INEQUALITIES FOR $n$–CONVEX FUNCTIONS

ILKO BRNETIĆ

Abstract. In this article new inequalities for $n$-convex functions are stated and proved and some applications of these results are given.

1. Introduction

The function $f$ is called $n$-convex on the interval $(a, b)$ if its $n$-th derivative $f^{(n)}(t)$ is positive for all $t \in (a, b)$. Using this terminology, convex function is called 2-convex function.

In [1] some results for convex and 3-convex functions (with applications to log-convex and 3-log convex functions) are obtained.

The aim of this article is to establish some basic results for $n$-convex functions which can be easily used for obtaining many other results.

2. Main results

Let’s state and prove the main result.

THEOREM 1. Let $f$ be $(n+1)$-convex function on $[a,b]$. Then, for each $x \in (a,b)$, the following inequalities hold

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \leq f(x) \leq \frac{f^{(n)}(b)}{n!}(x-a)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k. \quad (1)$$

If $n$ is odd, then

$$\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \leq f(x) \leq \frac{f^{(n)}(a)}{n!}(x-b)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k, \quad (2)$$

and if $n$ is even, it holds:

$$\frac{f^{(n)}(a)}{n!}(x-b)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \leq f(x) \leq \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k. \quad (3)$$


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\textbf{Proof.} Let us recall on Taylor’s formula:
\begin{equation}
 f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{f^{(n)}(c)}{n!} (x-a)^n,
\end{equation}
for some \( c \in (a,x) \) and similarly
\begin{equation}
 f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{f^{(n)}(c)}{n!} (x-b)^n
\end{equation}
for some \( c \in (x,b) \).

If \( f^{(n+1)} \) is convex on \([a,b] \), then \( f^{(n)} \) is increasing on \([a,b] \), i.e. \( f^{(n)}(a) \leq f^{(n)}(t) \leq f^{(n)}(b) \), for each \( t \in (a,b) \).

So, from (4), for each \( x \in (a,b) \), we have
\[
\frac{f^{(n)}(a)}{n!} (x-a)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \leq \frac{f^{(n)}(b)}{n!} (x-a)^n,
\]
or equivalently
\[
\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k \leq f(x) \leq \frac{f^{(n)}(b)}{n!} (x-a)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k.
\]

Also, from (5), for odd \( n \), for each \( x \in (a,b) \), we have
\[
\frac{f^{(n)}(b)}{n!} (x-b)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \leq \frac{f^{(n)}(a)}{n!} (x-b)^n,
\]
or equivalently
\[
\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!} (x-b)^k \leq f(x) \leq \frac{f^{(n)}(a)}{n!} (x-b)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k,
\]
and, for even \( n \), from (5) for each \( x \in (a,b) \), we have
\[
\frac{f^{(n)}(a)}{n!} (x-b)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \leq \frac{f^{(n)}(b)}{n!} (x-b)^n,
\]
or equivalently
\[
\frac{f^{(n)}(a)}{n!} (x-b)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \leq f(x) \leq \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!} (x-b)^k. \quad \blacksquare
\]

For \( n = 1 \), from (1) and (2) we obtain the following result:
COROLLARY 1. Let $f$ be convex on $(a,b)$. Then the following inequalities hold for all $x \in (a,b)$:

$$\max \{ f(a) + f'(a)(x-a), f(b) + f'(b)(x-b) \} \leq f(x) \leq \min \{ f(a) + f'(b)(x-a), f(b) + f'(a)(x-b) \}$$

COMMENT. The result from Corollary 1. is proved in [1] in a more elementary way.

For $n = 2$, from (1) and (3) we obtain the following result:

COROLLARY 2. Let $f$ be 3-convex on $[a,b]$. Then the following inequalities hold for all $x \in (a,b)$:

$$\max \left\{ f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2, f(b) + f'(b)(x-b) + \frac{f''(a)}{2}(x-b)^2 \right\} \leq f(x) \leq \min \left\{ f(a) + f'(b)(x-a) + \frac{f''(b)}{2}(x-a)^2, f(b) + f'(a)(x-b) + \frac{f''(b)}{2}(x-b)^2 \right\}$$

In [1] the result of Corollary 1. is used as a basic result from which many other results can be derived. In the same fashion, recall the following known result:

If $f$ is $(n+2)$-convex on $[a,b]$, then the function

$$G_n(x) = [x,x+h,x+2h,...,x+nh]f,$$

with $h < \frac{b-a}{n}$, is convex on $[a,b-nh]$ (result from [3], see also [2], Theorem 2.51., page 74) where:

$$[x_i]f = f(x_i)$$

and

$$[x_0,x_1,x_2,...,x_n]f = \frac{[x_1,x_2,...,x_n]f - [x_0,x_1,...,x_{n-1}]f}{x_n-x_0}$$

By induction it is easy to establish following formula:

$$G_n(x) = \frac{1}{n! \cdot h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n-i)h).$$

And now by applying Corollary 1 on the previous result, we obtain:

THEOREM 2. Let $f$ be $(n+2)$-convex on $[a,b]$ and $h$ real number such that
\( h < \frac{b-a}{n} \). Then, for all \( x \in (a, b - nh) \), the following inequalities hold:

\[
\max \left\{ \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(a + (n - i)h) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} f'(a + (n - i)h)(x - a), \right. \\
\left. \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(b - ih) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} f'(b - ih)(x - (b - nh)) \right\} \\
\leq \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - i)h) \\
\leq \min \left\{ \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(a + (n - i)h) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} f'(b - ih)(x - a), \right. \\
\left. \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(b - ih) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} f'(a + (n - i)h)(x - (b - nh)) \right\}.
\]

We state the special case of Theorem 2 for \( n = 1 \) in the following corollary:

**Corollary 3.** Let \( f \) be 3-convex on \( [a, b] \) and \( h \) real number such that \( h < b - a \). Then, for all \( x \in (a, b - h) \), the following inequalities hold:

\[
\max \{ f(a + h) - f(a) + (f'(a + h) - f'(a))(x - a), \\
f(b) - f(b - h) + (f'(b) - f'(b - h))(x - (b - h)) \} \\
\leq f(x + h) - f(x) \quad (6) \\
\leq \min \{ f(a + h) - f(a) + (f'(b) - f'(b - h))(x - a), \\
f(b) - f(b - h) + (f'(a + h) - f'(a))(x - (b - h)) \}.
\]

If we put \( h = \frac{b-a}{2} \) in (6), for a 3-convex function \( f \), for \( x \in (a, \frac{a+b}{2}) \), we obtain the following inequalities:

\[
\max \left\{ f \left( \frac{a+b}{2} \right) - f(a) + f' \left( \frac{a+b}{2} \right) - f'(a) \right\} (x - a), \\
f(b) - f \left( \frac{a+b}{2} \right) + f' \left( \frac{a+b}{2} \right) \left( x - \frac{a+b}{2} \right) \right\} \\
\leq f \left( \frac{x + b - a}{2} \right) - f(x) \\
\leq \min \left\{ f \left( \frac{a+b}{2} \right) - f(a) + f' \left( \frac{a+b}{2} \right) - f'(a) \right\} (x - a), \\
f(b) - f \left( \frac{a+b}{2} \right) + f' \left( \frac{a+b}{2} \right) - f'(a) \right\} \left( x - \frac{a+b}{2} \right) \right\}.
\]

**Comment.** In [1] some other related results are proved. For instance, if we use the fact that, for a 3-convex function \( f \) on \([a, b]\), the function \( F(x) = f(a + b - x) - f(x) \) is convex on \([a, \frac{a+b}{2}]\) (see [2], page 72), Corollary 1., and some easy algebraic
manipulation, we obtain the following inequalities (see Theorem 3. and Corollary 4. in [1]):

\[
\begin{align*}
    f(b) - f(a) - (f'(a) + f'(b))(x - a) &\leq f(a + b - x) - f(x) \\
    f(a) - f(b) + 2f'\left(\frac{a + b}{2}\right)(b - x) &\leq f(a + b - x) - f(x) \\
    2f'\left(\frac{a + b}{2}\right)\left(\frac{a + b}{2} - x\right) &\leq f(a + b - x) - f(x) \\
    f(a + b - x) - f(x) &\leq (f'(a) + f'(b))\left(\frac{a + b}{2} - x\right) \\
    f(a + b - x) - f(x) &\leq f(b) - f(a) - 2f'\left(\frac{a + b}{2}\right)(x - a) \\
    f(a + b - x) - f(x) &\leq f(a) - f(b) + (f'(a) + f'(b))(b - x)
\end{align*}
\]

for each \(x \in (a, \frac{a+b}{2})\).

REFERENCES