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ON AN INEQUALITY FOR CONVEX FUNCTIONS WITH SOME APPLICATIONS ON FRACTIONAL DERIVATIVES AND FRACTIONAL INTEGRALS

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Abstract. The main goal of the paper is to state and prove the new general inequality for convex and increasing functions. We introduce some new inequalities by involving some fractional integrals and fractional derivatives of Riemman-Liouville, Canavati, Hadamard and Erdelyi-Kóber type and apply our result to multidimensional setting to obtain new results involving mixed Riemman-Liouville fractional integrals.

1. Introduction

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive $\sigma-$finite measures, $k:\Omega_1\times\Omega_2\to\mathbb{R}$ be a non-negative function. Let $U(f)$ denotes the class of functions $g:\Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int_{\Omega_2} k(x,y)f(y)d\mu_2(y),\ x\in\Omega_1$$

where f is a measurable function on Ω_2 and

$$K(x)=\int_{\Omega_2} k(x,y)d\mu_2(y). \tag{1.1}$$

We suppose $K(x)>0$ a.e. on Ω_1. Our first approach is to prove the general inequality for convex functions and increasing functions. Such type of results have been proved in [11]. Now, we will generalize results for convex function and also prove new inequalities by involving some fractional integrals and fractional derivatives of Riemman-Liouville, Canavati, Hadamard, Erdelyi-Kóber and mixed Riemman-Liouville type.

In [7] the following result is given.

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THEOREM 1.1. *Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.1). Assume that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := \int_{\Omega_1} \frac{u(x)k(x,y)}{K(x)} d\mu_1(x) < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \leqslant \int_{\Omega_2} v(y)\phi\left(|f(y)|\right) d\mu_2(y),$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ and for all functions $g \in U(f)$.

If we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by $\dfrac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1,2)$ are measurable functions in Theorem 1.1 we obtain the following result.

THEOREM 1.2. *Let $f_i : \Omega_2 \to \mathbb{R}$ be measurable functions, $g_i \in U(f_i)$, $(i = 1,2)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{f_2(y)k(x,y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := f_2(y) \int_{\Omega_1} \frac{u(x)k(x,y)}{g_2(x)} d\mu_1(x) < \infty. \tag{1.2}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) d\mu_1(x) \leqslant \int_{\Omega_2} v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) d\mu_2(y),$$

holds.

REMARK 1.1. If ϕ is strictly convex and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then in Theorem 1.2 the inequality is strict.

REMARK 1.2. As a special case of Theorem 1.2 for $\Omega_1 = \Omega_2 = [a,b]$ and $d\mu_1(x) = dx$, $d\mu_1(y) = dy$ we obtain the result in[10] (see also [11, p. 236]).

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0 \cdot \infty, \frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a non-negative measurable function on the actual interval or more general set.

This paper is organized in this way:
After introduction, in Section 2, we give the new general inequality for convex and increasing functions and we give some new inequalities involving Riemann-Liouville fractional integrals, Canavati-type fractional derivative, Caputo fractional derivative and other fractional derivatives. We also apply our general result in multidimensional settings and conclude this paper with a generalization of Theorem 1.2 for convex functions of several variables.

2. New inequalities involving fractional integrals and derivatives

First, let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [1], [6], [12] and [13].

Let $0 < a < b \leqslant \infty$. By $C^m([a,b])$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leqslant \alpha < k+1$) and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geqslant \alpha\}$). By $L_1(a,b)$ we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Let us recall the well known definitions of the Riemann-Liouville fractional integrals, see [9].

Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^\alpha_{a^+} f$ and $I^\alpha_{b^-} f$ of order $\alpha > 0$ are defined by

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1} dy, \quad (x > a)$$

and

$$I^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(y)(y-x)^{\alpha-1} dy, \quad (x < b)$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals.
Some recent results involving Riemann-Liouville fractional integrals can be seen in e.g [7] and [8].

As a special case of Theorem 1.2 we obtain the following result.

COROLLARY 2.1. *Let u be a weight function on (a,b) and $\alpha > 0$. $I^\alpha_{b\_} g$ denotes the right-sided Riemann-Liouville fractional integral of g. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_a^y \frac{u(x)(y-x)^{\alpha-1}}{I^\alpha_{b\_} f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ *is convex and increasing function, then the inequality*

$$\int_a^b u(x)\phi\left(\left|\frac{I_{b-}^{\alpha} f_1(x)}{I_{b-}^{\alpha} f_2(x)}\right|\right) dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy. \tag{2.1}$$

holds.

Proof. Similar to the proof of Corollary 2.4 in [8].

REMARK 2.1. The result involving the left-sided Riemann-Liouville fractional integral is given in Corollary 2.4 in [8].

Next we give results with respect to the generalized Riemann–Liouville fractional derivative. Let us recall the definition, for details see [2].

Let $\nu > 0$ and $n = [\nu] + 1$ where $[\cdot]$ is the integral part and the *generalized Riemann-Liouville fractional derivative of* f of order ν by

$$D^{\nu} f(s) = \frac{d^n}{ds^n} I^{n-\nu} f(s) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{ds^n} \int_0^s (s-t)^{n-\nu-1} f(t)\, dt\,.$$

In addition, we stipulate

$$D^0 f := f =: I^0 f, \quad I^{-\nu} f := D^{\nu} f \text{ if } \nu > 0.$$

If $\nu \in \mathbb{N}$ then $D^{\nu} f = \frac{d^{\nu} f}{ds^{\nu}}$, the ordinary ν-order derivative.

The space $I^{\nu}(L(0,x))$ is defined as the set of all functions f on $[0,x]$ of the form $f = I^{\nu}\varphi$ for some $\varphi \in L(0,x)$, [12, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [12, p. 43], the latter characterization is equivalent to the condition

$$I^{n-\nu} f \in AC^n[0,x]\,, \tag{2.2}$$

$$\frac{d^j}{ds^j} I^{n-\nu} f(0) = 0, \quad j = 0,1,\ldots,n-1\,.$$

A function $f \in L(0,x)$ satisfying (2.2) is said to have an *integrable fractional derivative* $D^{\nu} f$, [12, Chapter 1, Definition 2.4].

The following lemma help us to prove the next result. For details see [2].

LEMMA 2.1. *Let* $\nu > \gamma \geqslant 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$. *Identity*

$$D^{\gamma} f(s) = \frac{1}{\Gamma(\nu-\gamma)} \int_0^s (s-t)^{\nu-\gamma-1} D^{\nu} f(t)\, dt\,, \quad s \in [0,x]\,. \tag{2.3}$$

is valid if one of the following conditions holds:

(*i*) $f \in I^{\nu}(L(0,x))$.

(ii) $I^{n-\nu}f \in AC^n[0,x]$ *and* $D^{\nu-k}f(0)=0$ *for* $k=1,\dots n$.

(iii) $D^{\nu-k}f \in C[0,x]$ *for* $k=1,\dots,n$, $D^{\nu-1}f \in AC[0,x]$ *and* $D^{\nu-k}f(0)=0$ *for* $k=1,\dots n$.

(iv) $f \in AC^n[0,x]$, $D^{\nu}f \in L(0,x)$, $D^{\gamma}f \in L(0,x)$, $\nu-\gamma \notin \mathbb{N}$, $D^{\nu-k}f(0)=0$ *for* $k=1,\dots,n$ *and* $D^{\gamma-k}f(0)=0$ *for* $k=1,\dots,m$.

(v) $f \in AC^n[0,x]$, $D^{\nu}f \in L(0,x)$, $D^{\gamma}f \in L(0,x)$, $\nu-\gamma=l \in \mathbb{N}$, $D^{\nu-k}f(0)=0$ *for* $k=1,\dots,l$.

(vi) $f \in AC^n[0,x]$, $D^{\nu}f \in L(0,x)$, $D^{\gamma}f \in L(0,x)$ *and* $f(0)=f'(0)=\cdots=f^{(n-2)}(0)=0$.

(vii) $f \in AC^n[0,x]$, $D^{\nu}f \in L(0,x)$, $D^{\gamma}f \in L(0,x)$, $\nu \notin \mathbb{N}$ *and* $D^{\nu-1}f$ *is bounded in a neighbourhood of* $t=0$.

COROLLARY 2.2. *Let* u *be a weight function on* (a,b) *and let the assumptions in Lemma* 2.1 *be satisfied. Define* v *on* (a,b) *by*

$$v(y)=\frac{D^{\nu}f_2(y)}{\Gamma(\nu-\gamma)}\int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D^{\gamma}f_2(x)}dx<\infty$$

If $\phi:(0,\infty)\to\mathbb{R}$ *is convex and increasing function, then the inequality*

$$\int_a^b u(x)\phi\left(\left|\frac{D^{\gamma}f_1(x)}{D^{\gamma}f_2(x)}\right|\right)dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{D^{\nu}f_1(y)}{D^{\nu}f_2(y)}\right|\right)dy, \tag{2.4}$$

holds.

Proof. Similar to the proof of Corollary 2.7 in [8].

The definition of Canavati-type fractional derivative is given in [1] but we will use the Canavati-type fractional derivative given in [3] with some new conditions. Now we define Canavati-type fractional derivative ($\nu-$fractional derivative of f), for details see [3]. We consider

$$C^{\nu}([0,1])=\{f\in C^n([0,1]):I_{1-\bar{\nu}}f^{(n)}\in C^1([0,1])\},$$

$\nu>0, n=[\nu], [.]$ is the integral part, and $\bar{\nu}=\nu-n, 0\leqslant\bar{\nu}<1$.
For $f\in C^{\nu}([0,1])$, the Canavati-ν fractional derivative of f is defined by

$$D^{\nu}f=DI_{1-\bar{\nu}}f^{(n)},$$

where $D=d/dx$.

LEMMA 2.2. *Let $\nu > \gamma > 0$, $n = [\nu]$, $m = [\gamma]$. Let $f \in C^{\nu}([0,1])$, be such that $f^{(i)}(0) = 0$, $i = m, m+1, \ldots, n-1$. Then*

(i) $f \in C^{\gamma}([0,1])$

(ii) $(D^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_0^x (x-t)^{\nu-\gamma-1} (D^{\nu} f)(t) dt,$

for every $x \in [a,b]$.

COROLLARY 2.3. *Let u be a weight function on (a,b) and let assumptions in Lemma 2.2 be satisfied. Define $v(y)$ on (a,b) by*

$$v(y) = \frac{D^{\nu} f_2(y)}{\Gamma(\nu-\gamma)} \int\limits_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D^{\gamma} f_2(x)} dx < \infty$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int\limits_a^b u(x) \phi\left(\left| \frac{D^{\gamma} f_1(x)}{D^{\gamma} f_2(x)} \right| \right) dx \leqslant \int\limits_a^b v(y) \phi\left(\left| \frac{D^{\nu} f_1(y)}{D^{\nu} f_2(y)} \right| \right) dy \tag{2.5}$$

holds.

Proof. Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_1(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a \leqslant y \leqslant x; \\ 0, & x < y \leqslant b. \end{cases}$$

and replacing f_i by $D^{\nu} f_i$, $i = 1,2$ we obtain (2.5).

Next, we define Caputo fractional derivative, for details see [1, p. 449]. The Caputo fractional derivative is defined as:

Let $\alpha \geqslant 0$, $n = \lceil \alpha \rceil$, $g \in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D^{\alpha}_{*a} g(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^x \frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}} dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

COROLLARY 2.4. *Let u be a weight function on (a,b) and $\alpha \geqslant 0$. $D^{\alpha}_{*a} f$ denotes the Caputo fractional derivative of f. Define $v(y)$ on (a,b) by*

$$v(y) = \frac{f_2^{(n)}(y)}{\Gamma(n-\alpha)} \int\limits_y^b \frac{u(x)(x-y)^{n-\alpha-1}}{D^{\alpha}_{*a} f_2(x)} dx < \infty$$

If $\phi : (0,\infty) \to \mathbb{R}$ *is convex and increasing, then the inequality*

$$\int_a^b u(x)\phi\left(\left|\frac{D_{*a}^{\alpha} f_1(x)}{D_{*a}^{\alpha} f2(x)}\right|\right)dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1^{(n)}(y)}{f_2^{(n)}(y)}\right|\right)dy \tag{2.6}$$

holds.

Proof. Similar to the proof of Corollary 2.3.

We continue with the following lemma that is given in [4].

LEMMA 2.3. *Let* $\nu > \gamma \geqslant 0,\ n = [\nu]+1,\ m = [\gamma]+1$ *and* $f \in AC^n([a,b])$. *Suppose that one of the following conditions hold:*

(a) $\nu,\gamma \notin \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m,...,n-1$.
(b) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m,...,n-2$.
(c) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m-1,...,n-1$.
(d) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m-1,...,n-2$.
Then

$$D_{*a}^{\gamma} f(x) = \frac{1}{\Gamma(\nu-\gamma)}\int_a^x (x-y)^{\nu-\gamma-1} D_{*a}^{\nu} f(y)dy$$

for all $a \leqslant x \leqslant b$.

COROLLARY 2.5. *Let* u *be a weight function on* (a,b) *and let assumptions in Lemma* 2.3 *be satisfied. Define* $v(y)$ *on* (a,b) *by*

$$v(y) = \frac{D_{*a}^{\nu} f_2(y)}{\Gamma(\nu-\gamma)}\int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D_{*a}^{\gamma} f_2(x)}dx < \infty$$

If $\phi : (0,\infty) \to \mathbb{R}$ *is convex and increasing, then the inequality*

$$\int_a^b u(x)\phi\left(\left|\frac{D_{*a}^{\gamma} f_1(x)}{D_{*a}^{\gamma} f_2(x)}\right|\right)dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{D_{*a}^{\nu} f_1(y)}{D_{*a}^{\nu} f_2(y)}\right|\right)dy \tag{2.7}$$

holds.

Proof. Similar to the proof of Corollary 2.3.

Now we continue with the definition of Hadamard-type fractional integrals. Let $[a,b]$ be finite or infinite interval of $\mathbb{R}_+$ and $\alpha > 0$. The left and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$(J_{a_+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y},\ \ x > a$$

and

$$(J_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{y}{x}\right)^{\alpha-1} \frac{f(y)dy}{y}, \;\; x < b$$

respectively.

COROLLARY 2.6. *Let u be a weight function and $\alpha > 0$. $J_{a_+}^{\alpha} f$ denotes the left-sided Hadamard-type fractional integral. Define*

$$v(y) = \frac{f_2(y)}{y\Gamma(\alpha)} \int_y^b u(x) \left(\log \frac{x}{y}\right)^{\alpha-1} \frac{1}{(J_{a+}^{\alpha} f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{J_{a_+}^{\alpha} f_1(x)}{J_{a_+}^{\alpha} f_2(x)}\right|\right) dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy, \tag{2.8}$$

holds.

Proof. Similar to the proof of Corollary 2.3.

Similarly we obtain the following Corollary.

COROLLARY 2.7. *Let u be a weight function and $\alpha > 0$. $J_{b_-}^{\alpha} f$ denotes the right-sided Hadamard-type fractional integral. Define*

$$v(y) = \frac{f_2(y)}{y\Gamma(\alpha)} \int_y^b u(x) \left(\log \frac{y}{x}\right)^{\alpha-1} \frac{1}{(J_{b-}^{\alpha} f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{J_{b_-}^{\alpha} f_1(x)}{J_{b_-}^{\alpha} f_2(x)}\right|\right) dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy.$$

holds.

Proof. Similar to the proof of Corollary 2.3.

Now will give the definition of Erdelyi-Kóber type fractional integrals. For details see [12] (also see [5, p, 154]).

Let $(a,b),(0 \leqslant a < b \leqslant \infty)$ be finite or infinite interval of $\mathbb{R}^+$. Let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. The left and right-sided Erdelyi-Kóber type fractional integral of order $\alpha > 0$ are defined by

$$(I^{\alpha}_{a+;\sigma;\eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)dt}{(x^{\sigma} - t^{\sigma})^{1-\alpha}}, \ \ (x > a)$$

and

$$(I^{\alpha}_{b-;\sigma;\eta} f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)dt}{(t^{\sigma} - x^{\sigma})^{1-\alpha}}, \ \ (x < b)$$

respectively.

COROLLARY 2.8. *Let u be a weight function, $I^{\alpha}_{a+;\sigma;\eta} f$ denotes the left-sided Erdelyi-Kóber type fractional integral of function f of order $\alpha > 0$. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_y^b \frac{u(x)\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^{\sigma} - y^{\sigma})^{1-\alpha} (I^{\alpha}_{a+;\sigma;\eta} f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{I^{\alpha}_{a+;\sigma;\eta} f_1(x)}{I^{\alpha}_{a+;\sigma;\eta} f_2(x)}\right|\right) dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy, \tag{2.9}$$

holds.

Proof. Similar to the proof of Corollary 2.3.

Similarly we obtain the following Corollary.

COROLLARY 2.9. *Let u be a weight function, $I^{\alpha}_{b-;\sigma;\eta} f$ denotes the right-sided Erdelyi-Kóber type fractional integral of function f. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_a^y \frac{u(x)\sigma x^{\sigma\eta} y^{\sigma(1-\alpha-\eta)-1}}{(y^{\sigma} - x^{\sigma})^{1-\alpha} (I^{\alpha}_{b-;\sigma;\eta} f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ be convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{I^{\alpha}_{b-;\sigma;\eta} f_1(x)}{I^{\alpha}_{b-;\sigma;\eta} f_2(x)}\right|\right) dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy,$$

holds.

Proof. Similar to the proof of Corollary 2.3.

In the previous corollaries we derived only inequalities over some subsets of $\mathbb{R}$. However, Theorem 1.2 covers much more general situations. We conclude this paper with multidimensional fractional integrals. Such type of fractional integrals are usually generalization of the corresponding one-dimensional fractional integral and fractional derivative.
For $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n)$, we use the following notations:

$$\Gamma(\alpha) = (\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)),\ [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

and by $\mathbf{x} > \mathbf{a}$ we mean $x_1 > a_1, ..., x_n > a_n$.
We define the mixed Riemann-Liouville fractional integrals of order $\alpha > 0$ as

$$(I^{\alpha}_{\mathbf{a}+} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} f(\mathbf{t})(\mathbf{x} - \mathbf{t})^{\alpha - 1} d\mathbf{t},\ (\mathbf{x} > \mathbf{a})$$

and

$$(I^{\alpha}_{\mathbf{b}-} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} f(\mathbf{t})(\mathbf{t} - \mathbf{x})^{\alpha - 1} d\mathbf{t},\ (\mathbf{x} < \mathbf{b}).$$

COROLLARY 2.10. *Let u be a weight function on $(\mathbf{a}, \mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{a}+} f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a}, \mathbf{b})$ by*

$$v(\mathbf{y}) := \frac{f_2(\mathbf{y})}{\Gamma(\alpha)} \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} u(\mathbf{x}) \frac{(\mathbf{x} - \mathbf{y})^{\alpha - 1}}{(I^{\alpha}_{\mathbf{a}+} f_2)(\mathbf{x})} d\mathbf{x} < \infty.$$

If $\phi : (0, \infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_{a_1}^{b_1} ... \int_{a_1}^{b_1} u(\mathbf{x}) \phi \left(\left| \frac{I^{\alpha}_{\mathbf{a}+} f_1(\mathbf{x})}{I^{\alpha}_{\mathbf{a}+} f_2(\mathbf{x})} \right| \right) d\mathbf{x} \leqslant \int_{a_1}^{b_1} ... \int_{a_1}^{b_1} v(\mathbf{y}) \phi \left(\left| \frac{f_1(\mathbf{y})}{f_2(\mathbf{y})} \right| \right) d\mathbf{y}, \tag{2.10}$$

holds.

Proof. Similar to the proof of Corollary 2.3.

Similarly we obtain the following result.

COROLLARY 2.11. *Let u be a weight function on $(\mathbf{a}, \mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{b}-} f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a}, \mathbf{b})$ by*

$$v(\mathbf{y}) := \frac{f_2(\mathbf{y})}{\Gamma(\alpha)} \int_{a_1}^{y_1} \cdots \int_{a_n}^{y_n} u(\mathbf{x}) \frac{(\mathbf{y} - \mathbf{x})^{\alpha - 1}}{(I^{\alpha}_{\mathbf{b}-} f_2)(\mathbf{x})} d\mathbf{x} < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ *is convex and increasing, then the inequality*

$$\int\limits_{a_1}^{b_1} \cdots \int\limits_{a_1}^{b_1} u(\mathbf{x})\phi\left(\left|\frac{I^{\alpha}_{\mathbf{b}_-} f_1(\mathbf{x})}{I^{\alpha}_{\mathbf{b}_-} f_2(\mathbf{x})}\right|\right) d\mathbf{x} \leqslant \int\limits_{a_1}^{b_1} \cdots \int\limits_{a_1}^{b_1} v(\mathbf{y})\phi\left(\left|\frac{f_1(\mathbf{y})}{f_2(\mathbf{y})}\right|\right) d\mathbf{y},$$

holds.

Note that Theorem 1.2 can be generalized for convex functions of several variables. We conclude this paper with the following result.

THEOREM 2.1. *Let* $g_i \in U(f_i)$, $(i = 1,2,3)$, *where* $g_2(x) > 0$ *for every* $x \in \Omega_1$. *Let* u *be a weight function on* Ω_1, k *be a non-negative measurable function on* $\Omega_1 \times \Omega_2$. *Let* v *be defined by* (1.2). *If* $\phi : (0,\infty) \to \mathbb{R}$ *is convex and increasing function, then the inequality*

$$\int\limits_{\Omega_1} u(x)\phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|, \left|\frac{g_3(x)}{g_2(x)}\right|\right) d\mu_1(x) \leqslant \int\limits_{\Omega_2} v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|, \left|\frac{f_3(y)}{f_2(y)}\right|\right) d\mu_2(y), \quad (2.11)$$

holds.

REMARK 2.2. Apply Theorem 2.1 with $\Omega_1 = \Omega_2 = [a,b]$ and $d\mu_1(x) = dx$, $d\mu_1(y) = dy$. Then

$$v(y) = f_2(y)\int_a^b \frac{u(x)k(x,y)}{g_2(x)}dx$$

and (2.11) reduces to

$$\int_a^b u(x)\phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|, \left|\frac{g_3(x)}{g_2(x)}\right|\right) dx \leqslant \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|, \left|\frac{f_3(y)}{f_2(y)}\right|\right) dy$$

This result is given in [10] (see also [11, p. 236]).

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