Journal of
Mathematical
Inequalities
Volume 5, Number 2 (2011), 213–217

SCHUR–CONVEXITY OF THE WEIGHTED ČEBIŠEV FUNCTIONAL

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Abstract. In this paper the weighted Čebišev functional $T(p;f,g;a,b)$ is regarded as a function of two variables

$$T(p;f,g;x,y)=\frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt}-\left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right),\ (x,y)\in[a,b]\times[a,b]$$

where f, g and $p>0$ are Lebesgue integrable functions. The property of Schur-covexity (Schur-concavity) of this function is proved.

1. Introduction

Let I be an interval with nonempty interior and $\mathbf{x}=(x_i,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ in I^n be two n-tuples such that $\mathbf{x}\prec\mathbf{y}$, i.e.

$$\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]},\ k=1,\ldots,n-1$$
$$\sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}y_{[i]},$$

where $x_{[i]}$ denotes the i th largest component in $\mathbf{x}$.

DEFINITION 1. Function $F:I^n\to\mathbb{R}$ is Schur-convex on I^n if

$$F(x_i,x_2,\ldots,x_n)\leqslant F(y_i,y_2,\ldots,y_n)$$

for each two n-tuples $\mathbf{x}$ and $\mathbf{y}$ such that it holds $\mathbf{x}\prec\mathbf{y}$ on I^n.

Function F is Schur-concave on I^n if and only if $-F$ is Schur-convex.

The next lemma gives us a necessery and sufficient condition for verifying the Schur-convexity property of F when $n=2$ ([4, p. 333], [3, p. 57]).

LEMMA A 1. *Let $F:I^2\to\mathbb{R}$ be a continuous function on I^2 and differentiable in interior of I^2. Then F is Schur-convex (Schur-concave) if and only if it is symmetric and*

$$\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right)(y-x)\geqslant 0 \tag{1}$$

holds (reverses) for all $x,y\in I$, $x\neq y$.

Mathematics subject classification (2010): Primary 26D15, Secondary 26D99.
Keywords and phrases: Convex functions, Schur-convex function, Čebišev functional, integral means.

The authors in [2] were inspired by some inequalities concerning gamma and digamma function and proved the following result for the integral arithmetic mean:

THEOREM A 1. *Let f be a continuous function on I. Then*
$$F(x,y)=\begin{cases}\frac{1}{y-x}\int_x^y f(t)dt, & x\neq y\\ f(x), & x=y\end{cases}$$
is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Also, in [2], applications to logarithmic mean are given.

The authors in [5] proved the Schur convexity of the weighted integral arithmetic mean of function f:

THEOREM A 2. *Let f be a continuous function on I, let p be a positive continuous weight on I. Then*
$$F_p(x,y)=\begin{cases}\frac{1}{\int_x^y p(t)dt}\int_x^y p(t)f(t)dt, & x,y\in I, x\neq y\\ f(x), & x=y\end{cases}$$
is Schur-convex (Schur-concave) on I^2 if and only if the inequality
$$\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\leqslant\frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)}$$
holds (reverses) for all x,y in I.

The Čebišev functional $T(f,g;a,b)$ is defined for two Lebesgue integrable f and g on interval $[a,b]\in\mathbb{R}$ as
$$T(f,g;a,b):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right).$$
Because the Čebišev functional can be express in the term of the integral arithmetic mean, we were inspired by above results in Theorem A1 and in [1] we generalized these results by proving the Schur-convexity of function
$$T(f,g;x,y)=\frac{1}{y-x}\int_x^y f(t)g(t)dt-\left(\frac{1}{y-x}\int_x^y f(t)dt\right)\left(\frac{1}{y-x}\int_x^y g(t)dt\right),$$
$(x,y)\in[a,b]\times[a,b]$.

THEOREM A 3. *Let f and g be Lebesgue integrable functions on $I=[a,b]$. If they are monotonic in the same sense (in the opposite sense) then $T(x,y):=T(f,g;x,y)$, $(x,y)\in[a,b]\times[a,b]\in\mathbb{R}^2$ is Schur-convex (Schur-concave) on $[a,b]\times[a,b]$.*

We used the well-known Čebišev inequality:

THEOREM A 4. *Let f and g be Lebesgue integrable on interval $[a,b]$. If f and g are monotonic in the same sense (in the opposite sense) then*

$$T(f,g;a,b) \geqslant 0 (\leqslant 0).$$

In this paper we will consider weighted Čebišev functional defined as

$$T(p;f,g;x,y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - \left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right),$$
$$(x,y) \in [a,b] \times [a,b],$$

for f and g Lebesgue integrable functions on $I = [a,b]$ and p a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on I.

Let we use the following notations:
$P(x,y) := \int_x^y p(t)dt$,
$\overline{f_p}(x,y) := \frac{1}{\int_x^y p(t)dt}\int_x^y p(t)f(t)dt$ and $\overline{g}_p(x,y) := \frac{1}{\int_x^y p(t)dt}\int_x^y p(t)g(t)dt$.
So, the $T(p;f,g;x,y)$ can be rewritten as

$$T(p;f,g;x,y) = \frac{\int_x^y p(t)f(t)g(t)dt}{P(x,y)} - \overline{f_p}(x,y)\cdot\overline{g}_p(x,y),\ (x,y) \in [a,b]\times[a,b].$$

In this paper we obtain corresponding result to Theorem A2 for weighted Čebišev functional and we will show the another proof of Theorem A3.

2. Results

THEOREM 2.1. *Let f and g be Lebesgue integrable functions on $I = [a,b]$ and let p be a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on $I = [a,b]$. Then $T(p;x,y) := T(p;f,g;x,y)$, is Schur-convex (Schur-concave) on $I^2 = [a,b]\times[a,b]$ if and only if the inequality*

$$T(p;x,y) \leqslant \frac{p(x)(\overline{f_p}(x,y)-f(x))(\overline{g_p}(x,y)-g(x))+p(y)(\overline{f_p}(x,y)-f(y))(\overline{f_g}(x,y)-g(y))}{p(x)+p(y)}. \tag{2}$$

holds (reverses) for all x,y in I.

Proof. To prove the Schur-convexity of $T(p;x,y)$ by Lemma A1, using the inequality (1) it is sufficient to prove $(\frac{\partial T(p;x,y)}{\partial y} - \frac{\partial T(p;x,y)}{\partial x})(y-x) \geqslant 0$, for all $x,y \in [a,b]$, since the function $T(p;x,y) := T(p;f,g;x,y)$ is evidently symmetric.

Now, we calculate $\frac{\partial T(p;x,y)}{\partial y}$ and $\frac{\partial T(p;x,y)}{\partial x}$:

$$\begin{aligned}
\frac{\partial T(p;x,y)}{\partial x} &= \frac{p(x)}{P(x,y)}[T(p;x,y) - (\overline{f_p} - f(x))(\overline{g}_p - g(x))];\\
\frac{\partial T(p;x,y)}{\partial y} &= \frac{p(y)}{P(x,y)}[-T(p;x,y) + (\overline{f_p} - f(y))(\overline{g}_p - g(y))].
\end{aligned}$$

Direct calculation yields that

$$\left(\frac{\partial T(p;x,y)}{\partial y}-\frac{\partial T(p;x,y)}{\partial x}\right)(y-x)$$
$$=\frac{[p(x)+p(y)]}{P(x,y)}$$
$$\left\{-T(p;x,y)+\frac{p(x)(\overline{f_p}-f(x))(\overline{g}_p-g(x))+p(y)(\overline{f_p}-f(y))(\overline{g}_p-g(y))}{p(x)+p(y)}\right\}(y-x).$$

Since $\frac{y-x}{P(x,y)}\geqslant 0$, then a necessary and sufficient condition for Schur-convexity of $T(p;x,y)$ is that holds

$$T(p;x,y)\leqslant\frac{p(x)(\overline{f_p}-f(x))(\overline{g}_p-g(x))+p(y)(\overline{f_p}-f(y))(\overline{g}_p-g(y))}{p(x)+p(y)}.$$

Similarly, we conclude the Schur-concavity of $T(p;x,y)$. □

For special choice $p(t)=1$, $t\in[a,b]$ and using the short notation for the integral means: $\overline{f}(x,y):=\frac{1}{y-x}\int_x^y g(t)dt$ and $\overline{g}(x,y):=\frac{1}{y-x}\int_x^y g(t)dt$ in Theorem 2.1 we can obtain the following result:

COROLLARY 2.1. *Let f and g be Lebesgue integrable functions on $I=[a,b]$. Then $T(x,y):=T(f,g;x,y)$, is Schur-convex (Schur-concave) on $I^2=[a,b]\times[a,b]$ if and only if the inequality*

$$T(x,y)\leqslant\frac{1}{2}(\overline{f}(x,y)-f(x))(\overline{g}(x,y)-g(x))+(\overline{f}(x,y)-f(y))(\overline{f}(x,y)-g(y))\qquad(3)$$

holds (reverses) for all x,y in I.

REMARK 2.1. Using Theorem 2.1 for special choice $p(t)=1$, $t\in[a,b]$, i.e. Corrolary 2.1, we can obtain result in Theorem A3 according conditions that functions f and g are monotonic in the same sense (in the opposite sense).

Another proof of Theorem A3:

There are three cases to be considered according monotonicity of functions.

Case 1. Let f and g be two increasing functions on $[a,b]$ and $x<y$. So, we have $f(x)\leqslant f(t)\leqslant f(y)$ and $g(x)\leqslant g(t)\leqslant g(y)$ and it yields

$$(f(y)-f(t))(f(t)-f(x))\geqslant 0,\qquad(4)$$
$$(g(y)-g(t))(g(t)-g(x))\geqslant 0.\qquad(5)$$

In the proof in [1] we showed that then the inequality (3) holds.

Then, Corrolary 2.1 implies the property of Schur-convexity of $T(f,g;x,y)$.

We have to remark that for $x > y$ the inequalities in (4) and (5) still are valid and we can find that

$$T(f,g;y,x) \leqslant \frac{1}{2}[(\overline{f}(x,y)-f(y))(\overline{g}(x,y)-g(y))+(f(x)-\overline{f}(x,y))(g(x)-\overline{g}(x,y))].$$

As the T is symmetric, it is obvious that

$$\begin{aligned} T(f,g;x,y) &\leqslant \frac{1}{2}[(\overline{f}(x,y)-f(y))(\overline{g}(x,y)-g(y))+(f(x)-\overline{f}(x,y))(g(x)-\overline{g}(x,y))] \\ &= \frac{1}{2}[(\overline{f}(x,y)-f(x))(\overline{g}(x,y)-g(x))+(f(y)-\overline{f}(x,y))(g(y)-\overline{g}(x,y))]. \end{aligned}$$

Again, the inequality (3) holds and Corollary 2.1 implies Shour-convexity of $T(f,g;x,y)$.

Simillary as in [1] for *Case 2*, we suppose that f and g are both decreasing functions on $[a,b]$ and $x < y$. Since $f(x) \geqslant f(t) \geqslant f(y)$ and $g(x) \geqslant g(t) \geqslant g(y)$ the inequalities in (4) and (5) again are valid and the proof is the same as in Case 1.

Case 3. Let f be an increasing function and g decreasing function. Note that we can consider Case 1. for function f and $-g$ and in [1] we proved reverse inequality in (3) for functions f and g. Similarly as in Case 1, according Corrolary 2.1 reverse inequality (3) implies the Schur-concavity of $T(f,g;x,y)$.

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(Received February 24, 2011)

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