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GENERALIZATION OF K–DIVERGENCE AND RELATED MEANS

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Abstract. In this paper, we give a generalization of K-divergence measure and related results by using some log-convexity criteria. Also we give related Cauchy means and prove monotonicity of these means.

1. Introduction

For a given function ϕ defined on an interval $I \subseteq \mathbb{R}$ the ϕ entropy of $\mathbf{x} \in I^n$ is defined by:

$$H_{n,\phi}(\mathbf{x}) = -\sum_{i=1}^{n} \phi(x_i).$$

For two vectors $\mathbf{x}, \mathbf{y} \in I^n$, the Jensen difference is defined by:

$$J_{n,\phi}(\mathbf{x},\mathbf{y}) = H_{n,\phi}\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) - \frac{1}{2}\left[H_{n,\phi}(\mathbf{x}) + H_{n,\phi}(\mathbf{y})\right]. \tag{1}$$

Several divergence measures are defined in the statistical literature to reflect the fact that some probability distributions are closer together than other and, consequently, that it may be easier to distinguish between the distribution of one pair then between those of the other. An important measure of divergence is $J_{n,\phi}$ defined in (1) also known as J-divergence [3]. This divergence has some interesting properties, se for example [4], p. 16. For applications of this divergence see [6, 8, 9]. Another important measure of divergence is K-divergence introduced by Burbea and Rao [3]. This measure is defined by:

$$K_{n,\phi}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} (x_i - y_i)\left(\frac{\phi(x_i)}{x_i} - \frac{\phi(y_i)}{y_i}\right), \tag{2}$$

where function ϕ is on an interval I not containing zero such that the function defined by $x \to \frac{\phi(x)}{x}$ is increasing function and $\mathbf{x}, \mathbf{y} \in I^n$.

In [3] we have an order relation between two divergence measures J and K defined above that is the following result.

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THEOREM 1.1. *For any* $\boldsymbol{x},\boldsymbol{y} \in R^n_+$

$$4J_{n,\phi}(\boldsymbol{x},\boldsymbol{y}) \geqslant K_{n,\phi}(\boldsymbol{x},\boldsymbol{y}) \tag{3}$$

if and only if the function defined by $x \to \frac{\phi(x)}{x}$ *is convex. The equality occurs if and only if* $\boldsymbol{x} = \boldsymbol{y}$.

In this paper without loss of generality we assume that J and K are n-dimensional so we write J_ϕ and K_ϕ instead of $J_{n,\phi}$ and $K_{n,\phi}$.

The special case of K-divergences for the function (cf. [3])

$$\phi_\alpha : \mathbb{R}^+ \to \mathbb{R},\ \alpha > 0\ , \text{ defined by}$$

$$\phi_\alpha(x) = \begin{cases} (\alpha-1)^{-1}(x^\alpha - x), & \alpha \neq 1; \\ x\log x, & \alpha = 1, \end{cases} \tag{4}$$

is denoted by $\mathfrak{K}_{n,\alpha}$ with respective forms

$$\mathfrak{K}_{n,\alpha}(\mathbf{x},\mathbf{y}) = \begin{cases} (\alpha-1)^{-1}\sum_{i=1}^n (x_i - y_i)(x_i^{\alpha-1} - y_i^{\alpha-1}), & \alpha \neq 1; \\ \sum_{i=1}^n (x_i - y_i)(\log x_i - \log y_i), & \alpha = 1. \end{cases} \tag{5}$$

By using positivity of (2), the following results are given in [5].

THEOREM 1.2. $\mathfrak{K}_{n,p}$ *defined by* (5) *is log-convex, that is,*

$$[\mathfrak{K}_{n,p}(\boldsymbol{x},\boldsymbol{y})]^{r-s} \leqslant [\mathfrak{K}_{n,s}(\boldsymbol{x},\boldsymbol{y})]^{r-p}[\mathfrak{K}_{n,r}(\boldsymbol{x},\boldsymbol{y})]^{p-s} \text{ for } -\infty < r < s < p < \infty. \tag{6}$$

COROLLARY 1.3. *For* $p,r,s,t \in \mathbb{R}$ *such that* $r \leqslant s$, $p \leqslant t$ *with* $r \neq p$, $s \neq t$ *the following inequality holds*

$$\left(\frac{\mathfrak{K}_{n,p}(\boldsymbol{x},\boldsymbol{y})}{\mathfrak{K}_{n,r}(\boldsymbol{x},\boldsymbol{y})}\right)^{\frac{1}{p-r}} \leqslant \left(\frac{\mathfrak{K}_{n,t}(\boldsymbol{x},\boldsymbol{y})}{\mathfrak{K}_{n,s}(\boldsymbol{x},\boldsymbol{y})}\right)^{\frac{1}{t-s}}.$$

In this paper we give extensions of results from [5] as well as some corresponding mean value theorems. We also introduce corresponding Cauchy means and establish monotonicity of the Cauchy means.

2. Results related to relation between J and K

In this section results related to difference of J and K divergence are discussed. Firstly Mean value theorems and convexity after which the corresponding Cauchy means are given.

LEMMA 2.1. *Let* $f \in C^2(I)$, *where* I *is compact interval in* $\mathbb{R}$ *does not containing zero and* m,M *be such that*

$$m \leqslant \frac{x^2 f''(x) - 2xf'(x) + 2f(x)}{x^3} \leqslant M. \tag{7}$$

If the functions ϕ_1, ϕ_2 *are defined on* I *by:*

$$\phi_1(x) = M\frac{x^3}{2} - f(x),$$

$$\phi_2(x) = f(x) - m\frac{x^3}{2}$$

then the functions $x \to \frac{\phi_1(x)}{x}$ *and* $x \to \frac{\phi_2(x)}{x}$ *are convex.*

Proof. Since $\left(\frac{\phi_1(x)}{x}\right)'' = M - \frac{x^2 f''(x) - 2xf'(x) + 2f(x)}{x^3} \geqslant 0$, $\frac{\phi_1(x)}{x}$ is convex function. Similarly we have that $\frac{\phi_2(x)}{x}$ is convex function too.

DEFINITION 2.2. Let I be a real interval not containing zero and $\phi : I \to \mathbb{R}$ such that the function defined by $x \to \frac{\phi(x)}{x}$ is convex. We define D-divergence $D(\mathbf{x},\mathbf{y},\phi)$ by:

$$D(\mathbf{x},\mathbf{y},\phi) = 4J_\phi(\mathbf{x},\mathbf{y}) - K_\phi(\mathbf{x},\mathbf{y}) \tag{8}$$

where $\mathbf{x},\mathbf{y} \in I^n$.

THEOREM 2.3. *If* $f \in C^2(I)$, *where* I *is a compact interval in* $\mathbb{R}$ *not containing zero, then there exists* ξ *such that the following equality is valid*

$$D(\boldsymbol{x},\boldsymbol{y},f) = \frac{\xi^2 f''(\xi) - 2\xi f'(\xi) + 2f(\xi)}{\xi^3} \times \sum_{i=1}^{n} \left[(x_i^3 + y_i^3) - \frac{1}{4}(x_i - y_i)^3 - (x_i - y_i)\left(\frac{x_i^2}{2} - \frac{y_i^2}{2}\right) \right]. \tag{9}$$

Proof. Suppose $\min_{x\in I}\left(\frac{f(x)}{x}\right)'' = m$ and $\max_{x\in I}\left(\frac{f(x)}{x}\right)'' = M$. By using ϕ_1 instead of f in (3) we obtain

$$\sum_{i=1}^{n} 4\left\{\frac{1}{2}(\phi_1(x_i) + \phi_1(y_i)) - \phi_1\left(\frac{x_i + y_i}{2}\right)\right\} - (x_i - y_i)\left(\frac{\phi_1(x_i)}{x_i} - \frac{\phi_1(y_i)}{y_i}\right) > 0$$

for $\mathbf{x} \neq \mathbf{y}$ because $\frac{\phi_1(x)}{x}$ is strictly convex. Therefore we have

$$\sum_{i=1}^{n} 4\left\{\frac{1}{2}\left(M\frac{x_i^3}{2} + M\frac{y_i^3}{2} - f(x_i) - f(y_i)\right) - \frac{M}{2}\left(\frac{x_i + y_i}{2}\right)^3 f\left(\frac{x_i + y_i}{2}\right)\right\}$$
$$-(x_i - y_i)\left(\frac{M}{2}(x_i^2 - y_i^2) - \frac{f(x_i)}{x_i} + \frac{f(y_i)}{y_i}\right) > 0$$

which implies the inequality

$$D(\mathbf{x},\mathbf{y},f) \geqslant M\sum_{i=1}^{n} \left[(x_i^3 + y_i^3) - \frac{1}{4}(x_i - y_i)^3 - (x_i - y_i)\left(\frac{x_i^2}{2} - \frac{y_i^2}{2}\right) \right]. \tag{10}$$

Similarly, using ϕ_2 in (3) instead of f we get

$$D(\mathbf{x},\mathbf{y},f) \leqslant m\sum_{i=1}^{n}\left[(x_i^3+y_i^3)-\frac{1}{4}(x_i-y_i)^3-(x_i-y_i)\left(\frac{x_i^2}{2}-\frac{y_i^2}{2}\right)\right]. \tag{11}$$

By combining inequalities (10) and (11) and using intermediate value theorem, that is for $m \leqslant \frac{x^2 f''(x)-2xf'(x)+2f(x)}{x^3} \leqslant M$ there exist $\xi \in I$ such that we get (9).

THEOREM 2.4. *Let $f,g \in C^2(I)$, where I is a compact interval in $\mathbb{R}$ not containing origin, then there exists ξ such that the following equality is valid*

$$\frac{D(\boldsymbol{x},\boldsymbol{y},f)}{D(\boldsymbol{x},\boldsymbol{y},g)} = \frac{\xi^2 f''(\xi)-2\xi f'(\xi)+2f(\xi)}{\xi^2 g''(\xi)-2\xi g'(\xi)+2g(\xi)} \tag{12}$$

provided the denominators are non zero.

Proof. The proof is similar to the proof of Theorem 3.3 in [2].

COROLLARY 2.5. *Let the function K defined by:*

$$K(\xi) = \frac{D(\boldsymbol{x},\boldsymbol{y},f)}{D(\boldsymbol{x},\boldsymbol{y},g)} = \frac{\xi^2 f''(\xi)-2\xi f'(\xi)+2f(\xi)}{\xi^2 g''(\xi)-2\xi g'(\xi)+2g(\xi)} \tag{13}$$

If K is invertible then

$$\xi = K^{-1}\left(\frac{D(\boldsymbol{x},\boldsymbol{y},f)}{D(\boldsymbol{x},\boldsymbol{y},g)}\right) \tag{14}$$

is a new mean provided that the denominator is non zero.

Proof. Since $\xi \in I$ and $\min x_i \leqslant \xi \leqslant \max x_i$ (14)implies

$$\min x_i \leqslant K^{-1}\left(\frac{D(\mathbf{x},\mathbf{y},f)}{D(\mathbf{x},\mathbf{y},g)}\right) \leqslant \max x_i\,.$$

LEMMA 2.6. *Let the function ψ_p defined by:*

$$\psi_p(x) = \begin{cases} \frac{x^{p+1}}{p(p-1)} & p\neq 0,1 \\ x^2\log x & p=1 \\ -x\log x & p=0. \end{cases} \tag{15}$$

Then the function defined by $x \to \frac{\psi_p(x)}{x}$ is convex for $x > 0$.

An important corollary of Theorem 2.4 is:

COROLLARY 2.7. *Let the function ψ_p be as defined above and $\frac{D(\boldsymbol{x},\boldsymbol{y},\psi_p)}{D(\boldsymbol{x},\boldsymbol{y},\psi_r)} = \xi^{p-r}$ for some $\xi \in I$ and $p \neq r$ such that $D(\boldsymbol{x},\boldsymbol{y},\psi_s) \neq 0$, $s = p,r$. Then we have another mean*

$$\xi = \left(\frac{D(\boldsymbol{x},\boldsymbol{y},\psi_p)}{D(\boldsymbol{x},\boldsymbol{y},\psi_r)}\right)^{\frac{1}{p-r}}.$$

In the following results exponential convexity and log-convexity of this new divergence measure D is discussed. First we give a definition and some results which will be needed.

DEFINITION 2.8. A function $f:(a,b)\to\mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} a_i a_j f(x_i+x_j) \geqslant 0 \tag{16}$$

for all $n\in\mathbb{N}$ and all choices $a_i\in\mathbb{R}, i=1,..,n$ such that $x_i+x_j\in(a,b)$, $1\leqslant i,j\leqslant n$.

For exponentially convex function $f:(a,b)\to\mathbb{R}$, (16) is valid for all $x_i+x_j\in(a,b)$, $1\leqslant i,j\leqslant n$ and it is also valid for all $\frac{x_i+x_j}{2}\in(a,b)$, $1\leqslant i,j\leqslant n$ that is

$$\sum_{i,j=1}^{n} a_i a_j f\left(\frac{x_i+x_j}{2}\right) \geqslant 0, \tag{17}$$

holds for every $a_i\in\mathbb{R}$ and every $x_i\in(a,b), 1\leqslant i\leqslant n$.

If f is exponentially convex then

$$\det\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0 \tag{18}$$

for every $n\in\mathbb{N}$, $x_i\in(a,b), i=1,..,n$.

THEOREM 2.9. *Let $D(\boldsymbol{x},\boldsymbol{y},\psi_p)$ be as defined above. Then:*

1. *The matrix $\left[D(\boldsymbol{x},\boldsymbol{y},\psi_{\frac{p_i+p_j}{2}})\right]_{i,j=1}^{n}$ is a positive semi definite matrix.*
2. *The function $t\mapsto D(\boldsymbol{x},\boldsymbol{y},\psi_t)$ is exponentially convex.*
3. *If $D(\boldsymbol{x},\boldsymbol{y},\psi_p)$ is positive then it is log-convex, that is*

$$[D(\boldsymbol{x},\boldsymbol{y},\psi_p)]^{r-s} \leqslant [D(\boldsymbol{x},\boldsymbol{y},\psi_r)]^{p-s}[D(\boldsymbol{x},\boldsymbol{y},\psi_s)]^{r-p}$$

for $-\infty<r<s<p<\infty$.

Proof.

1. Consider the function f defined by:

$$f(x) = \sum_{i,j=1}^{k} u_i u_j \psi_{t_{ij}}(x) \tag{A}$$

for $k = 1,..,n, x > 0, u_i \in \mathbb{R}$ and $t_{ij} = \frac{p_i+p_j}{2}$. Then

$$\frac{f(x)}{x} = \sum_{i,j=1}^{k} u_i u_j \frac{\psi_{t_{ij}}(x)}{x}$$

$$\left(\frac{f(x)}{x}\right)'' = \sum_{i,j=1}^{k} u_i u_j x^{t_{ij}-2} = \left(\sum_{i=1}^{n} u_i x^{\frac{p_i-1}{2}}\right)^2 \geqslant 0$$

that is the function defined by $x \to \frac{\psi_p(x)}{x}$ is convex. Using f defined as in (8) we obtain

$$\sum_{i=1}^{n} 4\left\{\frac{1}{2}(f(x_i)+f(y_i)) - f(\frac{x_i+y_i}{2})\right\} - (x_i - y_i)\left(\frac{f(x_i)}{x_i} - \frac{f(y_i)}{y_i}\right) \geqslant 0.$$

Now by substituting f as defined in (A) we obtain

$$\sum_{i,j=1}^{n} u_i u_j D(\mathbf{x}, \mathbf{y}, \psi_{t_{ij}}) \geqslant 0 \tag{19}$$

which implies positive semi definiteness.

2. We have $\lim_{p\to 0} D(\mathbf{x}, \mathbf{y}, \psi_p) = D(\mathbf{x}, \mathbf{y}, \psi_0)$, $\lim_{p\to 1} D(\mathbf{x}, \mathbf{y}, \psi_p) = D(\mathbf{x}, \mathbf{y}, \psi_1)$. This implies $D(\mathbf{x}, \mathbf{y}, \psi_p)$ is continuous for all p. Therefore by (17) we have exponential convexity of the function $D(\mathbf{x}, \mathbf{y}, \psi_p)$.

3. Now for $n = 2$, $\mathbf{D}(p) := D(\mathbf{x}, \mathbf{y}, \psi_p)$ is log-convex that is for $-\infty < r < s < p < \infty$ the following is valid

$$[D(\mathbf{x}, \mathbf{y}, \psi_p)]^{r-s} \leqslant [D(\mathbf{x}, \mathbf{y}, \psi_r)]^{p-s}[D(\mathbf{x}, \mathbf{y}, \psi_s)]^{r-p}.$$

DEFINITION 2.10. For $r, s \in \mathbb{R}$ we define generalized mean $M_{s,r}$ by

$$M_{s,r} = \left(\frac{D(\mathbf{x},\mathbf{y}, \psi_s)}{D(\mathbf{x},\mathbf{y}, \psi_r)}\right)^{\frac{1}{s-r}} \quad s \neq r \tag{20}$$

where $D(\mathbf{x},\mathbf{y},\psi_p) \neq 0$ for $p = r, s$.

Other cases can be obtained by taking limits as:

$$M_{s,0} = \left(\frac{-\sum_{i=1}^{n}\{4(\frac{x_i+y_i}{2})^{s+1} - 2(x_i^{s+1}+y_i^{s+1}) + (x_i-y_i)(x_i^s - y_i^s)\}}{s(s-1)\sum_{i=1}^{n}\{2(x_i+y_i)\log\frac{x_i+y_i}{2} - 2(x_i\log x_i + y_i\log y_i) + (x_i-y_i)(\log x_i - \log y_i)\}} \right)^{\frac{1}{s}}, \quad s \neq 0,1$$

$$M_{1,0} = \left(\frac{-\sum_{i=1}^{n}\{(x_i+y_i)^2\log(\frac{x_i+y_i}{2}) - 2(x_i^2\log x_i + y_i^2\log y_i) + (x_i-y_i)(x_i\log x_i - y_i\log y_i)\}}{\sum_{i=1}^{n}\{2(x_i+y_i)\log\frac{x_i+y_i}{2} - 2(x_i\log x_i + y_i\log y_i) + (x_i-y_i)(\log x_i - \log y_i)\}} \right),$$

$$M_{s,s} = \exp\left(\frac{\sum_{i=1}^{n}\{4(\frac{x_i+y_i}{2})^{s+1}\log(\frac{x_i+y_i}{2}) - 2(x_i^{s+1}\log x_i + y_i^{s+1}\log y_i) + (x_i-y_i)(x_i^s\log x_i - y_i^s\log y_i)\}}{\sum_{i=1}^{n}\{4(\frac{x_i+y_i}{2})^{s+1} - 2(x_i^{s+1}+y_i^{s+1}) + (x_i-y_i)(x_i^s - y_i^s)\}} - \frac{2s-1}{s(s-1)} \right),$$

$$s \neq 0,1$$

$$M_{0,0} = \exp\left(\frac{\sum_{i=1}^{n}\{2(x_i+y_i)\log^2(\frac{x_i+y_i}{2}) - 2(x_i\log^2 x_i + y_i\log^2 y_i) + (x_i-y_i)(\log^2 x_i - \log^2 y_i)\}}{2\sum_{i=1}^{n}\{2(x_i+y_i)\log(\frac{x_i+y_i}{2}) - 2(x_i\log x_i + y_i\log y_i) + (x_i-y_i)(\log x_i - \log y_i)\}} + 1 \right)$$

$$M_{1,1} = \exp\left(\frac{\sum_{i=1}^{n}\{(x_i+y_i)^2\log^2(\frac{x_i+y_i}{2}) - 2(x_i^2\log^2 x_i + y_i^2\log^2 y_i) + (x_i-y_i)(x_i\log^2 x_i - y_i\log^2 y_i)\}}{2\sum_{i=1}^{n}\{(x_i+y_i)^2\log(\frac{x_i+y_i}{2}) - 2(x_i^2\log x_i + y_i^2\log y_i) + (x_i-y_i)(x_i\log x_i - y_i\log y_i)\}} + 1 \right)$$

THEOREM 2.11. *For $p,r,s,t \in \mathbb{R}$ such that $r \leqslant s$ and $p \leqslant t$ the following inequality is valid*

$$M_{p,r} \leqslant M_{t,s}.$$

Proof. The proof is similar to the proof of Theorem 3.4 in [1].

3. Results related to K divergence

In this section we consider the following generalization of (2).

DEFINITION 3.1. For real valued functions h_i, $i = 1,..,n, f, h$ on some interval I such that h_i, $i = 1,..,n$ and $\frac{f}{h}$ are strictly increasing functions, we define T-divergence $T(\mathbf{x},\mathbf{y},f)$ by:

$$T(\mathbf{x},\mathbf{y},f) = \sum_{i=1}^{n} p_i(h_i(x_i) - h_i(y_i)) \left(\frac{f(x_i)}{h(x_i)} - \frac{f(y_i)}{h(y_i)} \right), \tag{21}$$

where $\mathbf{x},\mathbf{y} \in I^n$ and p_i are positive reals.

LEMMA 3.2. *Let $f,h \in C^1(I)$, where I is a compact interval in $\mathbb{R}$ does not contain zero, be such that the function defined by $x \to \frac{f}{h}(x)$ is in $C^1(I)$ and $\xi \in I$ be such that*

$$m \leqslant \frac{f'(\xi)h(\xi) - h'(\xi)f(\xi)}{h^2(\xi)} \leqslant M. \tag{22}$$

Let the functions ϕ_1, ϕ_2 defined by $\phi_1(x) = Mxh(x) - f(x)$, and $\phi_2(x) = f(x) - mxh(x)$, then the function defined by $x \to \frac{\phi_i}{h}(x)$ for $i = 1,2$ are increasing.

Proof. Since $\left(\frac{\phi_1(x)}{h(x)}\right)' = M - \frac{f'(x)h(x)-h'(x)f(x)}{h^2(x)} \geqslant 0$ therefore $\frac{\phi_1}{h}$ is increasing function. Similarly we have that $\frac{\phi_2}{h}$ is increasing function too.

By using this lemma we have the following result.

THEOREM 3.3. *If $\frac{f}{h} \in C^1(I)$, where I is a compact interval in $\mathbb{R}$ and $\frac{f}{h}$, h_i, $i=1,\ldots,n$ be strictly increasing functions then there exists ξ such that the following equality is valid*

$$T(\boldsymbol{x},\boldsymbol{y},f) = \frac{f'(\xi)h(\xi)-h'(\xi)f(\xi)}{h^2(\xi)} \sum_{i=1}^{n} p_i(h_i(x_i)-h_i(y_i))(x_i-y_i). \tag{23}$$

THEOREM 3.4. *Let $\frac{f}{h}$, $\frac{g}{h} \in C^1(I)$ and $\frac{f}{h}$, $\frac{g}{h}$, h_i, $i=1,..,n$ be strictly increasing. Then for $\boldsymbol{x} \neq \boldsymbol{y}$ there exists $\xi \in I$ such that the following equality is valid*

$$\frac{T(\boldsymbol{x},\boldsymbol{y},f)}{T(\boldsymbol{x},\boldsymbol{y},g)} = \frac{f'(\xi)h(\xi)-h'(\xi)f(\xi)}{g'(\xi)h(\xi)-h'(\xi)g(\xi)} \tag{24}$$

provided the denominators are not zero.

Proof. The proof is similar to the proof of Theorem 2.4.

COROLLARY 3.5. *Let the function $K(\xi)$ defined by*

$$K(\xi) = \frac{T(\boldsymbol{x},\boldsymbol{y},f)}{T(\boldsymbol{x},\boldsymbol{y},g)} = \frac{f'(\xi)h(\xi)-h'(\xi)f(\xi)}{g'(\xi)h(\xi)-h'(\xi)g(\xi)} \tag{25}$$

If K is invertible then

$$\xi = K^{-1}\left(\frac{T(\boldsymbol{x},\boldsymbol{y},f)}{T(\boldsymbol{x},\boldsymbol{y},g)}\right) \tag{26}$$

is a new mean provided that the denominator is non zero.

Proof. Since $\xi \in I$ and $\min x_i \leqslant \xi \leqslant \max x_i$, (26) implies

$$\min x_i \leqslant K^{-1}\left(\frac{T(\mathbf{x},\mathbf{y},f)}{T(\mathbf{x},\mathbf{y},g)}\right) \leqslant \max x_i\,.$$

LEMMA 3.6. *Consider the function*

$$\psi_p(x) = \begin{cases} \frac{x^p h(x)}{p} & p\neq 0 \\ h(x)\log x & p=0. \end{cases} \tag{27}$$

Then the function defined by $x \to \frac{\psi_p}{h}(x)$ is strictly increasing for $x>0$.

If we put $p=\alpha-1$ and $h_i(x)=x$ and denote $T(\mathbf{x},\mathbf{y},\psi_p)$ by $k_\alpha(\mathbf{x},\mathbf{y};\mathbf{p})$ then we have

$$k_\alpha(\mathbf{x},\mathbf{y};\mathbf{p}) = \begin{cases} \frac{1}{\alpha-1}\sum_{i=1}^{n} p_i(x_i-y_i)(x_i^{\alpha-1}-y_i^{\alpha-1}), & \alpha \neq 1, \\ \sum_{i=1}^{n} p_i(x_i-y_i)(\log x_i - \log y_i), & \alpha = 1. \end{cases}$$

REMARK 3.7. If we put $p_i = 1,\ i = 1,..,n$ in above equation. Then we get $\mathfrak{K}_\alpha(\mathbf{x},\mathbf{y})$ as defined above in (5).

Another corollary of Theorem 3.4 stated as.

COROLLARY 3.8. *Let the function ψ_p be as (27), and $\frac{T(\boldsymbol{x},\boldsymbol{y},\psi_p)}{T(\boldsymbol{x},\boldsymbol{y},\psi_r)} = \xi^{p-r}$ for some $\xi \in I$ and $p \neq r$ such that $T(\boldsymbol{x},\boldsymbol{y},\psi_s) \neq 0,\ s = p,r$ and so we have another mean*

$$\xi = \left(\frac{T(\boldsymbol{x},\boldsymbol{y},\psi_p)}{T(\boldsymbol{x},\boldsymbol{y},\psi_r)}\right)^{\frac{1}{p-r}}.$$

REMARK 3.9. If we put $p = \alpha - 1$ and $r = \beta - 1$ in Corollary 2.7 such that $k_\gamma(\mathbf{x},\mathbf{y};\mathbf{p}) \neq 0$, for $\gamma = \alpha,\beta$ then we have

$$\xi = \left(\frac{k_\alpha(\mathbf{x},\mathbf{y};\mathbf{p})}{k_\beta(\mathbf{x},\mathbf{y};\mathbf{p})}\right)^{\frac{1}{\alpha-\beta}}.$$

In the next result exponential convexity and log-convexity of this new divergence measure T is discussed.

THEOREM 3.10. *Let $T(\boldsymbol{x},\boldsymbol{y},\psi_p)$ be as defined above. Then: have*

1. *The matrix $\left[T(\boldsymbol{x},\boldsymbol{y},\psi_{\frac{p_i+p_j}{2}})\right]_{i,j=1}^{n}$ is a positive semi definite matrix.*
2. *The function $t \mapsto T(\boldsymbol{x},\boldsymbol{y},\psi_t)$ is exponentially convex.*
3. *If $T(\boldsymbol{x},\boldsymbol{y},\psi_p)$ is positive then it is log-convex, that is*

$$[T(\boldsymbol{x},\boldsymbol{y},\psi_p)]^{r-s} \leqslant [T(\boldsymbol{x},\boldsymbol{y},\psi_r)]^{p-s}[T(\boldsymbol{x},\boldsymbol{y},\psi_s)]^{r-p}$$

for $-\infty < r < s < p < \infty$.

Proof. The proof is similar to the proof of Theorem 2.9.

REMARK 3.11. If we put $p = \alpha - 1$, $r = \beta - 1$, $s = \gamma - 1$ in Theorem 3.14 (3) we get

$$[k_{n,\alpha}(\mathbf{x},\mathbf{y};\mathbf{p})]^{\beta-\gamma} \leqslant [k_{n,\beta}(\mathbf{x},\mathbf{y};\mathbf{p})]^{\alpha-\gamma}[k_{n,\gamma}(\mathbf{x},\mathbf{y};\mathbf{p})]^{\beta-\alpha}$$

which is an extension of (6).

DEFINITION 3.12. For $r,s \in \mathbb{R}$ we define generalized mean $M_{s,r}$ by

$$M_{s,r} = \left(\frac{T(\mathbf{x},\mathbf{y},\psi_s)}{T(\mathbf{x},\mathbf{y},\psi_r)}\right)^{\frac{1}{s-r}} \quad s \neq r \tag{28}$$

where $T(\mathbf{x},\mathbf{y},\psi_p) \neq 0$ for $p = r,s$.

Other cases can be obtained by taking limits as:

$$M_{s,s} = \exp\left(\frac{\sum_{i=1}^{n} p_i(h_i(x_i) - h_i(y_i))[s(x_i^s \log x_i - y_i^s \log y_i) - (x_i^s - y_i^s)]}{s\sum_{i=1}^{n} p_i(h_i(x_i) - h_i(y_i))(x_i^s - y_i^s)}\right), \ \ s \neq 0$$

$$M_{0,0} = \exp\left(\frac{\sum_{i=1}^{n} p_i(h_i(x_i) - h_i(y_i))((\log x_i)^2 - (\log y_i)^2)}{2\sum_{i=1}^{n} p_i(h_i(x_i) - h_i(y_i))(\log x_i - \log y_i)}\right).$$

REMARK 3.13. If we put $s = \alpha - 1$ and $r = \beta - 1$ and $h_i(x) = x$, $i = 1,..,n$ in (28) we get

$$\tilde{M}_{\alpha,\beta} =: M_{\alpha-1,\beta-1} = \left(\frac{k_\alpha(\mathbf{x,y;p})}{k_\beta(\mathbf{x,y;p})}\right)^{\frac{1}{\alpha-\beta}}$$

where $\alpha \neq \beta \neq 1$. For $\alpha, \beta \in \mathbb{R}$ we have

$$\tilde{M}_{\alpha,\beta} = \begin{cases} \left(\frac{k_{n,\alpha}(\mathbf{x,y;p})}{k_{n,\beta}(\mathbf{x,y;p})}\right)^{\frac{1}{\alpha-\beta}}, & \alpha \neq \beta; \\ \exp\left(\frac{\sum_{i=1}^{n} p_i(h_i(x_i)-h_i(y_i))[(\alpha-1)(x_i^{\alpha-1}\log x_i - y_i^{\alpha-1}\log y_i)-(x_i^{\alpha-1}-y_i^{\alpha-1})]}{(\alpha-1)\sum_{i=1}^{n} p_i(h_i(x_i)-h_i(y_i))(x_i^{\alpha-1}-y_i^{\alpha-1})}\right), & \alpha = \beta \neq 1; \\ \exp\left(\frac{\sum_{i=1}^{n} p_i(h_i(x_i)-h_i(y_i))((\log x_i)^2-(\log y_i)^2)}{2\sum_{i=1}^{n} p_i(h_i(x_i)-h_i(y_i))(\log x_i - \log y_i)}\right), & \alpha = 1; \end{cases}$$

THEOREM 3.14. *For $p, r, s, t \in R$ such that $r \leqslant s$ and $p \leqslant t$ we have*

$$M_{p,r} \leqslant M_{t,s}.$$

Proof. The proof is similar to the proof of Theorem 2.11.

REMARK 3.15. If we put $p = \alpha - 1$, $r = \beta - 1$, $s = \gamma - 1$, $t = \delta - 1$ in the above theorem we get

$$\tilde{M}_{\alpha,\beta} \leqslant \tilde{M}_{\delta,\gamma}$$

where $\alpha \neq \beta$, $\gamma \neq \delta$ and $\alpha \leqslant \delta$, $\beta \leqslant \gamma$

REMARK 3.16. If in our results we substitute $h_i(x) = x$ and $h_i(y) = y$ then we get results proved in [5].

REMARK 3.17. Integral version of generalized K-divergence can be defined as:

$$K_{\phi,n} = \int (h_1(x(s),s) - h_1(y(s),s))\left(\frac{f}{h}(x(s)) - \frac{f}{h}(y(s)\right) d\mu(s) \tag{29}$$

where x, y are real valued functions such that h_1 is strictly increasing in the first variable and $\frac{f}{h}$ is strictly increasing. Similar results can also be obtained for integral version.

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