

SOME EXTENSIONS OF HILBERT'S INTEGRAL INEQUALITY

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Abstract. In this paper we introduce a new extension of Hilbert's integral inequality with a best constant factor involving the hypergeometric function. The equivalent form and some examples will be given.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) > 0$, $0 < \int_0^\infty f^p(x) dx < \infty$, and $0 < \int_0^\infty g^q(x) dx < \infty$, then we have the following two equivalent inequalities as (see[1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x) dx \right\}^{\frac{1}{q}} \quad (1.1)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p \int_0^\infty f^p(x) dx \quad (1.2)$$

where the constant factors $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right]^p$ are the best possible in (1.1) and (1.2) respectively. Inequality (1.1) is called Hardy-Hilbert's integral inequality. During the last decade inequality (1.1) and (1.2) were generalized in many different ways, see for an example [2],[3]. In [4] the authors obtained the following extension of (1.1) and (1.2)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B(1 - pA_2, \lambda + pA_2 - 1) \left\{ \int_0^\infty x^{pqA_1-1} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_0^\infty y^{pqA_2-1} g^q(x) dx \right\}^{\frac{1}{q}}, \quad (1.3)$$

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$$\int_0^\infty y^{p-p^2A_2-1} \left[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \right]^p dy < [B(1-pA_2, \lambda+pA_2-1)]^p \int_0^\infty x^{pqA_1-1} f^p(x) dx \tag{1.4}$$

where $B(1-pA_2, \lambda+pA_2-1)$ and $[B(1-pA_2, \lambda+pA_2-1)]^p$ are the best possible constants ($B(x, y)$ is the Beta function), $\lambda > 0$, $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ and $pA_2 + qA_1 = 2 - \lambda$. Recently, in [5], M. Krinić obtained a Hilbert’s type inequality with a best constant factor involving the hypergeometric function.

In this paper we continue the research on Hilbert’s inequality by giving a new inequality with a best constant factor involving the Beta and the hypergeometric functions and from which inequalities (1.3) and (1.4) can be deduced.

2. Preliminaries and Lemmas

Recall that the hypergeometric function $F(\alpha, \beta; \gamma; x)$ is defined by (see [6])

$$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^\infty \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!}, \tag{2.1}$$

where $(\alpha)_r$ is the Pochhammer symbol defined by

$$(\alpha)_r = \alpha(\alpha + 1) \cdots (\alpha + r - 1) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}.$$

As it is known the series (2.1) converges for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$ the series converges if $\gamma > \alpha + \beta$ and in this case we have

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \beta - \alpha)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}. \tag{2.2}$$

The hypergeometric function satisfies the integral representation

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{\alpha-1} dt, \text{ if } \gamma > \beta > 0,$$

and the transformation

$$F(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{x}{x-1}\right). \tag{2.3}$$

We will need the following definition of the Beta function

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

and the Legendre duplication formula

$$\Gamma(2\delta) = \frac{2^{2\delta-1}}{\sqrt{\pi}} \Gamma(\delta) \Gamma\left(\delta + \frac{1}{2}\right). \tag{2.4}$$

In this paper we will assume the following: $a, c > 0$, $b^2 < ac$, $\lambda > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, u and v are differentiable nonnegative strictly increasing function on (a, b) ($-\infty \leq a < b \leq \infty$) and they satisfy the following conditions: $\lim_{t \rightarrow a^+} u(t) = \lim_{t \rightarrow a^+} v(t) = 0$, and $\lim_{t \rightarrow b^-} u(t) = \lim_{t \rightarrow b^-} v(t) = \infty$. For abbreviation we set $k_{2\lambda}(u(x), v(y)) := (au^2(x) + 2bu(x)v(y) + cv^2(y))^\lambda$.

LEMMA 2.1. *Suppose that $a, c > 0, b^2 < ac, 0 < \alpha < 2\lambda$. Then we have*

$$I_1 := \int_0^\infty \frac{x^{\alpha-1}}{(ax^2 + 2bx + c)^\lambda} dx = a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} C, \tag{2.5}$$

where

$$C = B(\alpha, 2\lambda - \alpha) F\left(\frac{\alpha}{2}, \lambda - \frac{\alpha}{2}, \lambda + \frac{1}{2}; 1 - \frac{b^2}{ac}\right). \tag{2.6}$$

Proof. Using the substitutions $t = \sqrt{\frac{a}{c}}x + \frac{b}{\sqrt{ac}}$, $\frac{\sqrt{ac}}{b}u = \frac{1}{t}$ respectively and then applying the binomial Theorem, we obtain

$$\begin{aligned} I_1 &= \int_0^\infty \frac{x^{\alpha-1}}{c^\lambda \left(\left(\sqrt{\frac{a}{c}}x + \frac{b}{\sqrt{ac}} \right)^2 + 1 - \frac{b^2}{ac} \right)^\lambda} dx \\ &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \int_{\frac{b}{\sqrt{ac}}}^\infty \frac{\left(t - \frac{b}{\sqrt{ac}} \right)^{\alpha-1}}{\left(t^2 + 1 - \frac{b^2}{ac} \right)^\lambda} dt \\ &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \left(\frac{ac}{b^2} \right)^{\lambda-\frac{\alpha}{2}} \int_0^1 \frac{(1-u)^{\alpha-1} u^{2\lambda-\alpha-1}}{\left(1 + \left(\frac{ac}{b^2} - 1 \right) u^2 \right)^\lambda} du \\ &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \left(\frac{ac}{b^2} \right)^{\lambda-\frac{\alpha}{2}} \int_0^1 (1-u)^{\alpha-1} u^{2\lambda-\alpha-1} \sum_{r=0}^\infty \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)} \left(1 - \frac{ac}{b^2} \right)^r u^{2r} du \\ &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \left(\frac{ac}{b^2} \right)^{\lambda-\frac{\alpha}{2}} \sum_{r=0}^\infty \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)} \left(1 - \frac{ac}{b^2} \right)^r \int_0^1 (1-u)^{\alpha-1} u^{2\lambda+2r-\alpha-1} du \\ &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \left(\frac{ac}{b^2} \right)^{\lambda-\frac{\alpha}{2}} \sum_{r=0}^\infty \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)} B(\alpha, 2\lambda + 2r - \alpha) \left(1 - \frac{ac}{b^2} \right)^r \\ &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \left(\frac{ac}{b^2} \right)^{\lambda-\frac{\alpha}{2}} \sum_{r=0}^\infty \frac{\Gamma(\lambda+r)\Gamma(\alpha)\Gamma(2\lambda+2r-\alpha)}{\Gamma(\lambda)\Gamma(2\lambda+2r)} \left(1 - \frac{ac}{b^2} \right)^r. \end{aligned}$$

Using Legendre duplication formula (2.4) in the last sum, we get

$$\begin{aligned}
 I &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} \frac{\Gamma(2\lambda-\alpha)\Gamma(\alpha)}{\Gamma(2\lambda)} \left(\frac{ac}{b^2}\right)^{\lambda-\frac{\alpha}{2}} \sum_{r=0}^{\infty} \frac{\Gamma(\lambda-\frac{\alpha}{2}+r)\Gamma(\lambda-\frac{\alpha}{2}+\frac{1}{2}+r)\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda-\frac{\alpha}{2})\Gamma(\lambda-\frac{\alpha}{2}+\frac{1}{2})\Gamma(\lambda+\frac{1}{2}+r)} \left(1-\frac{ac}{b^2}\right)^r \\
 &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} B(\alpha, 2\lambda-\alpha) \left(\frac{ac}{b^2}\right)^{\lambda-\frac{\alpha}{2}} \sum_{r=0}^{\infty} \frac{(\lambda-\frac{\alpha}{2})_r (\lambda-\frac{\alpha}{2}+\frac{1}{2})_r}{(\lambda+\frac{1}{2})_r} \left(1-\frac{ac}{b^2}\right)^r \\
 &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} B(\alpha, 2\lambda-\alpha) \left(\frac{ac}{b^2}\right)^{\lambda-\frac{\alpha}{2}} F\left(\lambda-\frac{\alpha}{2}, \lambda-\frac{\alpha}{2}+\frac{1}{2}, \lambda+\frac{1}{2}; 1-\frac{ac}{b^2}\right) \\
 &= a^{-\frac{\alpha}{2}} c^{\frac{\alpha}{2}-\lambda} B(\alpha, 2\lambda-\alpha) F\left(\frac{\alpha}{2}, \lambda-\frac{\alpha}{2}, \lambda+\frac{1}{2}; 1-\frac{b^2}{ac}\right).
 \end{aligned}$$

In the last equation we used the transformation (2.3). The Lemma is proved. \square

LEMMA 2.2. For $0 < \alpha < 2\lambda$, the following relation holds

$$B(\alpha, 2\lambda-\alpha) F\left(\frac{\alpha}{2}, \lambda-\frac{\alpha}{2}, \lambda+\frac{1}{2}; 1\right) = \frac{B(\frac{\alpha}{2}, \lambda-\frac{\alpha}{2})}{2}. \tag{2.7}$$

Proof. Using (2.2) and definition of the Beta function, we get

$$B(\alpha, 2\lambda-\alpha) F\left(\frac{\alpha}{2}, \lambda-\frac{\alpha}{2}, \lambda+\frac{1}{2}; 1\right) = \frac{\Gamma(\alpha)\Gamma(2\lambda-\alpha)}{\Gamma(2\lambda)} \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\lambda-\frac{\alpha}{2}+\frac{1}{2})}.$$

By the Legendre duplication formula (2.3) we have $\frac{\Gamma(2\lambda-\alpha)}{\Gamma(\lambda-\frac{\alpha}{2}+\frac{1}{2})} = \frac{2^{2\lambda-\alpha-1}}{\sqrt{\pi}}\Gamma(\lambda-\frac{\alpha}{2})$, and $\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)} = \frac{2^{1-2\lambda}\sqrt{\pi}}{\Gamma(\lambda)}$, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we obtain

$$\begin{aligned}
 B(\alpha, 2\lambda-\alpha) F\left(\frac{\alpha}{2}, \lambda-\frac{\alpha}{2}, \lambda+\frac{1}{2}; 1\right) &= 2^{1-2\lambda}\sqrt{\pi} \frac{\Gamma(\lambda-\frac{\alpha}{2})}{\Gamma(\lambda)} \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha}{2}+\frac{1}{2})} \\
 &= \frac{1}{2} \frac{\Gamma(\lambda-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\lambda)} = \frac{B(\frac{\alpha}{2}, \lambda-\frac{\alpha}{2})}{2}. \quad \square
 \end{aligned}$$

LEMMA 2.3. The weight coefficients $\omega(x)$ and $\omega(y)$, defined respectively by

$$\begin{aligned}
 \omega(x) &:= \int_a^b \frac{u(x)^{pA_1} u'(x)^{1-p} v'(y)}{k_{2\lambda} (u(x), v(y)) v(y)^{pA_2}} dy, \\
 \omega(y) &:= \int_a^b \frac{v(y)^{qA_2} v'(y)^{1-q} u'(x)}{k_{2\lambda} (u(x), v(y)) u(x)^{qA_1}} dx,
 \end{aligned}$$

satisfies the following relations

$$\omega(x) = \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} L_1, \tag{2.8}$$

$$\omega(y) = \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} L_2. \tag{2.9}$$

where $L_1 = a^{\frac{1-pA_2}{2}-\lambda} c^{\frac{pA_2-1}{2}} B(1-pA_2, 2\lambda+pA_2-1) F(\frac{1-pA_2}{2}, \lambda - \frac{1-pA_2}{2}, \lambda + \frac{1}{2}; 1 - \frac{b^2}{ac})$
 and $L_2 = a^{\frac{qA_1-1}{2}} c^{\frac{1-qA_1}{2}-\lambda} B(1-qA_1, 2\lambda+qA_1-1) F(\frac{1-qA_1}{2}, \lambda - \frac{1-qA_1}{2}, \lambda + \frac{1}{2}; 1 - \frac{b^2}{ac})$.

Proof. Setting $v(y) = u(x)t$ then by using Lemma 2.1 we get

$$\begin{aligned} \omega(x) &:= \int_a^b \frac{u(x)^{A_1 p} u'(x)^{1-p} v'(y)}{k_{2\lambda}(u(x), v(y)) v(y)^{pA_2}} dy \\ &= \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} \int_0^\infty \frac{t^{-pA_2}}{(a+2bt+ct^2)^\lambda} dy \\ &= \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} L_1. \end{aligned}$$

Similarly, we have $\omega(y) = \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} L_2$. \square

3. Main Results

THEOREM 3.1. *If $f, g \geq 0$ such that $0 < \int_a^b \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} f^p(x) dx < \infty$ and $0 <$*

$\int_a^b \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y) dy < \infty$, then

$$\begin{aligned} D &:= \int_a^b \int_a^b \frac{f(x)g(y)}{k_{2\lambda}(u(x), v(y))} dx dy \\ &\leq L_1^{\frac{1}{p}} L_2^{\frac{1}{q}} \left\{ \int_a^b \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y) dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\int_a^b v(y)^{(2\lambda-1)(p-1)+p(A_1-A_2)} v'(y) \left(\int_a^b \frac{f(x)}{k_{2\lambda}(u(x), u(y))} dx \right)^p dy \\ &\leq \left\{ L_1^{\frac{1}{p}} L_2^{\frac{1}{q}} \right\}^p \int_a^b \frac{(u(x))^{1-2\lambda+p(A_1-A_2)}}{(u'(x))^{p-1}} f^p(x) dx. \end{aligned} \tag{3.2}$$

Here, $A_1 \in (\frac{1-2\lambda}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1-2\lambda}{p}, \frac{1}{p})$. Inequalities (3.1) and (3.2) are equivalent.

Proof. By Hölder's inequality, taking into account Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned}
 D &= \int_a^b \int_a^b \frac{1}{k_{2\lambda}(u(x), v(y))} \frac{u(x)^{A_1} v'(y)^{\frac{1}{p}}}{v(y)^{A_2} u'(x)^{\frac{1}{q}}} f(x) \frac{v(y)^{A_2} u'(x)^{\frac{1}{q}}}{u(x)^{A_1} v'(y)^{\frac{1}{p}}} g(y) dx dy \\
 &\leq \left\{ \int_a^b \int_a^b \frac{1}{k_{2\lambda}(u(x), v(y))} \frac{u(x)^{pA_1} v'(y)}{v(y)^{pA_2} u'(x)^{p-1}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \int_a^b \int_a^b \frac{1}{k_{2\lambda}(u(x), v(y))} \frac{v(y)^{qA_2} u'(x)}{u(x)^{qA_1} v'(y)^{q-1}} g^q(y) dx dy \right\}^{\frac{1}{q}} \\
 &\leq L_1^{\frac{1}{p}} L_2^{\frac{1}{q}} \left\{ \int_a^b \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y) dx \right\}^{\frac{1}{q}}.
 \end{aligned}$$

Let us show that (3.1) and (3.2) are equivalent. Putting

$$g(y) = v(y)^{(2\lambda-1)(p-1)+p(A_1-A_2)} v'(y) \left(\int_a^b \frac{f(x)}{k_{2\lambda}(u(x), v(y))} dx \right)^{p-1},$$

using (3.1) we get

$$\begin{aligned}
 &\int_a^b \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y) dx \\
 &= \int_a^b \frac{v(y)^{(2\lambda-1)(p-1)+p(A_1-A_2)}}{v'(y)^{-1}} \left(\int_a^b \frac{f(x)}{k_{2\lambda}(u(x), v(y))} dx \right)^p \\
 &= \int_a^b \int_a^b \frac{f(x) g(y)}{k_{2\lambda}(u(x), v(y))} dx dy \\
 &\leq L_1^{\frac{1}{p}} L_2^{\frac{1}{q}} \left\{ \int_a^b \frac{u(x)^{1-2\lambda+p(A_1-A_2)}}{u'(x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y) dx \right\}^{\frac{1}{q}},
 \end{aligned}$$

from which we get (3.2). On the other hand, by Hölder's inequality

$$\begin{aligned}
 D &= \int_a^b \int_a^b \frac{f(x)g(y)}{k_{2\lambda}(u(x),v(y))} dx dy \\
 &= \int_a^b \left\{ v(y)^{-\frac{1-2\lambda+q(A_2-A_1)}{q}} v'(y)^{1-\frac{1}{q}} \int_a^b \frac{f(x)}{k_{2\lambda}(u(x),v(y))} dx \right\} \\
 &\quad \times v(y)^{\frac{1-2\lambda+q(A_2-A_1)}{q}} v'(y)^{-1+\frac{1}{q}} g(y) dy \\
 &\leq \left\{ \int_a^b \frac{v(y)^{(2\lambda-1)(p-1)+p(A_1-A_2)}}{v'(y)^{-1}} \left(\int_a^b \frac{f(x)}{k_{2\lambda}(u(x),v(y))} dx \right)^p dy \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \int_a^b \frac{v(y)^{1-2\lambda+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}
 \end{aligned}$$

By (3.2) we obtain (3.1). \square

THEOREM 3.2. *Under the conditions in Theorem 3.1, if the equation $pA_2 + qA_1 = 2 - 2\lambda$ holds, then (3.1) and (3.2) can be rewritten as*

$$D < L^* \left\{ \int_a^b \frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b \frac{v(y)^{pqA_2-1}}{v'(y)^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}, \tag{3.3}$$

$$\int_a^b v(y)^{p-p^2A_2-1} v'(y) \left(\int_a^b \frac{f(x)}{k_{2\lambda}(u(x),v(y))} dx \right)^p dy < \{L^*\}^p \int_a^b \frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}} f^p(x) dx. \tag{3.4}$$

In this case the constant factors in (3.3) and (3.4) are the best possible, where L^* is given by

$$L^* = a^{\frac{qA_1-1}{2}} c^{\frac{pA_2-1}{2}} B(1-pA_2, 2\lambda+pA_2-1) F\left(\frac{1-pA_2}{2}, \lambda - \frac{1-pA_2}{2}, \lambda + \frac{1}{2}; 1 - \frac{b^2}{ac}\right).$$

Proof. If the constant factor L^* is not the best possible, then there exists a positive constant K with $K < L^*$ for which (3.3) is still valid if we replace L^* by K . For $1 < \theta < 1 + p(1 - qA_1)$, setting f_θ and g_θ as $f_\theta(x) = 0$ for $x \in (a, a_1)$, $f_\theta(x) = u(x)^{\frac{1-\theta}{p}-qA_1} u'(x)$ for $x \in [a_1, b]$; $g_\theta(y) = 0$ on (a, a_2) , $g_\theta(y) = v(y)^{\frac{1-\theta}{q}-pA_2} v'(y)$ on $[a_2, b)$, where a_1 and a_2 are such that $u(a_1) = 1$ and $v(a_2) = 1$. Then we have(let

$\sigma = u(x)$ and $\tau = v(y)$)

$$\int_a^b \int_a^b \frac{f_\theta(x)g_\theta(y)}{k_{2\lambda}(u(x),v(y))} dx dy < K \left\{ \int_a^b u(x)^{-\theta} u'(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^b v(y)^{-\theta} v'(y) dy \right\}^{\frac{1}{q}}$$

$$= K \left\{ \int_1^\infty \sigma^{-\theta} d\sigma \right\}^{\frac{1}{p}} \left\{ \int_1^\infty \tau^{-\theta} d\tau \right\}^{\frac{1}{q}} = \frac{K}{\theta - 1}. \tag{3.5}$$

On the other hand, we find(let $u(x) = \sigma v(y)$ and $\tau = v(y)$)

$$D := \int_{a_2}^b \int_{a_1}^b \frac{u(x)^{\frac{1-\theta}{p}-qA_1} u'(x)v(y)^{\frac{1-\theta}{q}-pA_2} v'(y)}{k_{2\lambda}(u(x),v(y))} dx dy$$

$$= \int_1^\infty \tau^{-\theta} \int_{1/\tau}^\infty \frac{\sigma^{\frac{1-\theta}{p}-qA_1}}{(a\sigma^2 + 2b\sigma + c)^\lambda} d\sigma d\tau$$

$$= \int_1^\infty \tau^{-\theta} \left(\int_0^\infty \frac{\sigma^{\frac{1-\theta}{p}-qA_1}}{(a\sigma^2 + 2b\sigma + c)^\lambda} d\sigma - \int_0^{1/\tau} \frac{\sigma^{\frac{1-\theta}{p}-qA_1}}{(a\sigma^2 + 2b\sigma + c)^\lambda} d\sigma \right) d\tau$$

$$= \frac{1}{\theta - 1} [L^* + o(1)] - \int_1^\infty \tau^{-\theta} \left[\int_0^{1/\tau} \frac{\sigma^{\frac{1-\theta}{p}-qA_1}}{(a\sigma^2 + 2b\sigma + c)^\lambda} d\sigma \right] d\tau$$

$$> \frac{1}{\theta - 1} [L^* + o(1)] - \int_1^\infty \tau^{-\theta} \left[\int_0^{1/\tau} \sigma^{\frac{1-\theta}{p}-qA_1} d\sigma \right] d\tau$$

$$= \frac{1}{\theta - 1} [L^* + o(1)] - O(1).$$

Therefore, we get $\frac{1}{\theta - 1} [L^* + o(1)] - O(1) < \frac{K}{\theta - 1}$ or

$$[L^* + o(1)] - (\theta - 1)O(1) < K.$$

For $\theta \rightarrow 1^+$, it follows that $L^* \leq K$ which contradicts the fact that $K < L^*$. Hence the constant factor L^* in (3.3) is the best possible. Since (3.4) is equivalent to (3.3) then the constant factor in (3.4) is also the best possible. It remains to prove that the inequalities (3.3) and (3.4) are strict. If (3.3) takes the form of equality, then there exists constants M and N which are not all zero such that

$$\frac{M f^p(x)}{k_{2\lambda}(u(x),v(y))} \frac{u(x)^{pA_1} v'(y)}{v(y)^{pA_2} u'(x)^{p-1}} = \frac{N g^q(y)}{k_{2\lambda}(u(x),v(y))} \frac{v(y)^{qA_2} u'(x)}{u(x)^{qA_1} v'(y)^{q-1}}, \text{ a.e.in } (a,b) \times (a,b).$$

Hence, there exists a constant k such that

$$M \frac{u(x)^{pqA_1}}{u'(x)^p} f^p(x) = \frac{v(y)^{pqA_2}}{v'(y)^q} g^q(y) = k \text{ a.e. in } (a, b) \times (a, b).$$

We claim that $M = 0$. In fact if $M \neq 0$, then

$$\frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}} f^p(x) = \frac{ku'(x)}{Mu(x)} \text{ a.e. in } (a, b),$$

which contradicts the fact that $0 < \int_a^b \frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}} f^p(x) dx < \infty$. Thus, the theorem is proved. \square

4. Some Examples

In this section we shall consider the case for which the constant factor is the best possible, namely inequality (3.3).

1. If $u(x) = x^\alpha$, $v(y) = y^\beta$ where $\alpha, \beta > 0$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(ax^{2\alpha} + 2bx^\alpha y^\beta + cy^{2\beta})^\lambda} dx dy < \frac{L^*}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}} \left\{ \int_0^\infty x^{\alpha pqA_1 + p(1-\alpha)-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\beta pqA_2 + q(1-\beta)-1} g^q(y) dy \right\}^{\frac{1}{q}}. \quad (4.1)$$

In particular, from (4.1) we get the following particular cases:

(i) If $\alpha = \beta = 1$, then we obtain the following inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(ax^2 + 2bxy + cy^2)^\lambda} dx dy < L^* \left\{ \int_0^\infty x^{pqA_1-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{pqA_2-1} g^q(y) dy \right\}^{\frac{1}{q}}.$$

(ii) If $\alpha = \beta = \frac{1}{2}$, $a = c = 1$, $b = 0$, then by using Lemma 2.2 we can obtain inequality (1.3).

(iii) If $\alpha = \beta = 1$, $\lambda = \frac{1}{2}$, $A_1 = A_2 = \frac{1}{pq}$, we have $L^* = B\left(\frac{1}{p}, \frac{1}{q}\right) F\left(\frac{1}{2p}, \frac{1}{2q}, 1; 1 - \frac{b^2}{ac}\right)$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\sqrt{ax^2 + 2bxy + cy^2}} dx dy < \frac{\pi}{\sin \frac{\pi}{p}} F\left(\frac{1}{2p}, \frac{1}{2q}, 1; 1 - \frac{b^2}{ac}\right) \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y) dy \right\}^{\frac{1}{q}}.$$

2. Let $u(x) = \ln x$, $v(y) = \ln y$. We have

$$\int_1^{\infty} \int_1^{\infty} \frac{f(x)g(y)}{(a \ln^2 x + 2b \ln x \ln y + c \ln^2 y)^\lambda} dx dy$$

$$< L^* \left\{ \int_1^{\infty} \frac{(\ln x)^{pqA_1-1}}{x^{1-p}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_1^{\infty} \frac{(\ln y)^{pqA_2-1}}{y^{1-q}} g^q(y) dy \right\}^{\frac{1}{q}}.$$

3. Let $u(x) = \tan x$, $v(y) = \tan y$. We have

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{f(x)g(y)}{(a \tan^2 x + 2b \tan x \tan y + c \tan^2 y)^\lambda} dx dy$$

$$< L^* \left\{ \int_0^{\pi/2} \frac{(\tan x)^{pqA_1-1}}{(\sec^2 x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\pi/2} \frac{(\tan y)^{pqA_2-1}}{(\sec^2 y)^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}.$$

4. Let $u(x) = e^x$, $v(y) = e^y$. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(ae^{2x} + 2be^x e^y + ce^{2y})^\lambda} dx dy$$

$$< L^* \left\{ \int_{-\infty}^{\infty} \frac{(e^x)^{pqA_1-1}}{(e^x)^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} \frac{(e^y)^{pqA_2-1}}{(e^y)^{q-1}} g^q(y) dy \right\}^{\frac{1}{q}}.$$

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