} \quad \text{for all } n \geqslant n_0.$$

Then for all $n \geqslant n_0$ we have

$$\begin{aligned} d_{n+1} - d_n &= f(n+1) - \int_{n+1-\varepsilon}^{n+2-\varepsilon} f(t)dt \\ &\leqslant f(n+1) - \int_{n+1-\varepsilon}^{n+1-\frac{\varepsilon}{2}} f(t)dt \\ &\leqslant f(n+1) - \frac{\varepsilon}{2} f\left(n+1-\frac{\varepsilon}{2}\right) < 0. \end{aligned}$$

Consequently, the sequence $(d_n)_{n\geqslant n_0}$ is strictly decreasing. On the other hand, this sequence converges to γ_f because of (5), whence $d_n > \gamma_f$ for all $n \geqslant n_0$. This means that $F_n(1-\varepsilon) > 0$ for all $n \geqslant n_0$, F_n being the function defined by (6). Taking into account that F_n is strictly decreasing on $[0,\infty)$ and that $F_n(\omega_n) = 0$, it follows that

$$1-\varepsilon < \omega_n \leqslant 1 \quad \text{for all } n \geqslant n_0.$$

Since $\varepsilon \in (0,1)$ was arbitrarily chosen, we conclude that $\lim_{n\to\infty} \omega_n = 1$. $\square$

3. Monotonicity of the sequence (ω_n)

Surprisingly, also the monotonicity of the sequence (ω_n) defined by (7) is the same as that of the sequence (θ_n) defined by (2).

THEOREM 4. *Let $f : [1,\infty) \to (0,\infty)$ be a twice differentiable decreasing convex function satisfying* (5)*, and let $\omega_n \in \left[\frac{1}{2},1\right]$ be the unique real number defined by* (7)*. If the function f''/f' is monotone, then the sequence (ω_n) has the opposite monotonicity. Moreover, $\lim_{x\to\infty} \frac{f(x+t)}{f(x)}$ exists in $\mathbb{R}$ for every $t \in [0,1]$, whence* (20) *holds.*

Proof. Since the proof is similar to that of Theorem 6 in [7] we only sketch it by pointing out the differences. Assume, for instance, that f''/f' is increasing on $[1,\infty)$. Let $F : [1,\infty) \to [0,\infty)$ be the strictly increasing function defined by

$$F(x) := \int_1^x f(t)dt.$$

By (7) it follows that

$$F(n+\omega_n) - F(n-1+\omega_{n-1}) = f(n) \quad \text{for all } n \geqslant 2.$$

Further, let $\varphi : [1,\infty) \times \left[\frac{1}{2},1\right] \to \mathbb{R}$ be the function defined by

$$\varphi(x,y) := F(x+y) - F(x+y-1) - f(x).$$

For any fixed $x \in [1,\infty)$ the partial function $\varphi(x,\cdot)$ is decreasing and satisfies the inequality $\varphi\left(x,\frac{1}{2}\right) > 0 \geqslant \varphi(x,1)$, whence there is a unique $y \in \left[\frac{1}{2},1\right]$ such that $\varphi(x,y) = 0$. In other words, there is a unique function $\Theta : [1,\infty) \to \left[\frac{1}{2},1\right]$ such that $\varphi(x,\Theta(x)) = 0$, i.e.,

$$F(x+\Theta(x)) - F(x+\Theta(x)-1) = f(x) \quad \text{for all } x \in [1,\infty). \tag{22}$$

Furthermore, Θ is decreasing on $[1,\infty)$ (see [7, pp. 97–98] for details).

Next set

$$e_n := f(1) + f(2) + \cdots + f(n) - F(n+\Theta(n))$$

for every $n \in \mathbb{N}$. Taking into account that F is strictly increasing, by (22) it follows that

$$\begin{aligned} e_n - e_{n+1} &= F(n+1+\Theta(n+1)) - f(n+1) - F(n+\Theta(n)) \\ &= F(n+\Theta(n+1)) - F(n+\Theta(n)) \leqslant 0 \end{aligned}$$

because $\Theta(n+1) \leqslant \Theta(n)$. Therefore, the sequence (e_n) is increasing. On the other hand, (e_n) converges to γ_f because of (5), whence $e_n \leqslant \gamma_f$ for all $n \in \mathbb{N}$. Taking into account (7) and the definition of e_n, this inequality is equivalent to

$$f(1)+\cdots+f(n)-F(n+\Theta(n)) \leqslant f(1)+\cdots+f(n-1)-F(n-1+\omega_{n-1}),$$

i.e., to

$$F(n-1+\omega_{n-1}) \leqslant F(n+\Theta(n))-f(n)=F(n-1+\Theta(n)),$$

by virtue of (22). Since F is increasing and $\varphi(n,\cdot)$ is decreasing, we deduce that $\omega_{n-1} \leqslant \Theta(n)$, whence

$$\varphi(n,\omega_{n-1}) \geqslant \varphi(n,\Theta(n))=0.$$

This inequality implies that

$$\begin{aligned} 0 &\leqslant F(n+\omega_{n-1})-F(n-1+\omega_{n-1})-f(n) \\ &= F(n+\omega_{n-1})-F(n+\omega_n), \end{aligned}$$

whence $\omega_{n-1} \geqslant \omega_n$. Thus the sequence (ω_n) is decreasing.

The last statement of the theorem follows easily by l'Hôpital's rule (see [7]). □

REMARK 4. If $f(x)=\frac{1}{x}$ for all $x \in [1,\infty)$, then by Theorem 4 it follows that the sequence (ω_n), defined by (19), is decreasing (this monotonicity of (ω_n) seems to be new).

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