Journal of
Mathematical
Inequalities
Volume 5, Number 1 (2011), 141–153

STRONG CONVERGENCE THEOREMS OF MODIFIED MANN ITERATIVE PROCESS FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

HAIFANG LIU AND RUDONG CHEN

(*Communicated by M. A. Noor*)

Abstract. The purpose of this article is to modify normal Mann's iterative process to have strong convergence for nonexpansive mappings in the formework of Hilbert spaces. We prove the strong convergence of the proposed iterative algorithm to the fixed point of nonexpansive mappings which is the unique solution of a variational inequality, which is also the optimality condition for a minimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. Let C be a nonempty closed convex subset of H and recall that a self-mapping $T : C \to C$ is called a contraction mapping if there exists a constant $\alpha \in (0,1)$ such that

$$\|T(x) - T(y)\| \leqslant \alpha \|x - y\|. \tag{1.1}$$

If $\alpha = 1$, we call it is nonexpansive. We denote the set of fixed points of T by $\mathrm{Fix}(T)$. Namely, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

Let A be a bounded linear operator on H and assume that A is strongly positive with coefficient $\overline{\gamma}$; that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \geqslant \overline{\gamma} \|x\|^2 \quad \forall x \in H.$$

The Variational Inequality Problem [1, 2, 3] has been and will continue to be one of the central problems in nonlinear analysis and is defined as follows: given monotone operator $F : H \to H$ and closed convex set $C \subset H$,

$$\text{find } x^* \in C \text{ such that } \langle x - x^*, F(x^*) \rangle \geqslant 0 \text{ for all } x \in C. \tag{1.2}$$

This condition is the optimality condition of the convex optimization problem: min θ over C when $F = \theta'$. The simplest iterative procedure for the variational inequality problem (VIP) may be the well-known *projected gradient method* [4]: $x_{n+1} =$

Mathematics subject classification (2010): 47H09, 47H10, 47H05, 58E35.

Keywords and phrases: Nonexpansive mapping, strong positive operator, variational inequality, iterative algorithm.

Supported by the National Natural Science Foundation of China under grant (10771050).

Paper JMI-05-13

$P_C(x_n - \mu F(x_n))$ $(n = 0,1,2,\cdots)$ where P_C is the convex projection onto C and μ is a positive real number. This method requires repetitive use of P_C, although the closed form expression of P_C is not always known in many situations. To help resolve this problem, the following *hybrid steepest descent method* [4, 5, 6] for (1.2) when C is equal to the fixed point set $\mathrm{Fix}(T) := \{x \in H : T(x) = x\}$ of a nonexpansive mapping T has been established: $x_1 \in H$ and $x_{n+1} = T(x_n - \mu\alpha_n F(x_n))$ for every $n \in N$, where $\mu > 0, (\alpha_n)_{n\in N} \subset (0,1]$ is a slowly diminishing constant sequence, and $F : H \to H$ is strongly monotone and Lipschitz continuous. In this paper, we give a new algorithm of modified Mann's iterative process to appropriate the unique solution of the Variational Inequality.

Recall that the normal Mann's iterative process was introduced by Mann [7] in 1953. Since then, construction of fixed points for nonexpansive mappings via the normal Mann's iterative process has been extensively investigated by many authors.

The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in K,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ \forall n \geqslant 1, \tag{1.3}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $(0,1)$.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.3) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [7].)

Attempts to modify the normal Mann iterative method (1.3) for nonexpansive mappings so that strong convergence is guaranteed have recently been made; see, e. g., [8, 9, 10, 11, 12, 13, 14, 15, 22, 23] and the references therein.

Xu [18] consider the iterative process with viscosity approximation method.

$$\begin{cases} x_0 \in C \ \text{ is arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T x_n, \ \text{ for all } \ n \geqslant 1. \end{cases} \tag{1.4}$$

where C is a closed convex subset of a space X, if X is either Hilbert or uniformly smooth, then it is shown that, under certain appropriate conditions on α_n, $\{x_n\}$ converges strongly to a fixed point of T which solves the following variational inequality:

$$\langle (I-f)q, p-q\rangle \leqslant 0,\ \ \forall p \in F(T). \tag{1.5}$$

Kim and Xu [8] introduced the following iterative process.

$$\begin{cases} x_0 = x \in K \quad \text{arbitrarily chosen,} \\ y_n = \beta_n x_n + (1-\beta_n)T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \ \ n \geqslant 0. \end{cases} \tag{1.6}$$

where T is a nonexpansive mapping of K into itself, $u \in$ K is a given point. They proved the sequence $\{x_n\}$ defined by (1.6) converges strongly to a fixed point of T provided the control sequence α_n and β_n satisfy appropriate conditions.

Yao et al. [16] also modified Mann's iterative scheme (1.3) by using the so-called viscosity approximation method which was introduced by Moudafi. Yao et al introduced the following iterative algorithm.

$$\begin{cases} x_0 = x \in K \quad \text{arbitrarily chosen,} \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad n \geqslant 0. \end{cases} \tag{1.7}$$

In this paper, motivated by Xu [18], Kim and Xu [8], Yao [16], we introduce a new iterative scheme generated by:

$$\begin{cases} x_0 = x \in K \quad \text{arbitrarily chosen,} \\ y_n = P_K[\beta_n x_n + (1-\beta_n)Tx_n], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)y_n, \quad n \geqslant 0. \end{cases} \tag{1.8}$$

We prove under weaker hypotheses that the sequence $\{x_n\}$ converges strongly to the fixed point of the nonexpansive mapping T, which also solves the following variational inequality:

$$\langle \widehat{A}(q), q-p \rangle \leqslant 0, \ \ \forall p \in F(T). \tag{1.9}$$

where $\widehat{A} := A - \gamma f$.

2. Preliminaries

In this section, we collect some lemmas which will be used in the proof for the main result in next section.

LEMMA 2.1. *Let H be a real Hilbert space, then for any x, $y \in H$ we have*
(i) $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle$
(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x,y\rangle + \|y\|^2$
(iii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0,1]$

LEMMA 2.2. ([17]) *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leqslant (1-\gamma_n)\alpha_n + \delta_n, \quad n \geqslant 0,$$

where $\{\gamma_n\}$ is a sequence in (0, 1) and δ_n is a sequence in R such that

$$(1) \sum_{n=1}^{\infty} \gamma_n = \infty; (2) \limsup_{n\to\infty} \delta_n/\gamma_n \leqslant 0 \ or \sum_{n=1}^{\infty} |\delta_n| < \infty;$$

then $\lim_{n\to\infty} \alpha_n = 0$.

LEMMA 2.3. (Marino and Xu [12]) *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leqslant \|A\|^{-1}$. Then $\|I - \rho A\| \leqslant 1 - \rho\overline{\gamma}$.*

LEMMA 2.4. ([20]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space and let* $\{\beta_n\}$ *be a sequence of [0,1] such that* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \in N$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

LEMMA 2.5. (Browder and Pertyshyn [20]) *Let* $T : K \to H$ *be a nonexpansive mapping. Define* $S : K \to H$ *by* $Sx = \lambda x + (1-\lambda)Tx$ *for each* $x \in K$. *Then, as* $\lambda \in [k,1]$, *where* $k \in (0,1)$ *is a constant,* S *is a nonexpansive mapping such that* $F(S) = F(T)$.

LEMMA 2.6. [9] *Let* C *be a closed convex subset of a real Hilbert space* H. *Give* $x \in H$ *and* $y \in C$. *Then* $y = P_C x$ *if and only if there holds the inequality:* $\langle x-y, y-z\rangle \geqslant 0, \ \forall z \in C$.

LEMMA 2.7. Zhou [21] *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $T : K \to H$ *be a k-strict pseudo-contraction with* $F(T) \neq \emptyset$. *Then* $F(P_K T) = F(T)$.

LEMMA 2.8. *Let* H *be a Hilbert space,* C *be a closed convex subset of* H. *Let* A *be a strongly positive linear bounded self-adjoint operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $f : C \to C$ *be a contraction with the contractive constant* α $(0 < \alpha < 1)$. *Assume that* $0 < \gamma \leqslant \frac{\overline{\gamma}}{\alpha}$. *Let* $T : C \to C$ *be a nonexpansive mapping with fixed point* x_t *of contraction* $C \ni x \longmapsto t\gamma f(x) + \beta x + ((1-\beta)I - tA)Tx$ *and let* $\widehat{A} := A - \gamma f$. *Then* $\{x_t\}$ *converges strongly to a fixed point* $\widetilde{x}$ *of* T *as* $t \to 0$, *which solves the following variational inequality:*

$$\langle \widehat{A}(\widetilde{x}), \widetilde{x} - z\rangle \leqslant 0, \quad \forall z \in F(T).$$

Proof. First, we show that $x \longmapsto t\gamma f(x) + \beta x + ((1-\beta)I - tA)Tx$ is a contraction. $\forall x, y \in C$, we have

$$\begin{aligned}
&\|t\gamma f(x) + \beta x + ((1-\beta)I - tA)Tx - (t\gamma f(y) + \beta y + ((1-\beta)I - tA)Ty)\| \\
&\qquad = \|t\gamma(f(x) - f(y)) + \beta(x-y) + ((1-\beta)I - tA)(Tx - Ty)\| \\
&\qquad \leqslant t\gamma\alpha\|x-y\| + \beta\|x-y\| + (1-\beta-t\overline{\gamma})\|x-y\| \\
&\qquad \leqslant (1 - t(\overline{\gamma} - \gamma\alpha))\|x-y\|.
\end{aligned}$$

By the condition $0 < \gamma \leqslant \frac{\overline{\gamma}}{\alpha}$ we know that is a contraction. Let x_t denotes the unique fixed point of T.

Second we show that x_t is bounded.

Take $\forall q \in F(T)$,

$$\begin{aligned}
\|x_t - q\| &= \|t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)Tx_t - q\| \\
&= \|t(\gamma f(x_t) - Aq) + \beta(x_t - q) + ((1-\beta)I - tA)(Tx_t - q)\| \\
&\leqslant (1-\beta-t\overline{\gamma})\|x_t - q\| + \beta\|x_t - q\| + t\|\gamma f(x_t) - Aq\| \\
&\leqslant (1-t\overline{\gamma})\|x_t - q\| + t\gamma\alpha\|x_t - q\| + t\|\gamma f(q) - Aq\| \\
&\leqslant (1 - t(\overline{\gamma} - \gamma\alpha))\|x_t - q\| + t\|\gamma f(q) - Aq\| \\
\|x_t - q\| &\leqslant \frac{1}{\overline{\gamma} - \gamma\alpha}\|\gamma f(q) - Aq\|.
\end{aligned}$$

So $\|x_t - q\|$ is bounded.

Third, we prove $\lim_{t\to 0}\|x_t - Tx_t\| = 0$.

$$x_t = t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)Tx_t,$$
$$\|x_t - Tx_t\| = \frac{1}{1-\beta} t\|\gamma f(x_t) - ATx_t\| \to 0 \ \ as \ \ t \to 0.$$

Last, we prove $\{x_t\}$ converges strongly to a fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the following variational inequality:

$$\langle \widehat{A}(\widetilde{x}), \widetilde{x} - z\rangle \leqslant 0, \quad \forall z \in F(T).$$

Assume $t_n \to 0^+$, $x_{t_n} \rightharpoonup \overline{x}$. By the demi-closeness principle, we have $\overline{x} \in F(T)$. Then claim $x_{t_n} \to \overline{x}$, $\overline{x} = \widetilde{x}$.

$\forall x^* \in F(T)$,

$$\begin{aligned}
\|x_t - x^*\| &= \|t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)Tx_t - x^*\| \\
&= \|t(\gamma f(x_t) - Ax^*) + \beta(x_t - x^*) + ((1-\beta)I - tA)(Tx_t - x^*)\| \\
\|x_t - x^*\|^2 &= \langle t(\gamma f(x_t) - Ax^*) + \beta(x_t - x^*) + ((1-\beta)I - tA)(Tx_t - x^*), x_t - x^*\rangle \\
&\leqslant (1-\beta - t\overline{\gamma})\|x_t - x^*\|^2 + \beta\|x_t - x^*\|^2 + t\langle \gamma f(x_t) - Ax^*, x_t - x^*\rangle \\
&\leqslant (1 - t(\overline{\gamma} - \gamma\alpha))\|x_t - x^*\|^2 + t\langle \gamma f(x^*) - Ax^*, x_t - x^*\rangle \\
\|x_t - x^*\|^2 &\leqslant \frac{1}{\overline{\gamma} - \gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_t - x^*\rangle.
\end{aligned}$$

In particular, take $x^* = \overline{x}, x_t = x_{t_n}$, then we have

$$\|x_{t_n} - \overline{x}\|^2 \leqslant \frac{1}{\overline{\gamma} - \gamma\alpha}\langle \gamma f(\overline{x}) - A\overline{x}, x_{t_n} - \overline{x}\rangle \to 0 \; as \; t_n \to 0^+$$

So

$$x_{t_n} \to \overline{x} \; as \; t_n \to 0^+.$$

From $\|x_t - x^*\|^2 \leqslant \frac{1}{\overline{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_t - x^*\rangle$, we have $\frac{1}{\overline{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_t - x^*\rangle \geqslant 0 \Rightarrow \langle \gamma f(x^*) - Ax^*, x_{t_n} - x^*\rangle \geqslant 0 \Rightarrow \langle \gamma f(x^*) - Ax^*, \overline{x} - x^*\rangle \geqslant 0$.

By the equivalent dual inequality we have

$$\langle \widehat{A}(\overline{x}), \overline{x} - x^*\rangle = \langle (A - \gamma f)(\overline{x}), \overline{x} - x^*\rangle \leqslant 0, \;\; \forall x^* \in F(T).$$

So, by the uniqueness solution of the above variational inequality, we have $\overline{x} = \widetilde{x}$. □

LEMMA 2.9. ([4]) *Suppose that $F : H \to H$ is κ-Lipschitzian and η-strongly monotone over a closed convex set $C \neq \emptyset$. Then, VIP(F, C) has its unique solution $u^* \in C$.*

LEMMA 2.10. *Let H be a Hilbert space, C be a closed convex subset of H. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant α $(0 < \alpha < 1)$ such that $0 < \gamma \leqslant \frac{\overline{\gamma}}{\alpha}$. Then $\widehat{A} := A - \gamma f$ is $(\overline{\gamma} - \gamma\alpha)$-strongly monotone and $(L + \gamma\alpha)$-Lipschitzian continuous, where L satisfying $||A(x)|| \leqslant Lx$ for all $x \in H$.*

Proof. Take $\forall x, y \in C$, using the property of A and Schwarz inequality, we have

$$\begin{aligned}
\langle \widehat{A}(x) - \widehat{A}(y), x - y \rangle &= \langle (A - \gamma f)x - (A - \gamma f)y, x - y \rangle \\
&= \langle A(x - y), x - y \rangle - \gamma \langle f(x) - f(y), x - y \rangle \\
&\geqslant \overline{\gamma} \|x - y\|^2 - \gamma \|f(x) - f(y)\| \|x - y\| \\
&\geqslant \overline{\gamma} \|x - y\|^2 - \gamma\alpha \|x - y\|^2 \\
&\geqslant (\overline{\gamma} - \gamma\alpha) \|x - y\|^2,
\end{aligned}$$

which implies that $\widehat{A}$ is $(\overline{\gamma} - \gamma\alpha)$-strongly monotone over C.

$$\begin{aligned}
\|\widehat{A}(x) - \widehat{A}(y)\| &= \|(A - \gamma f)x - (A - \gamma f)y\| \\
&= \|(Ax - Ay) - \gamma(f(x) - f(y))\| \\
&\leqslant \|Ax - Ay\| + \gamma \|f(x) - f(y)\| \\
&\leqslant L\|x - y\| + \gamma\alpha \|x - y\| \\
&\leqslant (L + \gamma\alpha) \|x - y\|,
\end{aligned}$$

which implies that $\widehat{A}$ is $(L + \gamma\alpha)$-Lipschitz continuous over C. □

3. Main results

In this section, we prove our strong convergence theorem.

THEOREM 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ $(C \pm C := \{x \pm y, \forall x, y \in C\})$ and $T : C \to H$ be a nonexpansive mapping with a fixed point. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant α $(0 < \alpha < 1)$ such that $0 < \gamma \leqslant \frac{\overline{\gamma}}{\alpha}$. Let $\widehat{A} := A - \gamma f$ and $\{x_n\}$ be the sequence generated by (1.8). If the control sequence $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ satisfying:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < k \leqslant \beta_n \leqslant \delta < 1, \quad \forall n \geqslant 1$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converge strongly to a fixed point p of T, which solves the variational inequality (1.9).

Proof. Since $\alpha_n \to 0$, we shall assume that $\alpha_n \leqslant (1 - \beta)\|A\|^{-1}$ and $1 - \alpha_n(\overline{\gamma} - \alpha\gamma) > 0$.

Observe that, if $\|u\| = 1$, then

$$\langle((1-\beta)I-\alpha_nA)u,u\rangle = (1-\beta)-\alpha_n\langle Au,u\rangle \geqslant (1-\beta-\alpha_n\|A\|) \geqslant 0.$$

By Lemma 2.3, we have

$$\|(1-\beta)I-\alpha_nA\| \leqslant 1-\beta-\alpha_n\overline{\gamma}.$$

We shall divide the proof into several steps.

Step 1. The sequence $\{x_n\}$ is bounded. Let $p \in F(T)$, we have

$$\begin{aligned}\|y_n-p\| &= \|P_K[\beta_nx_n+(1-\beta_n)Tx_n]-p\|\\ &\leqslant \beta_n\|x_n-p\|+(1-\beta_n)\|Tx_n-p\|\\ &\leqslant \beta_n\|x_n-p\|+(1-\beta_n)\|x_n-p\|\\ &\leqslant \|x_n-p\|\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta x_n+((1-\beta)I-\alpha_nA)y_n-p\|\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta(x_n-p)+((1-\beta)I-\alpha_nA)(y_n-p)\|\\ &\leqslant (1-\beta-\alpha_n\overline{\gamma})\|x_n-p\|+\beta\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Ap\|\\ &\leqslant (1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\|\gamma f(x_n)-\gamma f(p)\|+\alpha_n\|\gamma f(p)-Ap\|\\ &\leqslant (1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &\leqslant (1-\alpha_n(\overline{\gamma}-\alpha\gamma)\|x_n-p\|+\alpha_n\frac{(\overline{\gamma}-\alpha\gamma)}{(\overline{\gamma}-\alpha\gamma)}\|\gamma f(p)-Ap\|\\ &\leqslant \max\left\{\|x_0-p\|,\frac{1}{(\overline{\gamma}-\alpha\gamma)}\|\gamma f(p)-Ap\|\right\}\end{aligned}$$

So $\{x_n\}$ is bounded.

Step 2. $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$.
Set $z_n = \frac{x_{n+1}-\beta x_n}{1-\beta}$, so

$$x_{n+1} = \beta x_n+(1-\beta)z_n \tag{3.1}$$

$$\begin{aligned}\|z_{n+1}-z_n\| &= \frac{1}{1-\beta}\|x_{n+2}-\beta x_{n+1}-(x_{n+1}-\beta x_n)\|\\ &= \frac{1}{1-\beta}\|\alpha_{n+1}\gamma f(x_{n+1})+\beta x_{n+1}+((1-\beta)I-\alpha_{n+1}A)y_{n+1}\\ &\quad -\beta x_{n+1}-(\alpha_n\gamma f(x_n)+\beta x_n+((1-\beta)I-\alpha_nA)y_n-\beta x_n)\|\\ &= \frac{1}{1-\beta}\|\gamma(\alpha_{n+1}f(x_{n+1})-\alpha_nf(x_n))+(1-\beta)(y_{n+1}-y_n) \qquad (3.2)\\ &\quad -A(\alpha_{n+1}y_{n+1}-\alpha_ny_n)\|\\ &= \|\frac{\gamma}{1-\beta}(\alpha_{n+1}f(x_{n+1})-\alpha_nf(x_n))+y_{n+1}-y_n\\ &\quad -\frac{1}{1-\beta}(\alpha_{n+1}Ay_{n+1}-\alpha_nAy_n)\|.\end{aligned}$$

Because $\{x_n\}$ is bounded, so $\{f(x_n)\}, \{f(x_{n+1})\}, \{Ay_n\}, \{Ay_{n+1}\}, \{Tx_n\}, \{Tx_{n+1}\}$ are all bounded.

So, $\exists\, M_1 > 0,\ \max\{\frac{\gamma}{1-\beta}\{f(x_n), f(x_{n+1})\}, \frac{1}{1-\beta}\{Ay_n, Ay_{n+1}\} : n \in \mathbb{N}\} \leqslant M_1$.

So (3.2) can be reduced to

$$\|z_{n+1} - z_n\| \leqslant \|y_{n+1} - y_n\| + M_1|\alpha_{n+1} - \alpha_n|. \tag{3.3}$$

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_K(\beta_{n+1}x_{n+1} + (1-\beta_{n+1})Tx_{n+1}) - P_K(\beta_n x_n + (1-\beta_n)Tx_n)\| \\ &\leqslant \|\beta_{n+1}x_{n+1} + (1-\beta_{n+1})Tx_{n+1} - (\beta_n x_n + (1-\beta_n)Tx_n)\| \\ &\leqslant \|x_{n+1} - x_n\| + \|\beta_{n+1}(x_{n+1} + Tx_{n+1}) - \beta_n(x_n + Tx_n)\| \\ &\leqslant \|x_{n+1} - x_n\| + M_2|\beta_{n+1} - \beta_n|,\end{aligned} \tag{3.4}$$

where $M_2 > 0$ is a constant, $M_2 := \sup\{\|x_{n+1} + Tx_{n+1}\|, \|x_n + Tx_n\| : n \in \mathbb{N}\}$.

Substituting (3.4) into (3.3), we have

$$\|z_{n+1} - z_n\| \leqslant \|x_{n+1} - x_n\| + M_1|\alpha_{n+1} - \alpha_n| + M_2|\beta_{n+1} - \beta_n|.$$

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leqslant \lim_{n\to\infty}(M_1|\alpha_{n+1} - \alpha_n| + M_2|\beta_{n+1} - \beta_n|) \to 0.$$

Therefore, by Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Because $x_{n+1} = \beta x_n + (1-\beta)z_n$,

So

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = (1-\beta)\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.5}$$

Step 3. We prove the following inequality holds

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle \leqslant 0.$$

Define a mapping $T_n x := P_K[\beta_n x + (1-\beta_n)Tx]$ for each $x \in K$.

Then $T_n : K \to K$ is nonexpansive. Indeed, by Lemma 2.1 (iii), we have for all $x, y \in K$

$$\begin{aligned}\|T_n x - T_n y\|^2 &= \|P_K[\beta_n x + (1-\beta_n)Tx] - P_K[\beta_n y + (1-\beta_n)Ty]\|^2 \\ &\leqslant \|\beta_n(x-y) + (1-\beta_n)(T_x - T_y)\|^2 \\ &\leqslant \beta_n\|x-y\|^2 + (1-\beta_n)\|T_x - T_y\|^2 \\ &\quad - \beta_n(1-\beta_n)\|(I-T)x - (I-T)y\|^2 \\ &\leqslant \|x-y\|^2.\end{aligned}$$

which implies that T_n is nonexpansive. Therefore, (1.8) reduces to

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)T_n x_n. \tag{3.6}$$

Combining (3.1) and (3.6), we have

$$\begin{aligned}x_{n+1} &= \alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)T_n x_n \\ &= \beta x_n + (1-\beta)z_n.\end{aligned}$$

$$\|T_n x_n - z_n\| = \alpha_n \frac{1}{1-\beta} \|AT_n x_n - \gamma f(x_n)\| \to 0 \text{ as } n \to \infty. \tag{3.7}$$

So, using (3.5) and (3.7) we have

$$\begin{aligned} \|x_n - T_n x_n\| &= \|x_n - z_n - (T_n x_n - z_n)\| \\ &\leqslant \|x_n - z_n\| + \|T_n x_n - z_n\| \to 0, \quad n \to \infty \end{aligned} \tag{3.8}$$

From the condition (ii) and (iii), we have $\beta_n \to \lambda$ as $n \to \infty$, where $\lambda \in [k,1)$. Define $S : K \to H$ by $Sx = \lambda x + (1-\lambda)Tx$. Then, S is nonexpansive with $F(S) = F(T)$ by Lemma 2.5. It follows from Lemma 2.7 that $F(P_K S) = F(S) = F(T)$. Notice that

$$\begin{aligned} \|P_K S x_n - x_n\| &\leqslant \|x_n - T_n x_n\| + \|T_n x_n - P_K S x_n\| \\ &\leqslant \|x_n - T_n x_n\| + \|\beta_n x_n + (1-\beta_n)Tx_n - (\lambda x_n + (1-\lambda)Tx_n)\| \\ &\leqslant \|x_n - T_n x_n\| + |\beta_n - \lambda| \|x_n - Tx_n\| \to 0. \end{aligned} \tag{3.9}$$

Now, we claim that

$$\limsup_{n\to\infty} \langle \widehat{A}(q), x_n - q \rangle \geqslant 0, \tag{3.10}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto t\gamma f(x) + \beta x + ((1-\beta)I - tA)P_K Sx.$$

(Replace T with $P_K S$ in Lemma 2.8.)

Then x_t solves the fixed point equation

$$x_t = t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)P_K S x_t.$$

Thus we have

$$\|x_t - x_n\| = \|t(\gamma f(x_t) - Ax_n) + \beta(x_t - x_n) + ((1-\beta)I - tA)(P_K S x_t - x_n)\|.$$

It follows from Lemma 2.1 that

$$\begin{aligned} \|x_t - x_n\|^2 &= \|t(\gamma f(x_t) - Ax_n) + \beta(x_t - x_n) + ((1-\beta)I - tA)(P_K S x_t - x_n)\|^2 \\ &\leqslant \|\beta(x_t - x_n) + ((1-\beta)I - tA)(P_K S x_t - x_n)\|^2 \\ &\quad + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\ &\leqslant (1-\beta-t\overline{\gamma})^2 \|P_K S x_t - x_n\|^2 + \beta^2 \|x_t - x_n\|^2 \\ &\quad + 2\beta(1-\beta-t\overline{\gamma})\langle P_K S x_t - x_n, x_t - x_n \rangle + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\ &\leqslant (1-\beta-t\overline{\gamma})^2 \|P_K S x_t - P_K S x_n + P_K S x_n - x_n\|^2 + \beta^2 \|x_t - x_n\|^2 \\ &\quad + 2\beta(1-\beta-t\overline{\gamma})\langle P_K S x_t - P_K S x_n + P_K S x_n - x_n, x_t - x_n \rangle \\ &\quad + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n \rangle \\ &\leqslant (1-\beta-t\overline{\gamma})^2 \|x_t - x_n\|^2 + \beta^2 \|x_t - x_n\|^2 \\ &\quad + 2(1-\beta-t\overline{\gamma})^2 \|P_K S x_n - x_n\| (\|P_K S x_n - x_n\| + \|x_t - x_n\|) \\ &\quad + 2\beta(1-\beta-t\overline{\gamma})\|x_t - x_n\|^2 + 2\beta(1-\beta-t\overline{\gamma})\|P_K S x_n - x_n\| \|x_t - x_n\| \\ &\quad + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle + 2t\langle Ax_t - Ax_n, x_t - x_n \rangle. \end{aligned} \tag{3.11}$$

Since A is strongly positive and bounded linear operator, we have

$$\langle Ax_t - Ax_n, x_t - x_n\rangle = \langle A(x_t - x_n), x_t - x_n\rangle \geqslant \overline{\gamma}\|x_t - x_n\|^2. \tag{3.12}$$

Put

$$\begin{aligned} f_n(t) &= 2(1-\beta-t\overline{\gamma})^2\|P_K S x_n - x_n\|(\|P_K S x_n - x_n\| + \|x_t - x_n\|) \\ &\quad + 2\beta(1-\beta-t\overline{\gamma})\|P_K S x_n - x_n\|\|x_t - x_n\| \\ &= \|P_K S x_n - x_n\|[2(1-\beta-t\overline{\gamma})^2(\|P_K S x_n - x_n\| + \|x_t - x_n\|) \\ &\quad + 2\beta(1-\beta-t\overline{\gamma})\|x_t - x_n\|] \to 0, \ as\ n \to \infty. \end{aligned} \tag{3.13}$$

Combing (3.11), (3.12) and (3.13), we have

$$\begin{aligned} 2t\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle &\leqslant (t^2\overline{\gamma}^2 - 2t\overline{\gamma})\|x_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle Ax_t - Ax_n, x_t - x_n\rangle \\ &\leqslant \overline{\gamma}t^2\langle Ax_t - Ax_n, x_t - x_n\rangle + f_n(t) \\ \langle Ax_t - \gamma f(x_t), x_t - x_n\rangle &\leqslant \frac{\overline{\gamma}t}{2}\langle Ax_t - Ax_n, x_t - x_n\rangle + \frac{1}{t}f_n(t). \end{aligned} \tag{3.14}$$

Let $n \to \infty$ in (3.14) and note that (3.13) yields

$$\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \leqslant \frac{t}{2}M_3, \tag{3.15}$$

where $M_3 > 0$ is a constant and $\overline{\gamma}\langle Ax_t - Ax_n, x_t - x_n\rangle \leqslant M_3$ for all $t \in (0,1)$ and $n \geqslant 1$.

Taking $t \to 0$ from (3.15), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \leqslant 0. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned} \langle \gamma f(q) - Aq, x_n - q\rangle &= \langle \gamma f(q) - Aq, x_n - q\rangle - \langle \gamma f(q) - Aq, x_n - x_t\rangle \\ &\quad + \langle \gamma f(q) - Aq, x_n - x_t\rangle - \langle \gamma f(q) - Ax_t, x_n - x_t\rangle \\ &\quad + \langle \gamma f(q) - Ax_t, x_n - x_t\rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle \\ &\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle &\leqslant \|\gamma f(q) - Aq\|\|x_t - q\| + \|A\|\|x_t - q\|\lim_{n\to\infty}\|x_n - x_t\| \\ &\quad + \gamma\alpha\|x_t - q\|\lim_{n\to\infty}\|x_n - x_t\| \\ &\quad + \limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle. \end{aligned}$$

Therefore, from (3.16), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle \\
&\leqslant \limsup_{t\to 0}\|\gamma f(q)-Aq\|\|x_t-q\| \\
&\quad +\limsup_{t\to 0}\|A\|\|x_t-q\|\lim_{n\to\infty}\|x_n-x_t\| \\
&\quad +\limsup_{t\to 0}\gamma\alpha\|x_t-q\|\lim_{n\to\infty}\|x_n-x_t\| \\
&\quad +\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle \\
&\leqslant 0.
\end{aligned}$$

Hence, (3.10) holds.

Finally, we prove $x_n\to q$, as $n\to\infty$.

From Lemma (2.1), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n(\gamma f(x_n)-Aq)+\beta(x_n-q)+((1-\beta)I-\alpha_nA)(y_n-q)\|^2 \\
&\leqslant \|\beta(x_n-q)+((1-\beta)I-\alpha_nA)(y_n-q)\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Aq,x_{n+1}-q\rangle \\
&\leqslant (1-\beta-\alpha_n\overline{\gamma})^2\|y_n-q\|^2+\beta^2\|x_n-q\|^2 \\
&\quad +2\beta(1-\beta-\alpha_n\overline{\gamma})\langle y_n-q,x_n-q\rangle \\
&\quad +2\alpha_n\langle\gamma f(x_n)-\gamma f(q),x_{n+1}-q\rangle+2\alpha_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle \\
&\leqslant (1-\beta-\alpha_n\overline{\gamma})^2\|x_n-q\|^2+\beta^2\|x_n-q\|^2 \\
&\quad +2\beta(1-\beta-\alpha_n\overline{\gamma})\|x_n-q\|^2 \\
&\quad +\alpha_n\gamma\alpha(\|x_n-q\|^2+\|x_{n+1}-q\|^2)+2\alpha_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle, \\
\|x_{n+1}-q\|^2 &\leqslant \frac{1-2\alpha_n\overline{\gamma}+\alpha_n\gamma\alpha+\alpha^2\overline{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle \\
&\leqslant [1-\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}]\|x_n-q\|^2 \\
&\quad +\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}[\frac{1}{\overline{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\alpha\gamma)}M_3],
\end{aligned}$$

where $M_3>0$ is an appropriate constant such that $M_3\geqslant\sup_{n\geqslant 1}\{\|x_n-q\|^2\}$. Put $j_n=\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}$ and $t_n=\frac{1}{\overline{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\alpha\gamma)}M_3$.

That is,

$$\|x_{n+1}-q\|^2\leqslant(1-j_n)\|x_n-q\|^2+j_nt_n. \tag{3.18}$$

It follows from condition (i) and (3.10) that $\lim_{n\to\infty}j_n=0$, $\sum_{n=1}^{\infty}j_n=\infty$ and

$\limsup_{n\to\infty} t_n \leqslant 0$. Apply Lemma2. 2 to (3.18) to conclude $x_n \to q$, as $n \to \infty$. This completes the proof. $\square$

Compared with Theorem 4.1 [3] which present an algorithm using a conjugate gradient direction(this method is defined by combining the ideas of the hybrid steepest descent method and the conjugate gradient method), we removed the condition (iv) that about the coefficient α_n, and the prove is different.

REMARK. Taking $\gamma = 1$ and $A = I$, the identity mapping, in Theorem 3.1, we have the following results immediately.

COROLLARY 3.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$ and $T : C \to H$ be a nonexpansive mapping with a fixed point. Let $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$. Let $\{x_n\}$ be a sequence generated by (1.8). If the control sequence $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ satisfying:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $k \leqslant \beta_n \leqslant \delta < 1$, $\forall n \geqslant 1$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of the variational inequality

$$\langle (f(q) - q, p - q\rangle \leqslant 0, \ \ \forall p \in F(T).$$

Acknowledgments

This paper was Supported by the National Natural Science Foundation of China under grant (10771050).

REFERENCES

[1] YONGHONG YAO, MUHAMMAD ASLAM NOOR, RUDONG CHEN, YEONG-CHENG LIOU, *Strong convergence of three-step relaxed hybrid steepest-descent methods for variational inequalities*, Applied Mathematics and Computation, **201**, 1–2 (2008), 175–183.

[2] YANRONG YU AND RUDONG CHEN, *Hybrid steepest-descent method for general variational inequalities*, Journal of Inequalities and Applications, **2007** (2007), Article ID 19270, 14 pp.

[3] HIDEAKI IIDUKA, ISAO YAMADA, *A use of conjugate direction for the convex optimization problem over the fixed point set of a nonexpansive mapping*, SLAM J. OPTIM., **19**, 4 (2009), 1881–1893.

[4] I. YAMADA, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in Inherently Parallel Algorithms for Feasibility and Optimization and Their Applications (D. Butnariu, Y. Censor and S. Reich, Eds.), Elsevier, 2001, pp. 473–504.

[5] I. YAMADA AND N. OGURA, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mapping*, Numer. Funct. Anal. Optim., **25** (2004), 619–655.

[6] I. YAMADA, N. OGURA, AND N. SHIRAKAWA, *A numerical robust hybrid steepest descent method for the convexly constrained generalized inverse problems*, Contemp. Math., **313** (2002), 269–305.

[7] S. REICH, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67** (1979), 274–276.

[8] T. H. KIM, H. K. XU, *Strong convergece of modified Mann iterations*, Nonlinear Anal., **61** (2005), 51–60.

[9] H. IIDUKA, W. TAKAHASHI, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal., **61** (2005), 341–350.

[10] G. MARINO, H. K. XU, *Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl., **329** (2007), 336–349.

[11] X. QIN, Y. SU, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl., **329** (2007), 415–424.

[12] G. MARINO, H. K. XU, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **318** (2006), 43–52.

[13] YISHENG SONG, RUDONG CHEN, *Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings*, Applied Mathematics and Computation, **180**, 1 (2006), 275–287.

[14] YISHENG SONG, RUDONG CHEN, *Weak and strong convergence of Mann's-type iterations for a countable family of nonexpansive mappings*, Journal of the Korean Mathematical Society, **45**, 5 (2008), 1393–1404.

[15] RUDONG CHEN, YUNYAN SONG, *Strong Convergence to Common Fixed Point of Nonexpansive Semigroups In Banach Space*, Journal of Computational and Applied Mathematics, **200**, 2 (2007), 566–575.

[16] Y. YAO, R. CHEN, J. C. YAO, *Strong convergence and certain control conditions for modified Mann iteration*, Nonlinear Anal., **68**, 6 (2008), 1687–1693.

[17] K. NAKAJO, W. TAKAHASHI, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279** (2003), 372–379.

[18] HONG-KUN XU, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298** (2004), 279–291.

[19] T. SUZUKI, *Strong convergence of Krasnoselskii and Mann's type sequence for one-parameter nonexpansive semigroups without Bochner intergral*, J. Math. Anal. Appl., **305**, 1 (2005), 227–239.

[20] F. E. BROWDER, W. V. PETRYSHYN, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228.

[21] H. ZHOU, *Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbet space*, Nonlinear Anal., **69**, 2 (2008), 456–462.

[22] H. K. XU, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116** (2003), 659–678.

[23] YONGHONG YAO, RUDONG CHEN, HAIYUN ZHOU, *Strong convergence to common fixed points of nonexpansive mappings without commutativity assumption*, Fixed Point Theory and Applications, Art. No. 89470, 2006.

(Received January 18, 2010)

RuDong Chen
Department of Mathematics
Tianjin Polytechnic University
Tianjin 300160
P.R. China
e-mail: tjcrd@yahoo.com.cn

HaiFang Liu
Department of Mathematics
Tianjin Polytechnic University
Tianjin 300160
P.R. China
e-mail: liuhaifang031@yahoo.cn