

## EXPONENTIAL CONVEXITY AND JENSEN'S INEQUALITY FOR DIVIDED DIFFERENCES

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*Abstract.* In this paper we obtain means which involve divided differences for  $n$ -convex functions. We examine their monotonicity property using exponentially convex functions.

### 1. Introduction

Let  $f$  be a real-valued function defined on the segment  $[a, b]$ . The divided difference of order  $n$  of the function  $f$  at distinct points  $x_0, \dots, x_n \in [a, b]$ , is defined recursively (see [3], [10] and [12]) by

$$f[x_i] = f(x_i) \text{ for } i = 0, \dots, n$$

and

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value  $f[x_0, \dots, x_n]$  is independent of the order of the points  $x_0, \dots, x_n$ . The definition may be extended to include the case that some (or all) of the points coincide. Assuming that  $f^{(j-1)}(x)$  exists, we define

$$f[\underbrace{x, \dots, x}_j] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

For divided difference, in the case of distinct points, the following holds:

$$f[x_0, \dots, x_n] = \sum_{i=0}^n \frac{f(x_i)}{\omega'(x_i)}, \text{ where } \omega(x) = \prod_{j=0}^n (x - x_j),$$

so we have that

$$f[x_0, \dots, x_n] = \sum_{i=0}^n \frac{f(x_i)}{\prod_{j=1, j \neq i}^n (x_i - x_j)}.$$

A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be  $n$ -convex if the  $n$ -th order divided difference of  $f$  satisfies

$$f[x_0, \dots, x_n] \geq 0 \text{ for all } a \leq x_0 < \dots < x_n \leq b.$$

The following theorem is proved (see [5]):

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**THEOREM 1.** *Let  $f$  be an  $(n + 2)$ -convex function on  $(a, b)$  and  $\mathbf{x} \in (a, b)^{n+1}$ . Then*

$$G(\mathbf{x}) = f[x_0, \dots, x_n]$$

*is a convex function of the vector  $\mathbf{x} = (x_0, \dots, x_n)$ . Consequently,*

$$f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \leq \sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] \tag{1.1}$$

*holds for all  $a_i \geq 0$  such that  $\sum_{i=0}^m a_i = 1$ .*

**Schur polynomial** in  $n + 1$  variables  $x_0, \dots, x_n$  of degree  $d = d_0 + \dots + d_n$  ( $d_0 \geq \dots \geq d_n$ ) is defined as

$$S_{(d_0, \dots, d_n)}(x_0, \dots, x_n) = \frac{\det \left[ x_i^{d_{n-j}+j} \right]_{i,j=0}^n}{\det \left[ x_i^j \right]_{i,j=0}^n}.$$

The numerator consists of alternating polynomials (they change the sign under any transposition of the variables) and so they are all divisible by the denominator which is Vandermonde determinant. Schur polynomial is also symmetric because the numerator and denominator are both alternating.

For  $n \geq 1$  using Schur polinomial and Vandermonde determinant (extended with logarithmic function)

$$V(\mathbf{x}; p, q) = \det \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} & x_0^p \ln^q x_0 \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} & x_1^p \ln^q x_1 \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} & x_2^p \ln^q x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & x_n^p \ln^q x_n \end{bmatrix}$$

we obtain:

**PROPOSITION 1.** *For monomial function  $h(x) = x^{n+k}$ , where  $k \geq 1$  is an integer, holds*

$$h[x_0, \dots, x_n] = S_{(\underbrace{k, 0, \dots, 0}_{n \text{ times}})}(x_0, \dots, x_n) = \frac{V(\mathbf{x}; n+k, 0)}{V(\mathbf{x}; n, 0)}.$$

*For potential function  $f(x) = x^p = x^{n+p-n}$ , where  $p$  is a real number, holds*

$$f[x_0, \dots, x_n] = \frac{V(\mathbf{x}; p, 0)}{V(\mathbf{x}; n, 0)}.$$

Further, for a partition  $\pi$  the Schur polynomial can be expressed by a sum

$$S_\pi(x_0, \dots, x_n) = \sum_T \mathbf{x}^T = \sum_T x_0^{t_0} \dots x_n^{t_n}.$$

The summation is over all semistandard Young tableaux  $T$  of shape  $\pi$  where the exponents  $t_0, \dots, t_n$  give the weight of  $T$  in which each  $t_j$  counts the occurrences of the number  $j$  in  $T$  (see [2] and [7]). So, we have the next proposition:

PROPOSITION 2. For monomial function  $h(x) = x^{n+k}$ , where  $k \geq 1$  is an integer, holds

$$h[x_0, \dots, x_n] = S_{(\underbrace{k, 0, \dots, 0}_{n \text{ times}})}(x_0, \dots, x_n) = \sum_{i_1=0}^n \sum_{i_2=0}^{i_1} \dots \sum_{i_k=0}^{i_{k-1}} x_{i_1} x_{i_2} \dots x_{i_k}.$$

The goal of this paper is to define means using the inequality (1.1). We will examine their monotonicity property using exponential convex functions. Also, we will consider some special cases.

### 2. Jensen means for divided differences

THEOREM 2. Let  $f \in C^{n+2}([a, b])$  and  $\mathbf{x}^i \in (a, b)^{n+1}$  for  $i = 0, \dots, m$ . If  $a_i \geq 0$  such that  $\sum_{i=0}^m a_i = 1$  and  $\sum_{j=0}^n \sum_{k=0}^j \left( \sum_{i=0}^m a_i x_j^i x_k^i - \sum_{i=0}^m a_i x_j^i \sum_{l=0}^m a_l x_k^l \right) \neq 0$ , then there exists  $\xi \in (a, b)$  such that

$$\begin{aligned} & \sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \\ &= \frac{f^{(n+2)}(\xi)}{(n+2)!} \sum_{j=0}^n \sum_{k=0}^j \left( \sum_{i=0}^m a_i x_j^i x_k^i - \sum_{i=0}^m a_i x_j^i \sum_{l=0}^m a_l x_k^l \right). \end{aligned} \tag{2.1}$$

*Proof.* Let us denote  $\alpha = \min f^{(n+2)}$  and  $\beta = \max f^{(n+2)}$ . We first consider the following function  $\phi_1(x) = \frac{\beta x^{n+2}}{(n+2)!} - f(x)$ . Then  $\phi_1^{(n+2)}(x) = \beta - f^{(n+2)}(x) \geq 0, x \in [a, b]$ , so  $\phi_1$  is an  $(n+2)$ -convex function. Applying Theorem 1 on an  $(n+2)$ -convex function  $\phi_1$  with  $\phi(x) = \frac{x^{n+2}}{(n+2)!}$  we have

$$\begin{aligned} & \beta \cdot \sum_{i=0}^m a_i \phi[x_0^i, \dots, x_n^i] - \sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] \\ & \geq \beta \cdot \phi \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] - f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right], \end{aligned}$$

i.e.

$$\begin{aligned} & \sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \\ & \leq \beta \left( \sum_{i=0}^m a_i \phi[x_0^i, \dots, x_n^i] - \phi \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \right). \end{aligned}$$

Similarly, a function  $\phi_2(x) = f(x) - \alpha \cdot \phi(x)$  is an  $(n + 2)$ -convex function. Inequality from the Theorem 1 with  $(n + 2)$ -convex function  $\phi_2$  becomes

$$\begin{aligned} & \sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - \alpha \cdot \sum_{i=0}^m a_i \phi[x_0^i, \dots, x_n^i] \\ & \geq f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] - \alpha \cdot \phi \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right], \end{aligned}$$

i.e.

$$\begin{aligned} & \sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \\ & \geq \alpha \left( \sum_{i=0}^m a_i \phi[x_0^i, \dots, x_n^i] - \phi \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \right). \end{aligned}$$

From Proposition 2 with  $k = 2$  and the fact that the function  $\phi(x)$  is  $(n + 2)$ -convex we have

$$\begin{aligned} & \sum_{i=0}^m a_i \phi[x_0^i, \dots, x_n^i] - \phi \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right] \\ & = \frac{1}{(n + 2)!} \left[ \sum_{j=0}^n \sum_{k=0}^j \left( \sum_{i=0}^m a_i x_j^i x_k^i - \sum_{i=0}^m a_i x_j^i \sum_{l=0}^m a_l x_k^l \right) \right] > 0. \end{aligned}$$

We can now conclude that there exists  $\xi \in (a, b)$  that we are looking for in (2.1).  $\square$

REMARK 1. Let us note that the left-hand side in equation (2.1) is greater than or equal to zero if  $f^{(n+2)} \geq 0$  which is the statement of Theorem 1.

COROLLARY 1. Let  $f, g \in C^{n+2}([a, b])$  and  $\mathbf{x}^i \in (a, b)^{n+1}$ . If  $a_i \geq 0$  and  $\sum_{i=0}^m a_i = 1$ , then there exists  $\xi \in (a, b)$  such that

$$\frac{\sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right]}{\sum_{i=0}^m a_i g[x_0^i, \dots, x_n^i] - g \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right]} = \frac{f^{(n+2)}(\xi)}{g^{(n+2)}(\xi)} \tag{2.2}$$

provided that both denominators not equal zero.

*Proof.* We use the following standard technique: Let us define the linear functional  $L(h) = \sum_{i=0}^m a_i h[x_0^i, \dots, x_n^i] - h \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right]$ . Next, we define  $\psi(t) = f(t)L(g) - g(t)L(f)$ . According to Theorem 2, applied on  $\psi$ , there exists  $\xi \in (a, b)$  so that

$$L(\psi) = \frac{\psi^{(n+2)}(\xi)}{(n + 2)!} \sum_{j=0}^n \sum_{k=0}^j \left( \sum_{i=0}^m a_i x_j^i x_k^i - \sum_{i=0}^m a_i x_j^i \sum_{l=0}^m a_l x_k^l \right).$$

From  $L(\psi) = 0$ , it follows  $f^{(n+2)}(\xi)L(g) - g^{(n+2)}(\xi)L(f) = 0$  and (2.2) is proved.  $\square$

The above corollary enables us to define various types of means, because if the function  $f^{(n+2)}/g^{(n+2)}$  has inverse, from (2.2) we can get

$$\xi = \left( \frac{f^{(n+2)}}{g^{(n+2)}} \right)^{-1} \left( \frac{\sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f[\sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i]}{\sum_{i=0}^m a_i g[x_0^i, \dots, x_n^i] - g[\sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i]} \right).$$

Specially, if we take substitutions  $f(t) = t^p$ ,  $g(t) = t^q$  with  $t > 0$  in the above identity and use the method of continuous extensions, we get **Jensen means for divided differences**. Let us use the following notation:

$$\mathbf{X} = \begin{bmatrix} x_0^0 & x_1^0 & x_2^0 & \dots & x_{n-1}^0 & x_n^0 \\ x_0^1 & x_1^1 & x_2^1 & \dots & x_{n-1}^1 & x_n^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_0^m & x_1^m & x_2^m & \dots & x_{n-1}^m & x_n^m \end{bmatrix}, \mathbf{a} = (a_0, \dots, a_m),$$

$$V\mathbf{x}pq = V(\mathbf{x}; p, q) \text{ and } N = \{0, 1, \dots, n+1\}$$

The Jensen means for divided differences is defined as

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) = \begin{cases} \left( \prod_{i=0}^{n+1} \frac{q-i}{p-i} \frac{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j p_0}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} p_0}{V\mathbf{aX} n_0}}{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j q_0}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} q_0}{V\mathbf{aX} n_0}} \right)^{\frac{1}{p-q}} & \text{for } p \neq q; p \notin N, q \notin N \\ \left( \prod_{\substack{i=0 \\ i \neq k}}^{n+1} \frac{(q-k)(q-i)}{k-i} \frac{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j k_1}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} k_1}{V\mathbf{aX} n_0}}{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j q_0}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} q_0}{V\mathbf{aX} n_0}} \right)^{\frac{1}{k-q}} & \text{for } p \neq q; p = k \in N, q \notin N \\ \left( \prod_{\substack{i=0 \\ i \neq l}}^{n+1} \frac{l-i}{(p-l)(p-i)} \frac{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j p_0}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} p_0}{V\mathbf{aX} n_0}}{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j l_1}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} l_1}{V\mathbf{aX} n_0}} \right)^{\frac{1}{p-l}} & \text{for } p \neq q; p \notin N, q = l \in N \\ \left( \prod_{\substack{i=0 \\ i \neq k, l}}^{n+1} \frac{l-i}{i-k} \frac{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j k_1}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} k_1}{V\mathbf{aX} n_0}}{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j l_1}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} l_1}{V\mathbf{aX} n_0}} \right)^{\frac{1}{k-l}} & \text{for } p \neq q; p = k \in N, q = l \in N \\ \exp \left( \sum_{i=0}^{n+1} \frac{1}{i-p} + \frac{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j p_1}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} p_1}{V\mathbf{aX} n_0}}{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j p_0}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} p_0}{V\mathbf{aX} n_0}} \right) & \text{for } p = q; p \notin N, q \notin N \\ \exp \left( \sum_{\substack{i=0 \\ i \neq k}}^{n+1} \frac{1}{i-k} + \frac{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j k_2}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} k_2}{V\mathbf{aX} n_0}}{\sum_{j=0}^m a_j \frac{V\mathbf{x}^j k_1}{V\mathbf{x}^j n_0} - \frac{V\mathbf{aX} k_1}{V\mathbf{aX} n_0}} \right) & \text{for } p = q; p = q = k \in N \end{cases} \tag{2.3}$$

A mapping  $E_{Jen} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function in two variables  $p$  and  $q$ . The skeleton of all expressions for this means is a fraction

$$\frac{\sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f[\sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i]}{\sum_{i=0}^m a_i g[x_0^i, \dots, x_n^i] - g[\sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i]} = \frac{\sum_{i=0}^m a_i \frac{V_{\mathbf{x}^i p 0}}{V_{\mathbf{x}^i n 0}} - \frac{V_{\mathbf{aX} p 0}}{V_{\mathbf{aX} n 0}}}{\sum_{i=0}^m a_i \frac{V_{\mathbf{x}^i q 0}}{V_{\mathbf{x}^i n 0}} - \frac{V_{\mathbf{aX} q 0}}{V_{\mathbf{aX} n 0}}}$$

which follows from Proposition 1. The most parts of expressions proceed from

$$\lim_{p \rightarrow k} \frac{\sum_{i=0}^m a_i \frac{V_{\mathbf{x}^i p 0}}{V_{\mathbf{x}^i n 0}} - \frac{V_{\mathbf{aX} p 0}}{V_{\mathbf{aX} n 0}}}{p - k} = \sum_{i=0}^m a_i \frac{V_{\mathbf{x}^i k 1}}{V_{\mathbf{x}^i n 0}} - \frac{V_{\mathbf{aX} k 1}}{V_{\mathbf{aX} n 0}} \quad \text{for } k \in N. \quad (2.4)$$

The quotient under limit becomes an indeterminate form when  $p \rightarrow k \in N$  because  $V_{\mathbf{x}k0} = 0$  for  $k = 0, 1, \dots, n - 1$  and  $\sum_{i=0}^m a_i \frac{V_{\mathbf{x}^i k 0}}{V_{\mathbf{x}^i n 0}} - \frac{V_{\mathbf{aX} k 0}}{V_{\mathbf{aX} n 0}} = 1 - 1 = 0$  for  $k = n$ . When  $p \rightarrow k = n + 1$ :

$$\sum_{i=0}^m a_i \frac{V_{\mathbf{x}^i k 0}}{V_{\mathbf{x}^i n 0}} - \frac{V_{\mathbf{aX} k 0}}{V_{\mathbf{aX} n 0}} = \sum_{i=0}^m a_i \sum_{j=0}^n x_j^i - \sum_{j=0}^n \sum_{i=0}^m a_i x_j^i = 0.$$

So, we can apply L'Hospital's rule and also the formula  $\frac{d}{dp} V_{\mathbf{x} p q} = V_{\mathbf{x} p q+1}$  to get result (2.4). If  $n = 0$  the expressions for means  $E_{Jen}(\mathbf{X}, \mathbf{a}; p, q)$  proceed from the continuous extensions of function

$$(p, q) \rightarrow \left( \frac{q \sum_{i=0}^m a_i (x_0^i)^p - (\sum_{i=0}^m a_i x_0^i)^p}{p \sum_{i=0}^m a_i (x_0^i)^q - (\sum_{i=0}^m a_i x_0^i)^q} \right)^{\frac{1}{p-q}}.$$

### 3. Monotonicity of Jensen means for divided differences

Now we can introduce exponentially convex functions that will play an important role in this section. First, we give here a definition of exponentially convex function as it was done originally by Bernstein in [4] (see also [1],[8], [9]).

DEFINITION 1. A function  $\varphi : (a, b) \rightarrow \mathbb{R}$  is exponentially convex if it is continuous and  $\sum_{i,j=1}^n \xi_i \xi_j \varphi(x_i + x_j) \geq 0$  for every  $n \in \mathbb{N}$  and all choices  $\xi_1, \dots, \xi_n \in \mathbb{R}$  and all choices  $x_1, \dots, x_n \in (a, b)$  so that  $x_i + x_j \in (a, b)$ .

In the rest of the paper we will rely on the following proposition and its corollaries.

PROPOSITION 3. Let  $\varphi : (a, b) \rightarrow \mathbb{R}$ . Then the following statements are equivalent:

- (i)  $\varphi$  is exponentially convex.
- (ii)  $\varphi$  is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j \varphi\left(\frac{x_i + x_j}{2}\right) \geq 0 \quad (3.1)$$

for every  $n \in \mathbb{N}$  and all choices  $\xi_1, \dots, \xi_n \in \mathbb{R}$  and all choices  $x_1, \dots, x_n \in (a, b)$ .

Using basic calculus, we have these two corollaries.

COROLLARY 2. *If  $\varphi$  is exponentially convex, then*

$$\det \left[ \varphi \left( \frac{x_i + x_j}{2} \right) \right]_{i,j=1}^n \geq 0$$

for every  $n \in \mathbb{N}$  and all choices  $x_1, \dots, x_n \in (a, b)$ .

COROLLARY 3. *Exponentially convex function  $\varphi : (a, b) \rightarrow (0, \infty)$  is log-convex:*

$$\varphi \left( \frac{x+y}{2} \right) \leq \sqrt{\varphi(x)\varphi(y)} \quad \text{for all } x, y \in (a, b).$$

Using exponential convexity we will prove monotonicity of Jensen means  $E(\mathbf{X}, \mathbf{a}; p, q)$  in both parameters  $p$  and  $q$ .

THEOREM 3. *Let  $x_0^i, x_1^i, \dots, x_n^i$  for  $i = 0, \dots, m$  be mutually different positive real numbers so that  $\sum_{j=0}^n \sum_{k=0}^j \left( \sum_{i=0}^m a_i x_j^i x_k^i - \sum_{i=0}^m a_i x_j^i \sum_{i=0}^m a_i x_k^i \right) \neq 0$  for all  $a_i \geq 0$  and  $\sum_{i=0}^m a_i = 1$ . Also let*

$$\varphi(\lambda) = \sum_{i=0}^m a_i f_\lambda [x_0^i, \dots, x_n^i] - f_\lambda \left[ \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right],$$

where  $f_\lambda$  is defined with

$$f_\lambda(t) = \begin{cases} \frac{t^\lambda}{\lambda(\lambda-1)\dots(\lambda-n-1)} & \text{if } \lambda \notin \{0, 1, \dots, n+1\}, \\ \frac{t^k \ln t}{(-1)^{n+1-k} k!(n+1-k)!} & \text{if } \lambda = k \in \{0, 1, \dots, n+1\}. \end{cases} \tag{3.2}$$

(i) *The function  $\varphi$  is exponentially convex.*

(ii) *For every  $n \in \mathbb{N}$  and all choices  $t_1, \dots, t_n \in \mathbb{R}$  the matrix  $\left[ \varphi \left( \frac{t_i + t_j}{2} \right) \right]_{i,j=1}^n$  is a positive semi-definite matrix. Particularly*

$$\det \left[ \varphi \left( \frac{t_i + t_j}{2} \right) \right]_{i,j=1}^n \geq 0. \tag{3.3}$$

*Proof.* (i) First, the expression

$$\varphi(\lambda) = \begin{cases} \frac{1}{\lambda(\lambda-1)\dots(\lambda-n-1)} \left[ \sum_{i=0}^m a_i \frac{V(\mathbf{x}^i; \lambda, 0)}{V(\mathbf{x}^i; n, 0)} - \frac{V(\mathbf{aX}; \lambda, 0)}{V(\mathbf{aX}; n, 0)} \right] & \text{if } \lambda \notin \{0, 1, \dots, n+1\}, \\ \frac{1}{(-1)^{n+1-k} k!(n+1-k)!} \left[ \sum_{i=0}^m a_i \frac{V(\mathbf{x}^i; k, 1)}{V(\mathbf{x}^i; n, 0)} - \frac{V(\mathbf{aX}; k, 1)}{V(\mathbf{aX}; n, 0)} \right] & \text{if } \lambda = k \in \{0, 1, \dots, n+1\}, \end{cases}$$

shows that  $\varphi$  is a continuous function. Further, let us consider the function  $f(t) = \sum_{i,j=1}^n \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(t)$ , where  $\xi_i \in \mathbb{R}$  and  $t_i \in \mathbb{R}$  for  $i = 1, \dots, n$ . Since  $f_\lambda^{(n+2)}(t) = t^{\lambda-n-2}$ , it follows

$$f^{(n+2)}(t) = \sum_{i,j=1}^n \xi_i \xi_j t^{\frac{t_i+t_j}{2}-n-2} = \left( \sum_{i=1}^n \xi_i t^{\frac{t_i-n-2}{2}} \right)^2 \geq 0.$$

This is equivalent to inequality  $\sum_{i=0}^m a_i f[x_0^i, \dots, x_n^i] - f[\sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i] \geq 0$ , i.e.

$$\sum_{i,j=1}^n \xi_i \xi_j \varphi\left(\frac{t_i+t_j}{2}\right) \geq 0. \tag{3.4}$$

We now conclude that  $\varphi$  is an exponentially convex function.

(ii) This follows from nonnegativity of quadratic form (3.4).  $\square$

LEMMA 1. *If  $\varphi : (a, b) \rightarrow (0, \infty)$  is log-convex and  $s, t, u, v \in (a, b)$  so that  $s \neq t, u \neq v, s \leq u, t \leq u$  (or  $s \leq v, t \leq v$ ); then*

$$\left(\frac{\varphi(t)}{\varphi(s)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\varphi(v)}{\varphi(u)}\right)^{\frac{1}{v-u}}.$$

COROLLARY 4. *Let  $p \leq u$  and  $q \leq v$ . Then*

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) \leq E_{Jen}(\mathbf{X}, \mathbf{a}; u, v) \tag{3.5}$$

for all  $x_0^i, x_1^i, \dots, x_n^i$  mutually different positive real numbers so that

$$\sum_{j=0}^n \sum_{k=0}^j \left( \sum_{i=0}^m a_i x_j^i x_k^i - \sum_{i=0}^m a_i x_j^i \sum_{l=0}^m a_l x_k^l \right) \neq 0 \text{ for all } a_i \geq 0 \text{ and } \sum_{i=0}^m a_i = 1.$$

*Proof.* Let us first observe that for  $p \neq q$  holds

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) = \left(\frac{\varphi(p)}{\varphi(q)}\right)^{\frac{1}{p-q}}. \tag{3.6}$$

According to Theorem 3  $\varphi$  is a positive exponentially convex function and therefore, according to Corollary 3, a log-convex function. So, we can apply Lemma 1 on  $\varphi$  and get

$$\left(\frac{\varphi(t)}{\varphi(s)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\varphi(v)}{\varphi(u)}\right)^{\frac{1}{v-u}} \tag{3.7}$$

for  $s, t, u, v \in \mathbb{R}$  so that  $s \neq t, u \neq v, s \leq u, t \leq u$  (or  $s \leq v, t \leq v$ ). Using continuous extensions of (3.6) and (3.7) we finally conclude that,  $p \leq u$  and  $q \leq v$  implies

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) \leq E_{Jen}(\mathbf{X}, \mathbf{a}; u, v). \quad \square$$



### 4. Special cases

If in (2.3) we put  $x_0^j = x^j$ ,  $x_j^j = x^j + h_j$  and  $a_i = \frac{p_i}{P_m}$ , where  $i = 0, \dots, m$ ,  $P_m = \sum_{i=0}^m p_i$ ,  $j = 1, \dots, n$  and  $x^j + h_j \in (a, b)$ , we get means related to inequality (see [16]):

$$\frac{1}{P_m} \sum_{i=0}^m p_i f[x^j, x^j + h_1, \dots, x^j + h_n] \geq f[\bar{x}, \bar{x} + h_1, \dots, \bar{x} + h_n],$$

where  $\bar{x} = \frac{1}{P_m} \sum_{i=0}^m p_i x^i$ .

If we put  $n = 1$ ,  $h = x_1^i - x_0^i$  and  $x_i = x_0^i$ ,  $y_i = x_1^i$  for  $i = 0, \dots, m$  in above case we get means related to inequality (see [13]):

$$\frac{1}{P_m} \sum_{i=0}^m p_i f(x_i) - f\left(\frac{1}{P_m} \sum_{i=0}^m p_i x_i\right) \leq \frac{1}{P_m} \sum_{i=0}^m p_i f(y_i) - f\left(\frac{1}{P_m} \sum_{i=0}^m p_i y_i\right).$$

If in (2.3) we put  $n = 1$ ,  $x_0^i = 2a - x_i$ ,  $x_1^i = x_i$ , where  $i = 0, \dots, m$  and  $x_i \in [0, 2a]$ , we get means related to inequality (see [15] and [12]):

$$\sum_{i=0}^m a_i \frac{f(x_i) - f(2a - x_i)}{x_i - a} \leq \frac{f(\sum_{i=0}^m a_i x_i) - f(2a - \sum_{i=0}^m a_i x_i)}{\sum_{i=0}^m a_i x_i - a}.$$

#### 4.1. Shur means

Let  $\mathbf{x} = (x_0, \dots, x_n)$  and  $\mathbf{y} = (y_0, \dots, y_n)$  denote two real  $(n + 1)$ -tuples. We say that  $\mathbf{x}$  majorizes  $\mathbf{y}$  and put  $\mathbf{x} \succ \mathbf{y}$  if

$$\sum_{i=0}^k x_{[i]} \geq \sum_{i=0}^k y_{[i]} \quad \text{for } k = 0, 1, \dots, n - 1$$

and

$$\sum_{i=0}^n x_i = \sum_{i=0}^n y_i,$$

where

$$x_{[0]} \geq x_{[1]} \geq \dots \geq x_{[n]} \quad \text{and} \quad y_{[0]} \geq y_{[1]} \geq \dots \geq y_{[n]}$$

are the nonincreasing ordered components of  $\mathbf{x}$  and  $\mathbf{y}$ .

An important tool in the study of majorization is a theorem (see [6]) which says that for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\mathbf{x} \succ \mathbf{y}$  if and only if  $\mathbf{y} = \mathbf{xP}$  for some double stochastic matrix  $\mathbf{P}$ . A square matrix  $\mathbf{P}$  is said to be *stochastic* if its elements are all nonnegative and all row sums are one. If in addition to being stochastic, all column sums are one, the matrix is said to be *double stochastic*.

If we put  $m = n$ ,

$$\mathbf{P} = \begin{bmatrix} a_0 & a_n & a_{n-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & a_n & \dots & a_3 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 & a_n \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{bmatrix},$$

and

$$\begin{aligned} \mathbf{x}^0 &= (x_0, x_1, \dots, x_{n-1}, x_n), \\ \mathbf{x}^1 &= (x_1, x_2, \dots, x_n, x_0), \\ &\dots, \\ \mathbf{x}^{n-1} &= (x_{n-1}, x_n, \dots, x_{n-3}, x_{n-2}), \\ \mathbf{x}^n &= (x_n, x_0, \dots, x_{n-2}, x_{n-1}) \end{aligned}$$

we have

$$\begin{aligned} \mathbf{y} &= (y_0, y_1, \dots, y_n) \\ &= (x_0, x_1, \dots, x_n) \cdot \begin{bmatrix} a_0 & a_n & a_{n-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & a_n & \dots & a_3 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 & a_n \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{bmatrix} \\ &= \left( \sum_{i=0}^m a_i x_0^i, \dots, \sum_{i=0}^m a_i x_n^i \right) = \mathbf{aX}. \end{aligned}$$

Now, using Theorem 1 we get inequality (see [14]):

$$f[x_0, \dots, x_n] \geq f[y_0, \dots, y_n],$$

i.e. function  $G$  defined in Theorem 1 is Schur-convex. So, (2.3) in this case are Schur means (see [11]).

The following result (see [5]) is a consequence of the previous result:

**THEOREM 4.** *Let  $f^{(n)}$  be convex in  $(a, b)$ ,  $a \leq x_0 \leq \dots \leq x_n \leq b$ , and define  $\bar{x} = \frac{1}{n+1} \sum_{i=0}^n x_i$ . Then*

$$\frac{f^{(n)}(\bar{x})}{n!} = f[\underbrace{\bar{x}, \dots, \bar{x}}_{n+1 \text{ times}}] \leq f[x_0, \dots, x_n]. \tag{4.1}$$

*If  $x_0 \neq x_n$ , then equality in (4.1) holds iff  $f \in \Pi_{n+1}$  where  $\Pi_{n+1}$  denotes the class of polynomials of degree at most  $n + 1$ .*

For the Schur means related with above inequality we have the following:

$$E_{Sch}(\mathbf{x}; p, q) = \left\{ \begin{array}{l} \left( \prod_{i=0}^{n+1} \frac{q-i}{p-i} \frac{V_{xp0}}{V_{xn0}} \frac{\prod_{i=0}^{n-1} (p-i) \bar{x}^p}{n! \cdot \bar{x}^n} \right)^{\frac{1}{p-q}} \quad \text{for } p \neq q; p \notin N, q \notin N \\ \left( \prod_{i=0}^{n+1} \frac{(q-k)(q-i)}{k-i} \frac{V_{xk1}}{V_{xn0}} \frac{\prod_{i=0}^{n-1} (p-i) \bar{x}^p}{n! \cdot \bar{x}^n} - \frac{\sum_{j=0}^{n-1} \prod_{i=0}^{n-1} (k-i) \bar{x}^k + \prod_{i=0}^{n-1} (k-i) \bar{x}^k \ln \bar{x}}{i \neq j}}{V_{xn0} - \frac{\prod_{i=0}^{n-1} (q-i) \bar{x}^q}{n! \cdot \bar{x}^n}} \right)^{\frac{1}{k-q}} \quad \text{for } p \neq q; p = k \in N, q \notin N \\ \left( \prod_{i=0}^{n+1} \frac{l-i}{(p-l)(p-i)} \frac{V_{xp0}}{V_{xn0}} \frac{\prod_{i=0}^{n-1} (p-i) \bar{x}^p}{n! \cdot \bar{x}^n} \right)^{\frac{1}{p-l}} \quad \text{for } p \neq q; p \notin N, q = l \in N \\ \left( \prod_{i=0}^{n+1} \frac{l-i}{i \neq k, l} \frac{V_{xk1}}{V_{xn0}} \frac{\prod_{i=0}^{n-1} (p-i) \bar{x}^p}{n! \cdot \bar{x}^n} - \frac{\sum_{j=0}^{n-1} \prod_{i=0}^{n-1} (k-i) \bar{x}^k + \prod_{i=0}^{n-1} (k-i) \bar{x}^k \ln \bar{x}}{i \neq j}}{V_{xn0} - \frac{\prod_{i=0}^{n-1} (l-i) \bar{x}^l + \prod_{i=0}^{n-1} (l-i) \bar{x}^l \ln \bar{x}}{i \neq j}} \right)^{\frac{1}{k-l}} \quad \text{for } p \neq q; p = k \in N, q = l \in N \\ \exp \left( \sum_{i=0}^{n+1} \frac{1}{i-p} + \frac{V_{xp1}}{V_{xn0}} \frac{\prod_{i=0}^{n-1} (p-i) \bar{x}^p + \prod_{i=0}^{n-1} (p-i) \bar{x}^p \ln \bar{x}}{i \neq j} - \frac{n! \cdot \bar{x}^n}{V_{xp0} - \frac{\prod_{i=0}^{n-1} (p-i) \bar{x}^p}{n! \cdot \bar{x}^n}} \right) \quad \text{for } p = q; p \notin N, q \notin N \\ \exp \left( \sum_{i=0}^{n+1} \frac{1}{i-k} + \frac{1}{2} \frac{V_{xk2}}{V_{xn0}} \frac{\sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \prod_{i=0}^{n-1} (k-i) \bar{x}^k + 2 \sum_{j=0}^{n-1} \prod_{i=0}^{n-1} (k-i) \bar{x}^k \ln \bar{x} + \prod_{i=0}^{n-1} (k-i) \bar{x}^k \ln^2 \bar{x}}{j_2 \neq j_1, i \neq j_1, i \neq j_2}}{V_{xn0} - \frac{\prod_{i=0}^{n-1} (k-i) \bar{x}^k + \prod_{i=0}^{n-1} (k-i) \bar{x}^k \ln \bar{x}}{i \neq j}} \right) \quad \text{for } p = q; p = q = k \in N \end{array} \right.$$

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