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INEQUALITIES FOR n–CONVEX FUNCTIONS

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Abstract. In this article new inequalities for n-convex functions are stated and proved and some applications of these results are given.

1. Introduction

The function f is called n-convex on the interval (a,b) if its n-th derivative $f^{(n)}(t)$ is positive for all $t \in (a,b)$. Using this terminology, convex function is called 2-convex function.

In [1] some results for convex and 3-convex functions (with applications to log-convex and 3-log convex functions) are obtained.

The aim of this article is to establish some basic results for n-convex functions which can be easily used for obtaining many other results.

2. Main results

Let's state and prove the main result.

THEOREM 1. *Let f be $(n+1)$-convex function on $[a,b]$. Then, for each $x \in (a,b)$, the following inequalities hold*

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \leqslant f(x) \leqslant \frac{f^{(n)}(b)}{n!}(x-a)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{1}$$

If n is odd, then

$$\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \leqslant f(x) \leqslant \frac{f^{(n)}(a)}{n!}(x-b)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k, \tag{2}$$

and if n is even, it holds:

$$\frac{f^{(n)}(a)}{n!}(x-b)^n + \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \leqslant f(x) \leqslant \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k. \tag{3}$$

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Proof. Let us recall on Taylor's formula:

$$f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k=\frac{f^{(n)}(c)}{n!}(x-a)^n, \tag{4}$$

for some $c\in(a,x)$ and similarly

$$f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k=\frac{f^{(n)}(c)}{n!}(x-b)^n \tag{5}$$

for some $c\in(x,b)$.

If $f^{(n+1)}$ is convex on $[a,b]$, then $f^{(n)}$ is increasing on $[a,b]$, i.e. $f^{(n)}(a)\leqslant f^{(n)}(t)\leqslant f^{(n)}(b)$, for each $t\in(a,b)$.

So, from (4), for each $x\in(a,b)$, we have

$$\frac{f^{(n)}(a)}{n!}(x-a)^n\leqslant f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\leqslant\frac{f^{(n)}(b)}{n!}(x-a)^n,$$

or equivalently

$$\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k\leqslant f(x)\leqslant\frac{f^{(n)}(b)}{n!}(x-a)^n+\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Also, from (5), for odd n, for each $x\in(a,b)$, we have

$$\frac{f^{(n)}(b)}{n!}(x-b)^n\leqslant f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\leqslant\frac{f^{(n)}(a)}{n!}(x-b)^n,$$

or equivalently

$$\sum_{k=0}^{n}\frac{f^{(k)}(b)}{k!}(x-b)^k\leqslant f(x)\leqslant\frac{f^{(n)}(a)}{n!}(x-b)^n+\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k,$$

and, for even n, from (5) for each $x\in(a,b)$, we have

$$\frac{f^{(n)}(a)}{n!}(x-b)^n\leqslant f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\leqslant\frac{f^{(n)}(b)}{n!}(x-b)^n,$$

or equivalently

$$\frac{f^{(n)}(a)}{n!}(x-b)^n+\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\leqslant f(x)\leqslant\sum_{k=0}^{n}\frac{f^{(k)}(b)}{k!}(x-b)^k. \quad \square$$

For $n=1$, from (1) and (2) we obtain the following result:

COROLLARY 1. *Let f be convex on (a,b). Then the following inequalities*

$$\max\{f(a)+f'(a)(x-a), f(b)+f'(b)(x-b)\} \leqslant f(x)$$
$$\leqslant \min\{f(a)+f'(b)(x-a), f(b)+f'(a)(x-b)\}$$

hold for all $x \in (a,b)$.

COMMENT. The result from Corollary 1. is proved in [1] in a more elementary way.

For $n = 2$, from (1) and (3) we obtain the following result:

COROLLARY 2. *Let f be 3-convex on $[a,b]$. Then the following inequalities hold for all $x \in (a,b)$:*

$$\max\left\{f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2, f(b)+f'(b)(x-b)+\frac{f''(a)}{2}(x-b)^2\right\}$$
$$\leqslant f(x)$$
$$\leqslant \min\left\{f(a)+f'(a)(x-a)+\frac{f''(b)}{2}(x-a)^2, f(b)+f'(b)(x-b)+\frac{f''(b)}{2}(x-b)^2\right\}$$

In [1] the result of Corollary 1. is used as a basic result from which many other results can be derived. In the same fashion, recall the following known result:

If f is $(n+2)$-convex on $[a,b]$, then the function

$$G_n(x) = [x, x+h, x+2h, \ldots, x+nh]f,$$

with $h < \frac{b-a}{n}$, is convex on $[a, b-nh]$ (result from [3], see also [2], Theorem 2.51., page 74) where:

$$[x_i]f = f(x_i)$$

and

$$[x_0, x_1, x_2, \ldots, x_n]f = \frac{[x_1, x_2, \ldots, x_n]f - [x_0, x_1, \ldots, x_{n-1}]f}{x_n - x_0}$$

By induction it is easy to establish following formula:

$$G_n(x) = \frac{1}{n! \cdot h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\left(x+(n-i)h\right).$$

And now by applying Corollary 1 on the previous result, we obtain:

THEOREM 2. *Let f be $(n+2)$-convex on $[a,b]$ and h real number such that*

$h < \frac{b-a}{n}$. *Then, for all* $x \in (a, b-nh)$, *the following inequalities hold:*

$$\max\left\{\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(a+(n-i)h\right)+\sum_{i=0}^{n}(-1)^i\binom{n}{i}f'\left(a+(n-i)h\right)(x-a),\right.$$
$$\left.\sum_{i=0}^{n}(-1)^i\binom{n}{i}f(b-ih)+\sum_{i=0}^{n}(-1)^i\binom{n}{i}f'(b-ih)\left(x-(b-nh)\right)\right\}$$
$$\leqslant \sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(x+(n-i)h\right)$$
$$\leqslant \min\left\{\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(a+(n-i)h\right)+\sum_{i=0}^{n}(-1)^i\binom{n}{i}f'(b-ih)(x-a),\right.$$
$$\left.\sum_{i=0}^{n}(-1)^i\binom{n}{i}f(b-ih)+\sum_{i=0}^{n}(-1)^i\binom{n}{i}f'\left(a+(n-i)h\right)\left(x-(b-nh)\right)\right\}.$$

We state the special case of Theorem 2 for $n = 1$ in the following corollary:

COROLLARY 3. *Let* f *be* 3*-convex on* $[a,b]$ *and* h *real number such that* $h < b-a$. *Then, for all* $x \in (a, b-h)$, *the following inequalities hold:*

$$\begin{aligned}
&\max\{f(a+h)-f(a)+(f'(a+h)-f'(a))(x-a),\\
&f(b)-f(b-h)+(f'(b)-f'(b-h))\left(x-(b-h)\right)\}\\
&\leqslant f(x+h)-f(x)\\
&\leqslant \min\{f(a+h)-f(a)+(f'(b)-f'(b-h))(x-a),\\
&f(b)-f(b-h)+(f'(a+h)-f'(a))\left(x-(b-h)\right)\}
\end{aligned} \tag{6}$$

If we put $h = \frac{b-a}{2}$ in (6), for a 3-convex function f, for $x \in (a, \frac{a+b}{2})$, we obtain the following inequalities:

$$\max\left\{f\left(\frac{a+b}{2}\right)-f(a)+\left(f'\left(\frac{a+b}{2}\right)-f'(a)\right)(x-a),\right.$$
$$\left.f(b)-f\left(\frac{a+b}{2}\right)+\left(f'(b)-f'\left(\frac{a+b}{2}\right)\right)\left(x-\frac{a+b}{2}\right)\right\}$$
$$\leqslant f\left(x+\frac{b-a}{2}\right)-f(x)$$
$$\leqslant \min\left\{f\left(\frac{a+b}{2}\right)-f(a)+\left(f'(b)-f'\left(\frac{a+b}{2}\right)\right)(x-a),\right.$$
$$\left.f(b)-f\left(\frac{a+b}{2}\right)+\left(f'\left(\frac{a+b}{2}\right)-f'(a)\right)\left(x-\frac{a+b}{2}\right)\right\}.$$

COMMENT. In [1] some other related results are proved. For instance, if we use the fact that, for a 3-convex function f on $[a,b]$, the function $F(x) = f(a+b-x) - f(x)$ is convex on $[a, \frac{a+b}{2}]$ (see [2], page 72), Corollary 1., and some easy algebraic manipulation, we obtain the following inequalities (see Theorem 3. and Corollary 4. in [1]):

$$f(b)-f(a)-(f'(a)+f'(b))(x-a) \leqslant f(a+b-x)-f(x)$$
$$f(a)-f(b)+2f'\left(\frac{a+b}{2}\right)(b-x) \leqslant f(a+b-x)-f(x)$$
$$2f'\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}-x\right) \leqslant f(a+b-x)-f(x)$$
$$f(a+b-x)-f(x) \leqslant (f'(a)+f'(b))\left(\frac{a+b}{2}-x\right)$$
$$f(a+b-x)-f(x) \leqslant f(b)-f(a)-2f'\left(\frac{a+b}{2}\right)(x-a)$$
$$f(a+b-x)-f(x) \leqslant f(a)-f(b)+(f'(a)+f'(b))(b-x)$$

for each $x \in (a, \frac{a+b}{2})$.

REFERENCES

[1] ILKO BRNETIĆ, *Inequalities for convex and 3-log convex functions*, submitted.
[2] JOSIP E. PEČARIĆ, FRANK PROSCHAN, Y. L. TONG, *Convex functions, partial ordering, and statistical applications*, Academic Press, 1992.
[3] D. ZWICK, *A divided difference inequality for n-convex functions*, J. Math. Anal. Appl. **104** (1984), 435–436.

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