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ON EXPONENTIAL CONVEXITY, JENSEN–STEFFENSEN–BOAS INEQUALITY, AND CAUCHY'S MEANS FOR SUPERQUADRATIC FUNCTIONS

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Abstract. In this paper we define new means of Cauchy's type using some recently obtained results that refine the Jensen-Steffensen-Boas inequality for convex and superquadratic functions [4],[5]. Applying so called *exp-convex method* established in [8],[9], we interpret results in the form of exponentially convex or (as a special case) logarithmically convex functions. We also present some related results which generalize results in [2].

1. Introduction

It is well known that for a convex function $\varphi : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, $\boldsymbol{x} = (x_1, \dots, x_n) \in I^n$ is any n-tuple and $\boldsymbol{a} = (a_1, \dots, a_n)$ any nonnegative n-tuple such that $A_n = \sum_{i=1}^n a_i > 0$, the Jensen inequality

$$\varphi\left(\frac{1}{A_n}\sum_{i=1}^n a_i x_i\right) \leqslant \frac{1}{A_n}\sum_{i=1}^n a_i \varphi(x_i) \tag{1.1}$$

holds (see [12, p. 43]).

The assumption "a non-negative n-tuple $\boldsymbol{a}$" can be relaxed at the expense of more restrictions on the n-tuple $\boldsymbol{x}$.

If $\boldsymbol{a}$ is a real n-tuple that satisfies

$$0 \leqslant A_j = \sum_{i=1}^j a_i \leqslant A_n, \quad j = 1, \dots, n, \quad A_n > 0, \tag{1.2}$$

and $\boldsymbol{x} \in I^n$ is any monotonic n-tuple, then (1.1) is still valid. Inequality (1.1) considered under conditions (1.2) is known as the Jensen-Steffensen inequality (see [12, p. 57]). In this case n-tuple $\boldsymbol{a}$ is called the Jensen-Steffensen coefficients.

The next integral variant of the Jensen-Steffensen inequality is given by R. P. Boas [11]. In the following we allways assume that $-\infty < \alpha < \beta < +\infty$.

THEOREM 1. (Jensen-Steffensen-Boas) *Let $f : [\alpha, \beta] \to (a, b)$ be a continuous and monotonic function, where $-\infty \leqslant a < b \leqslant +\infty$, and let $\varphi : (a, b) \to \mathbb{R}$ be a convex function. If $\lambda : [\alpha, \beta] \to \mathbb{R}$ is either continuous or of bounded variation satisfying*

$$\lambda(\alpha) \leqslant \lambda(x) \leqslant \lambda(\beta) \quad \textit{for all} \quad x \in [\alpha, \beta], \quad \lambda(\beta) - \lambda(\alpha) > 0, \tag{1.3}$$

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then

$$\varphi\left(\frac{\int_{\alpha}^{\beta} f(t)d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}\right) \leqslant \frac{\int_{\alpha}^{\beta} \varphi(f(t))\,d\lambda(t)}{\int_{\alpha}^{\beta} d\lambda(t)}.$$

We deal in this paper with inequalities related to superquadracity. Here is it's definition and some theorems related to it.

DEFINITION 1. [6, Definition 2.1] *A function* $\varphi : [0,b) \to \mathbb{R}$ *is superquadratic provided that for all* $0 \leqslant x < b$ *there exist a constant* $C(x) \in \mathbb{R}$ *such that*

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geqslant C(x)(y-x)$$

for all $0 \leqslant y < b$.

THEOREM 2. [6, Theorem 2.3] *The inequality*

$$\int \varphi(g(s))\mathrm{d}\mu \geqslant \varphi\left(\int g\mathrm{d}\mu\right) + \int \varphi\left(\left|g(s) - \int g\mathrm{d}\mu\right|\right)\mathrm{d}\mu \tag{1.4}$$

holds for all probability measures μ *and all non-negative* μ*-integrable functions* g *if and only if* φ *is superquadratic.*

The same inequality holds for the Jensen-Steffensen coefficients when φ' is superadditive, as proved in [5, Theorem 1] and [4, Theorem 1]. We quote the part of [5, Theorem 1] and [4, Theorem 1] that we use in the sequel.

THEOREM 3. [5, Theorem 1] *Let* $f : [\alpha,\beta] \to [0,b)$ *be continuous and monotonic and* $\lambda : [\alpha,\beta] \to \mathbb{R}$ *be either continuous or of bounded variation satisfying* (1.3). *Let* $\varphi : [0,b) \to \mathbb{R}$ *be continuously differentiable and* $\varphi' : [0,b) \to \mathbb{R}$ *be superadditive. If* $\varphi(0) \leqslant 0$, *then* φ *is superquadratic and*

$$\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} \varphi(f(t))\,\mathrm{d}\lambda(t) \geqslant \varphi\left(\bar{f}\right) + \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} \varphi\left(\left|f(t)-\bar{f}\right|\right)\mathrm{d}\lambda(t), \tag{1.5}$$

where $\bar{f} = \frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} f(t)\mathrm{d}\lambda(t)$.

THEOREM 4. [4, Theorem 1] *Let* $\varphi : [0,b) \to \mathbb{R}$ *be a continuously differentiable function and* $\varphi' : [0,b) \to \mathbb{R}$ *be superadditive. Let* $\boldsymbol{a}$ *be a real* n*-tuple satisfying* (1.2) *and* $\boldsymbol{x} \in [0,b)^n$ *be any monotonic* n*-tuple. If* $\varphi(0) \leqslant 0$, *then* φ *is superquadratic and*

$$\frac{1}{A_n}\sum_{i=1}^{n} a_i\varphi(x_i) \geqslant \varphi(\bar{x}) + \frac{1}{A_n}\sum_{i=1}^{n} a_i\varphi(|x_i - \bar{x}|), \tag{1.6}$$

where $\bar{x} = \frac{1}{A_n}\sum_{i=1}^{n} a_i x_i$.

REMARK 1. In the case when φ is strictly superquadratic and $x_i \neq 0,\ i = 1,..,n$, the inequality in (1.6) is strict unless one of the following two cases occurs:

$$(1)\text{ either } \bar{x} = x_1 \text{ or } \bar{x} = x_n, \tag{1.7}$$

(2) there exists $k \in \{3,...,n-2\}$ such that $\bar{x} = x_k$ and

$$\begin{cases} (\forall j \in \{1,...,k-1\}) & \left(A_j = 0 \vee x_j = x_{j-1}\right) \\ (\forall j \in \{k+1,...,n\}) & \left(\overline{A}_j = \sum_{i=j}^{n} a_i = 0 \vee x_j = x_{j-1}\right) \end{cases}$$

Specially, when φ is strictly superquadratic and $\boldsymbol{a} > 0$ the equality holds in (1.6) iff

$$x_1 = x_2 = \ldots = x_n \tag{1.8}$$

(see [7], [1]).

In the last section we use the next theorem together with Theorem 2 to prove some related results which generalize results in [2].

THEOREM 5. [10, Theorem 1] *Let* (Ω, A, μ) *be a measurable space with* $0 < \mu(\Omega) < \infty$ *and let* $\varphi : [0,b) \to \mathbb{R}$ *be a superquadratic function. If* $g : \Omega \to [a,b] \subseteq [0,b)$ *is such that* $g, \varphi \circ g \in L_1(\mu)$, *then with* $\bar{g} = \frac{1}{\mu(\Omega)} \int_\Omega g \mathrm{d}\mu$ *we have*

$$\begin{aligned} \frac{1}{\mu(\Omega)} \int_\Omega \varphi(g) \mathrm{d}\mu \leqslant \frac{b - \bar{g}}{b-a} \varphi(a) + \frac{\bar{g} - a}{b-a} \varphi(b) \\ - \frac{1}{\mu(\Omega)} \frac{1}{b-a} \int_\Omega \left[(b-g)\varphi(g-a) + (g-a)\varphi(b-g)\right] \mathrm{d}\mu. \end{aligned} \tag{1.9}$$

Now, we quote the definitions of exponential and logarithmic convexity and some propositions which can be found in [8] and [12], and that we use later.

DEFINITION 2. A function $\varphi : (a,b) \to \mathbb{R}$ is said to be exponentially convex if it is continuous and

$$\sum_{i,j=1}^{m} u_i u_j \varphi(x_i + x_j) \geqslant 0$$

holds for all $m \in \mathbb{N}$ and all choices $u_i \in \mathbb{R}, i = 1, 2, ..., m$ and $x_i \in (a,b)$ such that $x_i + x_j \in (a,b),\ 1 \leqslant i, j \leqslant m$.

DEFINITION 3. A function $\varphi : (a,b) \to \mathbb{R}_+$ is said to be logarithmically convex or log-convex if the function $\log \varphi$ is convex, or equivalently, if

$$\varphi\left((1-\lambda)x + \lambda y\right) \leqslant \varphi(x)^{1-\lambda} \varphi(y)^{\lambda}$$

holds for all $x, y \in (a,b), \lambda \in [0,1]$.

LEMMA 1. *Let* $\varphi : (a,b) \to \mathbb{R}_+$ *be a log-convex function. Then for any* $x_1,\ x_2,\ y_1,\ y_2 \in (a,b)$ *such that* $x_1 \leqslant y_1,\ x_2 \leqslant y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$ *the following is valid*

$$\left(\frac{\varphi(x_2)}{\varphi(x_1)}\right)^{\frac{1}{x_2 - x_1}} \leqslant \left(\frac{\varphi(y_2)}{\varphi(y_1)}\right)^{\frac{1}{y_2 - y_1}}.$$

PROPOSITION 1. *Let $\varphi : (a,b) \to \mathbb{R}$ be a function. The following propositions are equivalent:*

(i) *φ is exponentially convex.*

(ii) *φ is continuous and*

$$\sum_{i,j=1}^{m} u_i u_j \varphi\left(\frac{x_i+x_j}{2}\right) \geqslant 0$$

holds for all $m \in \mathbb{N}$ and all choices $u_i \in \mathbb{R}$ and every $x_i, x_j \in (a,b)$, $1 \leqslant i,j \leqslant m$.

COROLLARY 1. *If a function $\varphi : (a,b) \to \mathbb{R}_+$ is exponentially convex then φ is also log-convex.*

Although there are some similarities with the results in [3], this paper deals with the Jensen-Steffensen and the Jensen-Steffensen-Boas type cases and not as in [3] with the Jensen type cases.

2. Exp-convex method for superquadratic functions

Throughout the paper we denote with e_i, $i \in \mathbb{N}$, the function $\mathrm{e}_i : [0,b) \to \mathbb{R}$ defined by $\mathrm{e}_i(t) = t^i$.

Let L be a linear class of continuous functions $\varphi : [0,b) \to \mathbb{R}$. Let $f : [\alpha,\beta] \to (0,b)$ be continuous and monotonic and $\lambda : [\alpha,\beta] \to \mathbb{R}$ be either continuous or of bounded variation satisfying (1.3). We define the functional χ on L by

$$\chi(\varphi) = \frac{1}{\lambda(\beta)-\lambda(\alpha)} \int_{\alpha}^{\beta} \left[\varphi(f(t)) - \varphi\left(\left|f(t)-\bar{f}\right|\right)\right] \mathrm{d}\lambda(t) - \varphi(\bar{f}), \tag{2.1}$$

where $\bar{f} = \frac{1}{\lambda(\beta)-\lambda(\alpha)} \int_{\alpha}^{\beta} f(t)\mathrm{d}\lambda(t)$.

In the discrete case we define the functional Δ on L by

$$\Delta(\varphi) = \frac{1}{A_n} \sum_{i=1}^{n} a_i \left[\varphi(x_i) - \varphi(|x_i - \bar{x}|)\right] - \varphi(\bar{x}), \tag{2.2}$$

where $\boldsymbol{x} \in (0,b)^n$ is a monotonic n-tuple, $\boldsymbol{a}$ is a real n-tuple satisfying (1.2) and $\bar{x} = \frac{1}{A_n} \sum_{i=1}^{n} a_i x_i$.

We use the notation $\chi(\varphi) = \chi_\varphi$ and $\Delta(\varphi) = \Delta_\varphi$.

REMARK 2. If $\varphi \in L$ is *differentiable,* $\varphi(0) \leqslant 0$ and φ' is superadditive, then φ is superquadratic and

(i) by Theorem 3 it follows that $\chi_\varphi \geqslant 0$.

(ii) by Theorem 4 it follows that $\Delta_\varphi \geqslant 0$.

Similary to the proof of [6, Lemma 3.1] is:

LEMMA 2. *Let $\varphi : [0,b) \to \mathbb{R}$ be continuously differentiable and $\varphi(0) \leqslant 0$. If $\frac{\varphi'}{e_1}$ is increasing, then φ' is superadditive and φ is superquadratic.*

In the next lemma we introduce a new family of superquadratic functions which we frequently use in the sequel.

LEMMA 3. [3, Lemma 5] *Let* $s \in \mathbb{R}_+$. *We define the function* $\psi_s : [0,b) \to \mathbb{R}$ *by*

$$\psi_s(x) = \begin{cases} \frac{sxe^{sx} - e^{sx} + 1}{s^3}, & s \neq 0 \\ \frac{x^3}{3}, & s = 0 \end{cases}. \tag{2.3}$$

Then ψ_s *is superquadratic.*

Proof. Since $\psi_s(0) = 0$ and $\left(\frac{\psi_s'(x)}{x}\right)' = e^{sx} > 0$, by Lemma 2 it follows that ψ_s is superquadratic.

Applying the functional χ to ψ_s we have

$$\chi_{\psi_s} = \begin{cases} \frac{1}{s^3}\left[\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left(\mathscr{R}_s(f(t)) - \mathscr{R}_s(|f(t)-\bar{f}|)\right) d\lambda(t) - \mathscr{R}_s(\bar{f})\right], & s \neq 0 \\ \frac{1}{3}\left[\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left(f^3(t) - |f(t)-\bar{f}|^3\right) d\lambda(t) - \bar{f}^3\right], & s = 0 \end{cases}, \tag{2.4}$$

where we denote

$$\mathscr{R}_s(x) = sxe^{sx} - e^{sx} + 1. \tag{2.5}$$

Analogously, applying the functional Δ to ψ_s we have

$$\Delta_{\psi_s} = \begin{cases} \frac{1}{s^3}\left[\frac{1}{A_n}\sum_{i=1}^n a_i\left(\mathscr{R}_s(x_i) - \mathscr{R}_s(|x_i - \bar{x}|)\right) - \mathscr{R}_s(\bar{x})\right], & s \neq 0 \\ \frac{1}{3}\left[\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^3 - |x_i - \bar{x}|^3\right) - \bar{x}^3\right], & s = 0 \end{cases}.$$

THEOREM 6. *Let* χ_{ψ_s} *be defined as in* (2.4). *Then*
a) the function $s \mapsto \chi_{\psi_s}$ *is exponentially convex.*
b) if $\chi_{\psi_s} > 0$, *the function* $s \mapsto \chi_{\psi_s}$ *is log-convex.*

Proof. *a)* We can easily prove that $\lim_{s\to 0}\chi_{\psi_s} = \chi_{\psi_0}$, i.e. $s \mapsto \chi_{\psi_s}$ is continuous. Let $u_i \in \mathbb{R},\ p_i \in \mathbb{R}_+,\ i = 1,\dots,m,$ and $p_{ij} = \frac{p_i + p_j}{2},\ 1 \leqslant i,j \leqslant m$.
We consider the function $F : (0,b) \to \mathbb{R}$ defined by

$$F(x) = \sum_{i,j=1}^m u_i u_j \psi_{p_{ij}}(x),$$

where $\psi_{p_{ij}}$ is defined as in (2.3).

Then

$$\left(\frac{F'(x)}{x}\right)' = \sum_{i,j=1}^m u_i u_j \left(\frac{\psi_{p_{ij}}'(x)}{x}\right)' = \sum_{i,j=1}^m u_i u_j e^{p_{ij}x} = \left(\sum_{i=1}^m u_i e^{\frac{p_i}{2}x}\right)^2 \geqslant 0$$

and $F(0) = 0$. Therefore, by Lemma 2 it follows that F' is superadditive and F is superquadratic.

Applying inequality (1.5) to F we have that

$$\chi_F = \sum_{i,j=1}^{m} u_i u_j \chi_{\psi_{p_{ij}}} \geqslant 0 \tag{2.6}$$

holds for all $m \in \mathbb{N}$ and all choices of $u_i \in \mathbb{R},\ p_i \in \mathbb{R}_+,\ 1 \leqslant i \leqslant m$.

Since $s \mapsto \chi_{\psi_s}$ is continuous and (2.6) holds, by Proposition 1 it follows that $s \mapsto \chi_{\psi_s}$ is exponentially convex function.

$b)$ Since $\chi_{\psi_s} > 0$, by Corollary 1 it follows that $s \mapsto \chi_{\psi_s}$ is also log-convex.

We introduce another family of superquadratic functions which we use in the sequel.

LEMMA 4. [3, Lemma 3] *Let $s \in \mathbb{R}_+$. We define the function $\phi_s : [0,b) \to \mathbb{R}$ by*

$$\phi_s(x) = \begin{cases} \frac{x^s}{s(s-2)}, & s \neq 2 \\ \frac{x^2}{2}\log x, & s = 2 \end{cases} \tag{2.7}$$

with the convention $0\log 0 := 0$. Then ϕ_s is superquadratic.

Applying the functional χ to ϕ_s we have

$$\chi_{\phi_s} = \begin{cases} \frac{1}{s(s-2)}\left[\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left(f^s(t) - |\mathscr{Q}|^s\right) \mathrm{d}\lambda(t) - \bar{f}^s\right], s \neq 2 \\ \frac{1}{2}\left[\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \left(f^2(t)\log f(t) - \mathscr{Q}^2 \log|\mathscr{Q}|\right) \mathrm{d}\lambda(t) - \bar{f}^2 \log \bar{f}\right], s = 2 \end{cases}, \tag{2.8}$$

where

$$\mathscr{Q} = f(t) - \bar{f}. \tag{2.9}$$

Similarily, if we apply the functional Δ to ϕ_s we have

$$\Delta_{\phi_s} = \begin{cases} \frac{1}{s(s-2)}\left[\frac{1}{A_n}\sum_{i=1}^n a_i \left(x_i^s - |\mathscr{D}|^s\right) - \bar{x}^s\right], & s \neq 2 \\ \frac{1}{2}\left[\frac{1}{A_n}\sum_{i=1}^n a_i \left(x_i^2 \log x_i - \mathscr{D}^2 \log|\mathscr{D}|\right) - \bar{x}^2 \log \bar{x}\right], & s = 2 \end{cases},$$

where

$$\mathscr{D} = x_i - \bar{x}. \tag{2.10}$$

The next theorem can be proved in a similar way as Theorem 6.

THEOREM 7. *Let χ_{ϕ_s} be defined as in (2.8). Then*
$a)$ the function $s \mapsto \chi_{\phi_s}$ is exponentially convex.
$b)$ if $\chi_{\phi_s} > 0$, the function $s \mapsto \chi_{\phi_s}$ is log-convex.

THEOREM 8. *Theorems 6 and 7 are still valid if instead of χ_{ψ_s} and χ_{ϕ_s} we choose Δ_{ψ_s} and Δ_{ϕ_s}, respectively.*

3. Mean value theorems and Cauchy's means

In this section we present Lagrange's and Cauchy's type of Mean value theorem and introduce new means of Cauchy's type. The next two theorems are special cases of theorems in [3]. We state the proofs as they are somewhat different than those in [3]. We denote $I_1 = \left[\min_{\alpha\leqslant t\leqslant\beta} f(t), \max_{\alpha\leqslant t\leqslant\beta} f(t)\right] \subset (0,b)$.

THEOREM 9. *Let χ be the functional on L defined by (2.1) and suppose that $\chi_{e_3} \neq 0$. If $\varphi \in L$ is such that $\varphi(0) = 0$ and $\frac{\varphi'}{e_1} \in C^1(I_1)$, then there exists $\xi \in I_1$ such that*

$$\chi_\varphi = \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\chi_{e_3}. \tag{3.1}$$

Proof. Since $\frac{\varphi'}{e_1} \in C^1(I_1)$, there exist $m = \min_{x\in I_1}\left(\frac{\varphi'(x)}{x}\right)'$ and $M = \max_{x\in I_1}\left(\frac{\varphi'(x)}{x}\right)'$ such that $m \leqslant \left(\frac{\varphi'(x)}{x}\right)' = \frac{x\varphi''(x)-\varphi'(x)}{x^2} \leqslant M$ for each $x \in I_1$.

We consider the functions $\varphi_1, \varphi_2 \in L$ defined as $\varphi_1 = \frac{M}{3}e_3 - \varphi$ and $\varphi_2 = \varphi - \frac{m}{3}e_3$.
Then $\varphi_1(0) = \varphi_2(0) = 0$ and $\frac{\varphi_1'}{e_1}, \frac{\varphi_2'}{e_1} \in C^1(I_1)$.
Also $\left(\frac{\varphi_1'(x)}{x}\right)', \left(\frac{\varphi_2'(x)}{x}\right)' \geqslant 0$, i.e., the functions $\frac{\varphi_1'}{e_1}, \frac{\varphi_2'}{e_1}$ are increasing on I_1.
Then by Lemma 2, φ_1', φ_2' are superadditive and φ_1, φ_2 are superquadratic on I_1.
Applying inequality (1.5) to φ_1 and φ_2 we have

$$0 \leqslant \tfrac{M}{3}\chi_{e_3} - \chi_\varphi \quad \text{and} \quad \chi_\varphi - \tfrac{m}{3}\chi_{e_3} \geqslant 0.$$

Since $\chi_{e_3} \neq 0$, by combining the above two inequalities we get

$$m \leqslant 3\frac{\chi_\varphi}{\chi_{e_3}} \leqslant M.$$

Then there exists $\xi \in I_1$ such that $\left(\frac{\varphi'(\xi)}{\xi}\right)' = \frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2} = 3\frac{\chi_\varphi}{\chi_{e_3}}$.

THEOREM 10. *Let χ be the functional on L defined by (2.1) and suppose that $\chi_{e_3} \neq 0$. If $\varphi, \psi \in L$ are such that $\varphi(0) = \psi(0) = 0$ and $\frac{\varphi'}{e_1}, \frac{\psi'}{e_1} \in C^1(I_1)$, then there exists $\xi \in I_1$ such that*

$$\chi_\psi\left(\xi\varphi''(\xi) - \varphi'(\xi)\right) = \chi_\varphi\left(\xi\psi''(\xi) - \psi'(\xi)\right). \tag{3.2}$$

Proof. We consider the function $k \in L$ defined as $k = \chi_\psi\varphi - \chi_\varphi\psi$.
Since $k(0) = 0$ and $\frac{k'}{e_1} \in C^1(I_1)$, we can apply Theorem 9 and we get

$$\chi_k = \tfrac{1}{3}\frac{\xi k''(\xi)-k'(\xi)}{\xi^2}\chi_{e_3}.$$

Therefore, since $\chi_k = 0$, we have $\frac{\xi k''(\xi)-k'(\xi)}{\xi^2} = 0$, i.e.

$$\frac{\chi_\psi(\xi\varphi''(\xi)-\varphi'(\xi))}{\xi^2} - \frac{\chi_\varphi(\xi\psi''(\xi)-\psi'(\xi))}{\xi^2} = 0$$

from which it follows (3.2).

Theorem 10 enables us to define new means. If we choose $\varphi = \phi_s$ and $\psi = \phi_r$, where $r,s \in \mathbb{R}_+$, $r \neq s$, $r,s \neq 2$, then from (3.2) it follows $\chi_{\phi_r}\xi^{s-3} = \chi_{\phi_s}\xi^{r-3}$, i.e. $\xi = \left(\frac{\chi_{\phi_s}}{\chi_{\phi_r}}\right)^{\frac{1}{s-r}}$ and we have

$$\min_{\alpha\leqslant t\leqslant\beta} f(t) \leqslant \left(\frac{\chi_{\phi_s}}{\chi_{\phi_r}}\right)^{\frac{1}{s-r}} \leqslant \max_{\alpha\leqslant t\leqslant\beta} f(t).$$

We denote

$$M_{s,r}(f;\lambda) = \left(\frac{\chi_{\phi_s}}{\chi_{\phi_r}}\right)^{\frac{1}{s-r}}.$$

For $r,s \in \mathbb{R}_+$ we can extend this mean to the excluded cases as follows:

$$M_{s,r}(f;\lambda) = \left(\frac{r(r-2)\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta (f^s(t)-|\mathscr{Q}|^s)\mathrm{d}\lambda(t)-\overline{f}^s\right)}{s(s-2)\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta (f^r(t)-|\mathscr{Q}|^r)\mathrm{d}\lambda(t)-\overline{f}^r\right)}\right)^{\frac{1}{s-r}}, \quad r\neq s,\ r,s\neq 2,$$

$$M_{r,r}(f;\lambda) = \exp\left(\frac{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta (f^r(t)\log f(t)-|\mathscr{Q}|^r\log|\mathscr{Q}|)\mathrm{d}\lambda(t)-\overline{f}^r\log\overline{f}}{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta (f^r(t)-|\mathscr{Q}|^r)\mathrm{d}\lambda(t)-\overline{f}^r} - \frac{2r-2}{r(r-2)}\right), r\neq 2,$$

$$M_{2,2}(f;\lambda) = \exp\left(\frac{\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta (f^2(t)\log^2 f(t)-\mathscr{Q}^2\log^2|\mathscr{Q}|)\mathrm{d}\lambda(t)-\overline{f}^2\log^2\overline{f}}{2\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta (f^2(t)\log f(t)-\mathscr{Q}^2\log|\mathscr{Q}|)\mathrm{d}\lambda(t)-\overline{f}^2\log\overline{f}\right)} - \frac{1}{2}\right),$$

where $\mathscr{Q}$ is as in (2.9).

We can easily check that these means are symmetric and the special cases are limits of the general case, i.e.

$$M_{r,r}(f;\lambda) = \lim_{s\to r} M_{s,r}(f;\lambda),$$
$$M_{2,2}(f;\lambda) = \lim_{r\to 2} M_{r,r}(f;\lambda).$$

Now we prove the monotonicity of these means.

THEOREM 11. *Let $r,s,u,v \in \mathbb{R}_+$ such that $r \leqslant u$, $s \leqslant v$. Then*

$$M_{s,r}(f;\lambda) \leqslant M_{v,u}(f;\lambda). \tag{3.3}$$

Proof. By Theorem 7 it follows that the function $s \mapsto \chi_{\phi_s}$ is log-convex. Then by Lemma 1, for any $r,s,u,v \in \mathbb{R}_+$, such that $r \leqslant u$, $s \leqslant v$, $r \neq s$, $u \neq v$, we have

$$\left(\frac{\chi_{\phi_s}}{\chi_{\phi_r}}\right)^{\frac{1}{s-r}} \leqslant \left(\frac{\chi_{\phi_v}}{\chi_{\phi_u}}\right)^{\frac{1}{v-u}}$$

which is equivalent to (3.3). For $r = s$ and $u = v$ we consider limiting case.

In the same way we can derive discrete cases of the previous means. For $r,s \in \mathbb{R}_+$, $r \neq s,\ r,s \neq 2,$ we define $\xi = \left(\frac{\Delta_{\phi_s}}{\Delta_{\phi_r}}\right)^{\frac{1}{s-r}}$ and we have

$$\min_{i \in \{1,\dots,n\}} \{x_i\} \leqslant \left(\frac{\Delta_{\phi_s}}{\Delta_{\phi_r}}\right)^{\frac{1}{s-r}} \leqslant \max_{i \in \{1,\dots,n\}} \{x_i\}.$$

We use notation

$$M_{s,r}(\boldsymbol{x};\boldsymbol{a}) = \left(\frac{\Delta_{\phi_s}}{\Delta_{\phi_r}}\right)^{\frac{1}{s-r}}.$$

We can extend this mean in other cases. For $r,s \in \mathbb{R}_+$ we define:

$$M_{s,r}(\boldsymbol{x};\boldsymbol{a}) = \left(\frac{r(r-2)\left(\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^s - |\mathscr{D}|^s\right) - \bar{x}^s\right)}{s(s-2)\left(\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^r - |\mathscr{D}|^r\right) - \bar{x}^r\right)}\right)^{\frac{1}{s-r}}, r \neq s,\ r,s \neq 2,$$

$$M_{r,r}(\boldsymbol{x};\boldsymbol{a}) = \exp\left(\frac{\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^r \log x_i - |\mathscr{D}|^r \log|\mathscr{D}|\right) - \bar{x}^r \log \bar{x}}{\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^r - |\mathscr{D}|^r\right) - \bar{x}^r} - \frac{2r-2}{r(r-2)}\right), r \neq 2,$$

$$M_{2,2}(\boldsymbol{x};\boldsymbol{a}) = \exp\left(\frac{\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^2 \log^2 x_i - \mathscr{D}^2 \log^2|\mathscr{D}|\right) - \bar{x}^2 \log^2 \bar{x}}{2\left(\frac{1}{A_n}\sum_{i=1}^n a_i\left(x_i^2 \log x_i - \mathscr{D}^2 \log|\mathscr{D}|\right) - \bar{x}^2 \log \bar{x}\right)} - \frac{1}{2}\right),$$

where $\mathscr{D}$ is as in (2.10).

We can easily check that these means are also symmetric and the special cases are limits of the general case. Similarily as before, in the next theorem we state the monotonicity of these means without proof.

THEOREM 12. *Let $r,s,u,v \in \mathbb{R}_+$ such that $r \leqslant u,\ s \leqslant v$. Then*

$$M_{s,r}(\boldsymbol{x};\boldsymbol{a}) \leqslant M_{v,u}(\boldsymbol{x};\boldsymbol{a}).$$

4. Related results

In this section we present some related results which generalize results in [2]. We use the same technique as used there.

Let $g : \Omega \to [a,b] \subseteq (0,b)$ be such that $g \in L_1(\mu)$ and $\bar{g} = \frac{1}{\mu(\Omega)}\int_\Omega g\,\mathrm{d}\mu$. Let $\widetilde{L}$ be a linear class of functions $\varphi : [0,b) \to \mathbb{R}$ such that $\varphi \circ g \in L_1(\mu)$. We define the functionals Γ and $\widetilde{\Gamma}$ on $\widetilde{L}$ by

$$\Gamma(\varphi) = \frac{1}{\mu(\Omega)}\int_\Omega \left(\varphi(g) - \varphi\left(|g-\bar{g}|\right)\right)\mathrm{d}\mu - \varphi(\bar{g}) \tag{4.1}$$

$$\widetilde{\Gamma}(\varphi) = \frac{b-\bar{g}}{b-a}\varphi(a) + \frac{\bar{g}-a}{b-a}\varphi(b) - \frac{1}{\mu(\Omega)}\int_\Omega \left[\varphi(g) + \frac{1}{b-a}\left((b-g)\varphi(g-a) + (g-a)\varphi(b-g)\right)\right]\mathrm{d}\mu. \tag{4.2}$$

We use the notation $\Gamma(\varphi) = \Gamma_\varphi$ and $\widetilde{\Gamma}(\varphi) = \widetilde{\Gamma}_\varphi$.

REMARK 3. If $\varphi \in \widetilde{L}$ is superquadratic, it is obvious that from (1.4) and (1.9) it follows that $\Gamma_\varphi, \widetilde{\Gamma}_\varphi \geqslant 0$.

If we suppose that $\psi_s \in \widetilde{L}$, where ψ_s is given by (2.3), then applying Γ and $\widetilde{\Gamma}$ we have

$$\Gamma_{\psi_s} = \begin{cases} \frac{1}{s^3}\left[\frac{1}{\mu(\Omega)}\int_\Omega \left(\mathscr{R}_s(g) - \mathscr{R}_s(|g-\bar{g}|)\right) \mathrm{d}\mu - \mathscr{R}_s(\bar{g})\right], s \neq 0 \\ \frac{1}{3}\left[\frac{1}{\mu(\Omega)}\int_\Omega \left(g^3 - |g-\bar{g}|^3\right) \mathrm{d}\mu - \bar{g}^3\right], s = 0 \end{cases}$$

$$\widetilde{\Gamma}_{\psi_s} = \begin{cases} \frac{1}{s^3}\left[\frac{\overline{\mathscr{B}}\mathscr{R}_s(a)+\overline{\mathscr{A}}\mathscr{R}_s(b)}{b-a} - \frac{1}{\mu(\Omega)}\int_\Omega \left(\mathscr{R}_s(g) + \frac{1}{b-a}\left[\mathscr{B}\mathscr{R}_s(\mathscr{A}) + \mathscr{A}\mathscr{R}_s(\mathscr{B})\right]\right) \mathrm{d}\mu\right], s \neq 0 \\ \frac{1}{3}\left[\frac{\overline{\mathscr{B}}a^3+\overline{\mathscr{A}}b^3}{b-a} - \frac{1}{\mu(\Omega)}\int_\Omega \left(g^3 + \frac{1}{b-a}\left[\mathscr{B}\mathscr{A}^3 + \mathscr{A}\mathscr{B}^3\right]\right) \mathrm{d}\mu\right], s = 0 \end{cases}$$

where $\mathscr{R}_s$ is defined as in (2.5) and

$$\mathscr{A} = g - a, \;\; \mathscr{B} = b - g, \;\; \overline{\mathscr{A}} = \bar{g} - a \;\text{ and }\; \overline{\mathscr{B}} = b - \bar{g}. \tag{4.3}$$

THEOREM 13. *Theorem 6 is still valid if instead of* χ_{ψ_s} *we choose* Γ_{ψ_s} *and* $\widetilde{\Gamma}_{\psi_s}$.

If we suppose that $\phi_s \in \widetilde{L}$, where ϕ_s is given by (2.7), then applying Γ and $\widetilde{\Gamma}$ we have

$$\Gamma_{\phi_s} = \begin{cases} \frac{1}{s(s-2)}\left[\frac{1}{\mu(\Omega)}\int_\Omega \left(g^s - (|g-\bar{g}|)^s\right) \mathrm{d}\mu - \bar{g}^s\right], s \neq 2 \\ \frac{1}{2}\left[\frac{1}{\mu(\Omega)}\int_\Omega \left(g^2 \log g - (g-\bar{g})^2 \log|g-\bar{g}|\right) \mathrm{d}\mu - \bar{g}^2 \log \bar{g}\right], s = 2 \end{cases}$$

$$\widetilde{\Gamma}_{\phi_s} = \begin{cases} \frac{1}{s(s-2)}\left[\frac{\overline{\mathscr{B}}a^s+\overline{\mathscr{A}}b^s}{b-a} - \frac{1}{\mu(\Omega)}\int_\Omega \left(g^s + \frac{1}{b-a}\left[\mathscr{B}\mathscr{A}^s + \mathscr{A}\mathscr{B}^s\right]\right) \mathrm{d}\mu\right], s \neq 2 \\ \frac{1}{2}\left[\frac{\overline{\mathscr{B}}a^2\log a+\overline{\mathscr{A}}b^2\log b}{b-a}\right. \\ \quad \left. - \frac{1}{\mu(\Omega)}\int_\Omega \left(g^2 \log g + \frac{1}{b-a}\left[\mathscr{B}\mathscr{A}^2 \log\mathscr{A} + \mathscr{A}\mathscr{B}^2 \log\mathscr{B}\right]\right) \mathrm{d}\mu\right], s = 2 \end{cases}$$

where $\mathscr{A}$, $\overline{\mathscr{A}}$, $\mathscr{B}$, $\overline{\mathscr{B}}$ are as in (4.3).

THEOREM 14. *Theorem 7 is still valid if instead of* χ_{ϕ_s} *we choose* Γ_{ϕ_s} *and* $\widetilde{\Gamma}_{\phi_s}$.

In a similar way as in Section 3 we derive new means of Cauchy's type. For $r,s \in \mathbb{R}_+$, $r \neq s$, $r,s \neq 2$, we define $\xi = \left(\frac{\Gamma_{\phi_s}}{\Gamma_{\phi_r}}\right)^{\frac{1}{s-r}}$ and $\widetilde{\xi} = \left(\frac{\widetilde{\Gamma}_{\phi_s}}{\widetilde{\Gamma}_{\phi_r}}\right)^{\frac{1}{s-r}}$, means on the segment $[a,b]$. We use the notation

$$M_{s,r}(g;\mu) = \left(\frac{\Gamma_{\phi_s}}{\Gamma_{\phi_r}}\right)^{\frac{1}{s-r}}$$

and

$$\widetilde{M}_{s,r}(g;\mu)=\left(\frac{\widetilde{\Gamma}_{\phi_s}}{\widetilde{\Gamma}_{\phi_r}}\right)^{\frac{1}{s-r}}.$$

We can extend these means in other cases. For $r,s\in\mathbb{R}_+$ we define:

$$M_{s,r}(g;\mu)=\left(\frac{r(r-2)\left(\frac{1}{\mu(\Omega)}\int_\Omega(g^s-|g-\bar{g}|^s)\mathrm{d}\mu-\bar{g}^s\right)}{s(s-2)\left(\frac{1}{\mu(\Omega)}\int_\Omega(g^r-|g-\bar{g}|^r)\mathrm{d}\mu-\bar{g}^r\right)}\right)^{\frac{1}{s-r}},\ r\neq s,\ r,s\neq 2,$$

$$M_{r,r}(g;\mu)=\exp\left(\frac{\frac{1}{\mu(\Omega)}\int_\Omega(g^r\log g-|g-\bar{g}|^r\log|g-\bar{g}|)\mathrm{d}\mu-\bar{g}^r\log\bar{g}}{\frac{1}{\mu(\Omega)}\int_\Omega(g^r-|g-\bar{g}|^r)\mathrm{d}\mu-\bar{g}^r}-\frac{2r-2}{r(r-2)}\right),\ r\neq 2,$$

$$M_{2,2}(g;\mu)=\exp\left(\frac{\frac{1}{\mu(\Omega)}\int_\Omega\left(g^2\log^2 g-(g-\bar{g})^2\log^2|g-\bar{g}|\right)\mathrm{d}\mu-\bar{g}^2\log^2\bar{g}}{2\left(\frac{1}{\mu(\Omega)}\int_\Omega\left(g^2\log g-(g-\bar{g})^2\log|g-\bar{g}|\right)\mathrm{d}\mu-\bar{g}^2\log\bar{g}\right)}-\frac{1}{2}\right),$$

and

$$\widetilde{M}_{s,r}(g;\mu)=\left(\frac{r(r-2)\left(\frac{\overline{\mathscr{B}}a^s+\overline{\mathscr{A}}b^s}{b-a}-\frac{1}{\mu(\Omega)}\int_\Omega\left(g^s+\frac{1}{b-a}[\mathscr{B}\mathscr{A}^s+\mathscr{A}\mathscr{B}^s]\right)\mathrm{d}\mu\right)}{s(s-2)\left(\frac{\overline{\mathscr{B}}a^r+\overline{\mathscr{A}}b^r}{b-a}-\frac{1}{\mu(\Omega)}\int_\Omega\left(g^r+\frac{1}{b-a}[\mathscr{B}\mathscr{A}^r+\mathscr{A}\mathscr{B}^r]\right)\mathrm{d}\mu\right)}\right)^{\frac{1}{s-r}},\ r\neq s,\ r,s\neq 2,$$

$$\widetilde{M}_{r,r}(g;\mu)=\exp\left(\frac{\frac{\overline{\mathscr{B}}a^r\log a+\overline{\mathscr{A}}b^r\log b}{b-a}-\frac{1}{\mu(\Omega)}\int_\Omega\left(g^r\log g+\frac{1}{b-a}[\mathscr{B}\mathscr{A}^r\log\mathscr{A}+\mathscr{A}\mathscr{B}^r\log\mathscr{B}]\right)\mathrm{d}\mu}{\frac{\overline{\mathscr{B}}a^r+\overline{\mathscr{A}}b^r}{b-a}-\frac{1}{\mu(\Omega)}\int_\Omega\left(g^r+\frac{1}{b-a}[\mathscr{B}\mathscr{A}^r+\mathscr{A}\mathscr{B}^r]\right)\mathrm{d}\mu}-\frac{2r-2}{r(r-2)}\right),$$
$$r\neq 2,$$

$$\widetilde{M}_{2,2}(g;\mu)=\exp\left(\frac{\frac{\overline{\mathscr{B}}a^2\log^2 a+\overline{\mathscr{A}}b^2\log^2 b}{b-a}-\frac{1}{\mu(\Omega)}\int_\Omega\left(g^2\log^2 g+\frac{1}{b-a}\left[\mathscr{B}\mathscr{A}^2\log^2\mathscr{A}+\mathscr{A}\mathscr{B}^2\log^2\mathscr{B}\right]\right)\mathrm{d}\mu}{2\left(\frac{\overline{\mathscr{B}}a^2\log a+\overline{\mathscr{A}}b^2\log b}{b-a}-\frac{1}{\mu(\Omega)}\int_\Omega\left(g^2\log g+\frac{1}{b-a}\left[\mathscr{B}\mathscr{A}^2\log\mathscr{A}+\mathscr{A}\mathscr{B}^2\log\mathscr{B}\right]\right)\mathrm{d}\mu\right)}-\frac{1}{2}\right),$$

where $\mathscr{A}$, $\overline{\mathscr{A}}$, $\mathscr{B}$, $\overline{\mathscr{B}}$ are as in (4.3).

These means are symmetric and the special cases are limits of the general case. They are also monotonic as we express it in the next theorem.

THEOREM 15. *Let $r,s,u,v\in\mathbb{R}_+$ such that $r\leqslant u,\ s\leqslant v$. Then*
(i) $M_{s,r}(g;\mu)\leqslant M_{v,u}(g;\mu)$.
(ii) $\widetilde{M}_{s,r}(g;\mu)\leqslant\widetilde{M}_{v,u}(g;\mu)$.

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