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NOTE ON AN INEQUALITY OF GAUSS

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Abstract. In this paper a functional defined as the difference between the left-hand and the right-hand side of an extension of the Gauss inequality given in [H. Alzer, On an inequality of Gauss, Rev. Mat. Complut. 4(2) (1991), 179–183.] is studied. Related analogous of the Lagrange and the Cauchy mean value theorems are obtained. Furthermore, Gauss means are generated and their monotonicity property is proven.

1. Introduction

Let us recall the inequality of Gauss (see, [8, p. 195]):
Let $f:[0,\infty)\to\mathbb{R}$ be a decreasing function, then, for all real numbers $k>0$,

$$k^2\int_k^\infty f(x)dx \leqslant \frac{4}{9}\int_0^\infty x^2 f(x)dx. \tag{1.1}$$

H. Alzer proved in 1991 (see [1]) that an application of the following theorem leads to a new proof and to a converse of inequality (1.1):

THEOREM 1.1. *Let $g:[a,b]\to\mathbb{R}$ be strictly increasing, convex and differentiable, and let $f:I\to\mathbb{R}$ be decreasing. Then*

$$\int_a^b f(s(x))g'(x)dx \leqslant \int_{g(a)}^{g(b)} f(x)dx \leqslant \int_a^b f(t(x))g'(x)dx, \tag{1.2}$$

where

$$s(x)=\frac{g(b)-g(a)}{b-a}(x-a)+g(a) \tag{1.3}$$

and

$$t(x)=g'(x_0)(x-x_0)+g(x_0),\ x_0\in[a,b]. \tag{1.4}$$

($I\subseteq\mathbb{R}$ is an interval containing $a,b,g(a),g(b),t(a)$ and $t(b)$.)

If either g is convave (instead of convex) or f is increasing, then the reversed inequalities hold.

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In this paper we study a functional defined as the difference between the left-hand and the right-hand side of an extension of the Gauss inequality (given in [1]) and obtain related analogous of the Lagrange and the Cauchy mean value theorems. Furthermore, we define new Gauss means and prove their monotonicity property.

The paper is organized as follows. After this introduction, in Section 2 we prove the Lagrange and the Cauchy-type mean value theorems and study exponential and logarithmic convexity of the difference between the left-hand and the right-hand side of the inequality (1.2). In Section 3 we introduce new Gauss means and prove their monotonicity property.

First, let us recall some notions; log denotes the natural logarithm function, an interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$ and by dx we denote the Lebesgue measure on $\mathbb{R}$.

Now, we introduce some necessary notation and recall some basic facts about convex, log-convex functions (see e.g. [3], [7]) as well as exponentially convex functions (see e.g [2], [5], [6]).

LEMMA 1.1. *Let $h:(a,b)\to\mathbb{R}$. The following statements are equivalent:*

(i) h is exponentially convex,

(ii) h is continuous and

$$\sum_{i,j=1}^{n} t_i t_j h\left(\frac{x_i+x_j}{2}\right)\geqslant 0,$$

for every $n\in\mathbb{N}$, $t_i\in\mathbb{R}$ and every $x_i\in(a,b)$, $1\leqslant i\leqslant n$.

Condition from Lemma 1.1, part (ii) is equivalent with positive semi-definitness of matrices

$$\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}, \quad \text{for all } n\in\mathbb{N}.$$

Let us recall two useful lemmas from the convexity and the log-convexity theory.

LEMMA 1.2. *A function Φ is log-convex on an interval I, if and only if for all $a,b,c\in I$, $a<b<c$, it holds*

$$[\Phi(b)]^{c-a}\leqslant[\Phi(a)]^{c-b}[\Phi(c)]^{b-a}.$$

LEMMA 1.3. *Let f be log-convex on $I\subseteq\mathbb{R}$ and let $a_1,a_2,b_1,b_2\in I$ be such that $a_1\leqslant b_1$, $a_2\leqslant b_2$ and $a_1\neq a_2$, $b_1\neq b_2$. Then the following inequality is valid*

$$\left[\frac{f(a_2)}{f(a_1)}\right]^{\frac{1}{a_2-a_1}}\leqslant\left[\frac{f(b_2)}{f(b_1)}\right]^{\frac{1}{b_2-b_1}}.$$

2. Mean value theorems

First, let us define linear functionals $L_1, L_2 : C^1(I) \to \mathbb{R}$ by

$$L_1(f) = \int_a^b f(s(x))g'(x)dx - \int_{g(a)}^{g(b)} f(x)dx, \tag{2.1}$$

$$L_2(f) = \int_{g(a)}^{g(b)} f(x)dx - \int_a^b f(t(x))g'(x)dx, \tag{2.2}$$

where $g : [a,b] \to \mathbb{R}$ is strictly increasing, convex and differentiable function, s is defined by (1.3), t is defined by (1.4), and $a, b, g(a), g(b), t(a), t(b) \in I$.

Moreover, $L_1(f) \geqslant 0$, $L_2(f) \geqslant 0$ for all increasing functions f and $L_1(f) \leqslant 0$, $L_2(f) \leqslant 0$ for all decreasing functions f.

Furthermore, we state and prove the Lagrange-type mean value theorems related to L_1 and L_2.

THEOREM 2.1. *Let $g : [a,b] \to \mathbb{R}$ be strictly increasing, convex and differentiable, and s be defined by* (1.3). *Let I be compact interval such that $a, b, g(a)$, $g(b) \in I$, $h_2 : I \to \mathbb{R}$ be increasing and continuous, $J = h_2(I)$, and $h_1 \in C^1(J)$. Then there exists $\xi \in J$ such that*

$$\int_a^b h_1(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} h_1(h_2(x))dx = h_1'(\xi)\left[\int_a^b h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2(x)dx\right] \tag{2.3}$$

holds, that is,

$$L_1(h_1 \circ h_2) = h_1'(\xi)L_1(h_2),$$

where L_1 is defined by (2.1).

Proof. Since h_1' is continuous on compact interval J there exist $m = \min_{x\in J} h_1'(x)$ and $M = \max_{x\in J} h_1'(x)$ both real numbers. Now we consider functions $\Phi_1, \Phi_2 : J \to \mathbb{R}$ defined by

$$\Phi_1(x) = Mx - h_1(x) \text{ and } \Phi_2(x) = h_1(x) - mx.$$

Since $\Phi_1, \Phi_2 \in C^1(J)$, we have $\Phi_1'(x) = M - h_1'(x) \geqslant 0$ and $\Phi_2'(x) = h_1'(x) - m \geqslant 0$. Hence, functions Φ_1 and Φ_2 are increasing. Applying Theorem 1.1 on an increasing function $\Phi_1 \circ h_2$, we obtain

$$\int_a^b h_1(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} h_1(h_2(x))dx \leqslant M\left[\int_a^b h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2(x)dx\right],$$

that is, $L_1(h_1 \circ h_2) \leqslant ML_1(h_2)$. Similarly, if we consider an increasing function $\Phi_2 \circ h_2$ we obtain

$$m\left[\int_a^b h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2(x)dx\right] \leqslant \int_a^b h_1(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} h_1(h_2(x))dx,$$

that is, $L_1(h_1 \circ h_2) \geqslant mL_1(h_2)$. Combining those two results we obtain

$$mL_1(h_2) \leqslant L_1(h_1 \circ h_2) \leqslant ML_1(h_2).$$

If $L_1(h_2) = 0$, then $L_1(h_1 \circ h_2) = 0$, so (2.3) holds for all $\xi \in J$. Otherwise,

$$\min_{x\in J} h_1'(x) = m \leqslant \frac{L_1(h_1 \circ h_2)}{L_1(h_2)} \leqslant M = \max_{x\in J} h_1'(x), \text{ so } \frac{L_1(h_1 \circ h_2)}{L_1(h_2)} \in h_1'(J).$$

Since h_1' is continuous there exists $\xi \in J$ such that $\frac{L_1(h_1\circ h_2)}{L_1(h_2)} = h_1'(\xi)$, so the proof is completed. □

COROLLARY 2.1. *Let $g : [a,b] \to \mathbb{R}$ be strictly increasing, convex and differentiable, and s be defined by* (1.3). *Let I be compact interval such that $a,b,g(a)$, $g(b) \in I$, and $h_1 \in C^1(I)$. Then there exists $\xi \in I$ such that*

$$\int_a^b h_1(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_1(x)dx = h_1'(\xi)\left[\int_a^b s(x)g'(x)dx - \frac{g^2(b)-g^2(a)}{2}\right]$$

holds.

Proof. Apply Theorem 2.1 for $h_2(x) = x$. □

THEOREM 2.2. *Let $g : [a,b] \to \mathbb{R}$ be strictly increasing, convex and differentiable, and t be defined by* (1.4). *Let I be compact interval such that $a,b,g(a),g(b),t(a)$, $t(b) \in I$, $h_2 : I \to \mathbb{R}$ be increasing and continuous, $J = h_2(I)$, and $h_1 \in C^1(J)$. Then there exists $\xi \in J$ such that*

$$L_2(h_1 \circ h_2) = h_1'(\xi)L_2(h_2),$$

where L_2 is defined by (2.2).

Proof. Similar to the proof of Theorem 2.1. □

COROLLARY 2.2. *Let $g : [a,b] \to \mathbb{R}$ be strictly increasing, convex and differentiable, and t be defined by* (1.4). *Let I be compact interval such that $a,b,g(a),g(b)$, $t(a)$, $t(b) \in I$, and $h_1 \in C^1(I)$. Then there exists $\xi \in I$ such that*

$$\int_{g(a)}^{g(b)} h_1(x)dx - \int_a^b h_1(t(x))g'(x)dx = h_1'(\xi)\left[\frac{g^2(b)-g^2(a)}{2} - \int_a^b t(x)g'(x)dx\right]$$

holds.

Proof. Apply Theorem 2.2 for $h_2(x) = x$. □

We continue with the Cauchy-type mean value theorems related to L_1 and L_2.

THEOREM 2.3. *Let $g : [a,b] \to \mathbb{R}$ be strictly increasing, convex and differentiable function, and s be defined by (1.3). Let I be compact interval such that $a, b, g(a)$, $g(b) \in I$, $h_2 : I \to \mathbb{R}$ be increasing and continuous, and $J = h_2(I)$. Let $F, H \in C^1(J)$, $H'(x) \neq 0$ for every $x \in J$. Then there exists $\xi \in J$ such that*

$$\frac{\int_a^b F(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} F(h_2(x))dx}{\int_a^b H(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} H(h_2(x))dx} = \frac{F'(\xi)}{H'(\xi)}, \tag{2.4}$$

that is,

$$\frac{L_1(F \circ h_2)}{L_1(H \circ h_2)} = \frac{F'(\xi)}{H'(\xi)}$$

holds, where L_1 is defined by (2.1).

Proof. Set $\Phi(t) = F(t)L_1(H \circ h_2) - H(t)L_1(F \circ h_2)$. Note that $\Phi'(t) = F'(t)L_1(H \circ h_2) - H'(t)L_1(F \circ h_2)$. Obviously, $L_1(\Phi \circ h_2) = 0$. On the other hand, by Theorem 2.1 there exists $\xi \in J$ such that

$$L_1(\Phi \circ h_2) = \Phi'(\xi) \left[\int_a^b h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2(x)dx \right].$$

Since $L_1(h_2) \neq 0$, we have that

$$\Phi'(\xi) = F'(\xi)L_1(H \circ h_2) - H'(\xi)L_1(F \circ h_2) = 0.$$

By assumption $H'(\xi) \neq 0$, so Theorem 2.1 assures that $L_1(H \circ h_2) \neq 0$. Hence (2.4) follows. □

COROLLARY 2.3. *Let $g : [a,b] \to \mathbb{R}$ be strictly increasing, convex and differentiable function, and s be defined by (1.3). Let I be compact interval such that $a, b, g(a)$, $g(b) \in I$. Let $F, H \in C^1(I)$, $H'(x) \neq 0$ for every $x \in I$. Then there exists $\xi \in I$ such that*

$$\frac{\int_a^b F(s(x))g'(x)dx - \int_{g(a)}^{g(b)} F(x)dx}{\int_a^b H(s(x))g'(x)dx - \int_{g(a)}^{g(b)} H(x)dx} = \frac{F'(\xi)}{H'(\xi)}. \tag{2.5}$$

Proof. Apply Theorem 2.3 for $h_2(x) = x$. □

THEOREM 2.4. *Let $g:[a,b]\to\mathbb{R}$ be strictly increasing, convex and differentiable function, and t be defined by* (1.4). *Let I be compact interval such that $a,b,g(a),g(b)$, $t(a)$, $t(b)\in I$, $h_2:I\to\mathbb{R}$ be increasing and continuous, and $J=h_2(I)$. Let $F,H\in C^1(J)$, $H'(x)\neq 0$ for every $x\in J$. Then there exists $\xi\in J$ such that*

$$\frac{\int_{g(a)}^{g(b)} F(h_2(x))dx-\int_a^b F(h_2(t(x)))g'(x)dx}{\int_{g(a)}^{g(b)} H(h_2(x))dx-\int_a^b H(h_2(t(x)))g'(x)dx}=\frac{F'(\xi)}{H'(\xi)}, \tag{2.6}$$

that is,

$$\frac{L_2(F\circ h_2)}{L_2(H\circ h_2)}=\frac{F'(\xi)}{H'(\xi)}$$

holds, where L_2 is defined by (2.2).

Proof. Similar to the proof of Theorem 2.3. □

COROLLARY 2.4. *Let $g:[a,b]\to\mathbb{R}$ be strictly increasing, convex and differentiable function, and t be defined by* (1.4). *Let I be compact interval such that $a,b,g(a),g(b),t(a)$, $t(b)\in I$. Let $F,H\in C^1(I)$, $H'(x)\neq 0$ for every $x\in I$. Then there exists $\xi\in I$ such that*

$$\frac{\int_{g(a)}^{g(b)} F(x)dx-\int_a^b F(t(x))g'(x)dx}{\int_{g(a)}^{g(b)} H(x)dx-\int_a^b H(t(x))g'(x)dx}=\frac{F'(\xi)}{H'(\xi)}. \tag{2.7}$$

Proof. Apply Theorem 2.4 for $h_2(x)=x$. □

COROLLARY 2.5. *Let $k>0$, $F,H\in C^1(\mathbb{R}^+)$, $H'(x)\neq 0$ for every $x\in\mathbb{R}^+$. Then there exists $\xi\in\mathbb{R}^+$ such that*

$$\frac{3\int_0^k x^2F(x+k)dx-k^2\int_k^{2k} F(x)dx}{3\int_0^k x^2H(x+k)dx-k^2\int_k^{2k} H(x)dx}=\frac{F'(\xi)}{H'(\xi)}. \tag{2.8}$$

Proof. To prove (2.8) apply Theorem 2.3 with $a=0$, $b=k$, $g(x)=\frac{1}{k^2}x^3+k$. □

COROLLARY 2.6. *Let $k>0$, $F,H \in C^1(\mathbb{R}^+)$, $H'(x) \neq 0$ for every $x \in \mathbb{R}^+$. Then there exists $\xi \in \mathbb{R}^+$ such that*

$$\frac{k^2 \int_k^\infty F(x)dx - \frac{4}{9}\int_0^\infty x^2 F(x)dx}{k^2 \int_k^\infty H(x)dx - \frac{4}{9}\int_0^\infty x^2 H(x)dx} = \frac{F'(\xi)}{H'(\xi)}. \tag{2.9}$$

Proof. To prove (2.9) apply Theorem 2.4 with $a=0$, $x_0 = \frac{k}{2^{1/3}}$, $g(x) = \frac{1}{k^2}x^3 + k$, $b \to \infty$ and $g(b) \to \infty$. $\square$

REMARK 2.1. Corollary 2.6 was also obtained in [4] but from different Cauchy-type mean value theorem.

For $u \in \mathbb{R}$, let the function $\varphi_u : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\varphi_u(x) = \begin{cases} \frac{x^u}{u}, & u \neq 0; \\ \log x, & u = 0. \end{cases} \tag{2.10}$$

Then $\varphi_u'(x) = x^{u-1}$ for all $u \in \mathbb{R}$, that is, φ_u is an increasing function on $\mathbb{R}^+$. If we consider $L_i(\varphi_u \circ h_2)$ with L_i as in (2.1) or (2.2) and h_2 increasing, we have that $L_i(\varphi_u \circ h_2) \geqslant 0$ for $i = 1,2$ and for all $u \in \mathbb{R}$.

Properties of the mapping $u \mapsto L_i(\varphi_u \circ h_2)$, $i = 1,2$ are given in the following theorem:

THEOREM 2.5. *For L_i as in (2.1) and (2.2), h_2 increasing, and φ_u as in (2.10) we have the following:*

(i) the mapping $u \mapsto L_i(\varphi_u \circ h_2)$ is continuous on $\mathbb{R}$,

(ii) for every $n \in \mathbb{N}$ and $u_i \in \mathbb{R}, u_{ij} = \frac{u_i+u_j}{2}$, $i,j = 1,2,...,n$, the matrix $[L_i(\varphi_{u_{ij}} \circ h_2)]_{i,j=1}^n$ is positive semi-definite, that is

$$\det[L_i(\varphi_{u_{ij}} \circ h_2)]_{i,j=1}^n \geqslant 0,$$

(iii) the mapping $u \mapsto L_i(\varphi_u \circ h_2)$ is exponentially convex,

(iv) the mapping $u \mapsto L_i(\varphi_u \circ h_2)$ is log-convex,

(v) for $u_i \in \mathbb{R}$, $i = 1,2,3$, $u_1 < u_2 < u_3$,

$$[L_i(\varphi_{u_2} \circ h_2)]^{u_3-u_1} \leqslant [L_i(\varphi_{u_1} \circ h_2)]^{u_3-u_2}[L_i(\varphi_{u_3} \circ h_2)]^{u_2-u_1}.$$

Proof.

(i) Notice that

$$L_1(\varphi_u \circ h_2) = \begin{cases} \frac{1}{u}\left[\int_a^b h_2^u(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^u(x)dx\right], & u \neq 0; \\ \int_a^b \log(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} \log(h_2(x))dx, & u = 0. \end{cases}$$

It is obviously continuous on $\mathbb{R}\setminus\{0\}$. Suppose $u \to 0$:

$$\lim_{u\to 0} L_1(\varphi_u \circ h_2) = \lim_{u\to 0} \frac{1}{u}\left[\int_a^b h_2^u(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^u(x)dx\right] \tag{2.11}$$

from L'Hospital rule limit (2.11) is equal to $L_1(\varphi_0 \circ h_2)$. Hence, the mapping $u \mapsto L_1(\varphi_u \circ h_2)$ is continuous.

Similarly, the mapping $u \mapsto L_2(\varphi_u \circ h_2)$ is continuous.

(ii) Let $n \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, be arbitrary. Define the function $h : \mathbb{R}^+ \to \mathbb{R}$ by

$$h(x) = \sum_{i,j=1}^{n} t_i t_j \varphi_{u_{ij}}(x).$$

Then

$$h'(x) = \sum_{i,j=1}^{n} t_i t_j x^{u_{ij}-1} = \left(\sum_{i=1}^{n} t_i x^{\frac{u_i-1}{2}}\right)^2 \geqslant 0,$$

so h is an increasing function on $\mathbb{R}^+$. We can apply (1.2) on an increasing function $h \circ h_2$ and obtain

$$\int_a^b h(h_2(s(x)))g'(x)dx \geqslant \int_{g(a)}^{g(b)} h(h_2(x))dx,$$

that is,

$$\sum_{i,j=1}^{n} t_i t_j L_1(\varphi_{u_{ij}} \circ h_2) \geqslant 0.$$

So the matrix $[L_1(\varphi_{u_{ij}} \circ h_2)]_{i,j=1}^n$ is positive semi-definite.

(iii), (iv) and (v) are consequences of (i), (ii) and definition of exponentially convex and log-convex functions. □

3. Gauss means

Theorem 2.3 enables us to define various types of means, because if F'/H' has an inverse, from (2.4) we have

$$\xi = \left(\frac{F'}{H'}\right)^{-1} \left(\frac{\int_a^b F(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} F(h_2(x))dx}{\int_a^b H(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} H(h_2(x))dx} \right).$$

Specially, if we take substitutions $F(t) = t^{p-1}$, $H(t) = t^{q-1}$ in (2.4) we consider the following expression

$$M(h_2, g, s; a, b; p, q) = \left(\frac{q-1}{p-1} \cdot \frac{\int_a^b h_2^{p-1}(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{p-1}(x)dx}{\int_a^b h_2^{q-1}(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{q-1}(x)dx} \right)^{\frac{1}{p-q}}, \quad (3.1)$$

where $(p-q)(p-1)(q-1) \neq 0$.

Notice that, (3.1) can be written as

$$M(h_2, g, s; a, b; p, q) = \left(\frac{L_1(\varphi_{p-1} \circ h_2)}{L_1(\varphi_{q-1} \circ h_2)} \right)^{\frac{1}{p-q}}.$$

Moreover, we can extend these means to excluded cases. Taking a limit we can define

$$\begin{aligned} M(h_2, g, s; a, b; p, 1) &= \left(\frac{1}{p-1} \cdot \frac{\int_a^b h_2^{p-1}(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{p-1}(x)dx}{\int_a^b \log h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} \log h_2(x)dx} \right)^{\frac{1}{p-1}} \\ &= M(h_2, g, s; a, b; 1, p), \quad p \neq 1 \end{aligned}$$

for $p \neq 1$ $\quad M(h_2, g, s; a, b; p, p) =$

$$\exp\left(\frac{\int_a^b h_2^{p-1}(s(x)) \log h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{p-1}(x) \log h_2(x)dx}{\int_a^b h_2^{p-1}(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{p-1}(x)dx} - \frac{1}{p-1} \right)$$

$$M(h_2,g,s;a,b;1,1)=\exp\left(\frac{\int_a^b \log^2 h_2(s(x))g'(x)dx-\int_{g(a)}^{g(b)}\log^2 h_2(x)dx}{2\left(\int_a^b \log h_2(s(x))g'(x)dx-\int_{g(a)}^{g(b)}\log h_2(x)dx\right)}\right)$$

We continue with the following result.

THEOREM 3.1. *$M(h_2,g,s;a,b;p,q)$ is monotonous in each argument, that is*

$$M(h_2,g,s;a,b;p,q)\leqslant M(h_2,g,s;a,b;r,t) \tag{3.2}$$

holds for $p,q,r,t\in\mathbb{R}$, $p\leqslant r$, $q\leqslant t$.

Proof. From Theorem 2.5 we have that L_1 is log-convex, so we can apply Lemma 1.3 for $f=L_1$, $p\leqslant r, q\leqslant t$, $p\neq q, r\neq t$ to deduce that

$$\left(\frac{L_1(\varphi_{p-1}\circ h_2)}{L_1(\varphi_{q-1}\circ h_2)}\right)^{\frac{1}{p-q}}\leqslant\left(\frac{L_1(\varphi_{r-1}\circ h_2)}{L_1(\varphi_{t-1}\circ h_2)}\right)^{\frac{1}{r-t}}.$$

Since $(p,q)\mapsto M(h_2,g,s;a,b;p,q)$ is continuous we have (3.2) for $p\leqslant r, q\leqslant t$. □

Corollary 2.3 enables us to define various types of means, because if F'/H' has an inverse, from (2.5) we have

$$\xi=\left(\frac{F'}{H'}\right)^{-1}\left(\frac{\int_a^b F(s(x))g'(x)dx-\int_{g(a)}^{g(b)}F(x)dx}{\int_a^b H(s(x))g'(x)dx-\int_{g(a)}^{g(b)}H(x)dx}\right).$$

Specially, if we take substitutions $F(t)=t^{p-1}$, $H(t)=t^{q-1}$ in (2.5) we consider the following expression

$$M(g,s;a,b;p,q)=\left(\frac{q-1}{p-1}\cdot\frac{\int_a^b s^{p-1}(x)g'(x)dx-\frac{g^p(b)-g^p(a)}{p}}{\int_a^b s^{q-1}(x)g'(x)dx-\frac{g^q(b)-g^q(a)}{q}}\right)^{\frac{1}{p-q}}, \tag{3.3}$$

where $(p-q)(p-1)(q-1)pq\neq 0$.

Notice that, (3.3) can be written as

$$M(g,s;a,b;p,q)=\left(\frac{L_1(\varphi_{p-1})}{L_1(\varphi_{q-1})}\right)^{\frac{1}{p-q}}.$$

Moreover, we can extend these means to excluded cases. Taking a limit we can define

$$M(g,s;a,b;p,1)=\left(\frac{L_1(\varphi_{p-1})}{L_1(\varphi_0)}\right)^{\frac{1}{p-1}}=M(g,s;a,b;1,p),\quad p\neq 1$$

$$M(g,s;a,b;p,0)=\left(\frac{L_1(\varphi_{p-1})}{L_1(\varphi_{-1})}\right)^{\frac{1}{p}}=M(g,s;a,b;0,p),\quad p\neq 0$$

for $p\neq 0,1$ $\quad M(g,s;a,b;p,p)=$

$$\exp\left(\frac{\int_a^b s^{p-1}(x)\log s(x)g'(x)dx-\frac{pg^p(b)\log g(b)-pg^p(a)\log g(a)-g^p(b)+g^p(a)}{p^2}}{\int_a^b h_2^{p-1}(s(x))g'(x)dx-\int_{g(a)}^{g(b)} h_2^{p-1}(x)dx}-\frac{1}{p-1}\right)$$

$$M(g,s;a,b;1,1)=$$

$$\exp\left(\frac{\frac{1}{2}\left[\int_a^b \log^2 s(x)g'(x)dx-g(b)\log^2 g(b)+g(a)\log^2 g(a)\right]}{\int_a^b \log s(x)g'(x)dx-g(b)\log g(b)+g(a)\log g(a)+g(b)-g(a)}\right)$$
$$\times\exp\left(\frac{g(b)\log g(b)-g(a)\log g(a)-g(b)+g(a)}{\int_a^b \log s(x)g'(x)dx-g(b)\log g(b)+g(a)\log g(a)+g(b)-g(a)}\right)$$

$$M(g,s;a,b;0,0)=\exp\left(1+\frac{\log^2 g(b)-\log^2 g(a)-\int_a^b \frac{g'(x)\log s(x)}{s(x)}dx}{\log\frac{g(b)}{g(a)}-\int_a^b \frac{g'(x)}{s(x)}dx}\right)$$

THEOREM 3.2. *$M(g,s;a,b;p,q)$ is monotonous in each argument, that is*

$$M(g,s;a,b;p,q)\leqslant M(g,s;a,b;r,t)$$

holds for $p,q,r,t\in\mathbb{R}$, $p\leqslant r$, $q\leqslant t$.

Proof. Similar to the proof of Theorem 3.1. □

Corollary 2.5 enables us to define various types of means, because if F'/H' has an inverse, from (2.8) we have

$$\xi = \left(\frac{F'}{H'}\right)^{-1}\left(\frac{3\int_0^k x^2F(x+k)dx - k^2\int_k^{2k}F(x)dx}{3\int_0^k x^2H(x+k)dx - k^2\int_k^{2k}H(x)dx}\right).$$

Specially, if we take substitutions $F(t)=t^{p-1}$, $H(t)=t^{q-1}$ in (2.8) we consider the following expression

$$M(a,b;p,q) = \left(\frac{q(q+1)(q+2)}{p(p+1)(p+2)}\cdot\frac{k^{p+2}(2^{p-1}p+p-2^{p+2}+4)}{k^{q+2}(2^{q-1}q+q-2^{q+2}+4)}\right)^{\frac{1}{p-q}}, \tag{3.4}$$

where $(p-q)(p+1)(p+2)(q+1)(q+2)pq \neq 0$.

REMARK 3.1. Theorem 2.4, Corollary 2.4 and Corollary 2.6 enable us to define various types of means, but here we omit the details.

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