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ON SLATER'S INTEGRAL INEQUALITY

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Abstract. In this paper we give a generalization of results given by Pečarić and Adil (2010). We use a log-convexity criterion and establish improvements and reverses of Slater's and related inequalities.

1. Introduction

In paper [10] M. L. Slater has proved an interesting companion inequality to Jensen's inequality. Here we cite his result.

THEOREM 1. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $\phi : (a,b) \to \mathbb{R}$ *be increasing convex function defined on open interval* (a,b)*. If* $f : \Omega \to (a,b)$ *is such that* $\phi(f), \phi'_+(f)$ *and* $\phi'_+(f)f$ *are all in* $L^1(\mu)$*, then*

$$\frac{1}{\mu(\Omega)} \int_\Omega \phi(f)d\mu \leqslant \phi\left(\frac{\int_\Omega f\phi'_+(f)d\mu}{\int_\Omega \phi'_+(f)d\mu}\right) \tag{1}$$

holds whenever $\int_\Omega \phi'_+(f)d\mu > 0$*. In the case when* ϕ *is strictly convex, we have equality in* (1) *if and only if* f *is constant almost everywhere on* Ω*.*

Here ϕ'_+ denotes the right derivative of ϕ and similarly ϕ'_- denotes the left derivative of ϕ. The inequality in (1) remains valid if any occurrence of $\phi'_+(x)$ is replaced by any value from the interval $[\phi'_-(x), \phi'_+(x)]$. In [7] Pečarić noted that (1) remains true if we drop the assumption about monotonicity of ϕ, provided that

$$\int_\Omega \phi'_+(f)d\mu \neq 0, \ \frac{\int_\Omega \phi'_+(f)fd\mu}{\int_\Omega \phi'_+(f)d\mu} \in (a,b).$$

In [8] the interested reader can find the multidimensional case of Slater's inequality. For different types of converses of Jensen's inequality see [5]. The following converse of Jensen's inequality is the the integral analogue of theorem given in [4].

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THEOREM 2. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $\phi : (a,b) \to \mathbb{R}$ *be differentiable convex function defined on interval* (a,b). *If* $f : \Omega \to (a,b)$ *is such that* $f, \phi(f), \phi'(f)$ *and* $\phi'(f)f$ *are all in* $L^1(\mu)$, *and if*

$$\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu, \quad \overline{g} = \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu.$$

Then the inequalities

$$0 \leqslant \overline{g} - \phi(\overline{f}) \leqslant \frac{1}{\mu(\Omega)} \int_\Omega \phi'(f)(f - \overline{f}) d\mu \tag{2}$$

hold. In the case when ϕ *is strictly convex, we have equalities in* (2) *if and only if* f *is constant almost everywhere on* Ω.

The following inequality is due to M. Matić and J. Pečarić in [6] which implies inequalities (1) and (2).

THEOREM 3. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $\phi : (a,b) \to \mathbb{R}$ *be differentiable convex function defined on interval* (a,b). *If* $f : \Omega \to (a,b)$ *is such that* $\phi(f), \phi'(f)$ *and* $\phi'(f)f$ *are in* $L^1(\mu)$, *then for any* $d \in (a,b)$ *one has*

$$\overline{g} \leqslant \phi(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f - d)\phi'(f) d\mu, \tag{3}$$

where $\overline{g} = \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu$. *Also, when* ϕ *is strictly convex we have equality in* (3) *if and only if* $f = d$ *almost everywhere on* Ω.

REMARK 1. Let ϕ, f and $\overline{f}$ be stated as in Theorem 2, $\int_\Omega \phi'(f) d\mu \neq 0$ and let $\overline{\overline{f}} = \frac{\int_\Omega \phi'(f) f d\mu}{\int_\Omega \phi'(f) d\mu} \in (a,b)$. If we put $d = \overline{\overline{f}}$ in (3) we immediately obtain Slater's inequality (1). On the other hand, if we put $d = \overline{f}$ in (3) we immediately obtain (2).

The following refinement of (2) is also valid (see [6]).

THEOREM 4. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $\phi : (a,b) \to \mathbb{R}$ *be a differentiable, strictly convex function on interval* (a,b). *If* $f : \Omega \to (a,b)$ *is such that* $f, \phi(f), \phi'(f)$ *and* $\phi'(f)f$ *are all in* $L^1(\mu)$, *and if* $\overline{f}, \overline{g}$ *are defined as in* Theorem 2, *then there is exactly one* $\overline{d} \in (a,b)$ *such that*

$$\phi'(\overline{d}) = \frac{1}{\mu(\Omega)} \int_\Omega \phi'(f) d\mu$$

holds, and

$$\overline{g} \leqslant \phi(\overline{d}) + \frac{1}{\mu(\Omega)} \int_\Omega \phi'(f)(f - \overline{d}) d\mu, \tag{4}$$

$$0 \leqslant \overline{g} - \phi(\overline{f}) \leqslant \phi(\overline{d}) + \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f)(f - \overline{d})d\mu - \phi(\overline{f}) \leqslant \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f)(f - \overline{f})d\mu. \tag{5}$$

The third inequality in (5) *is strict unless* $\overline{f} = \overline{d}$.

This paper is organized in the following manner: in section 2 we prove mean value theorems and introduce generalized Cauchy type means. We give some important applications of these generalized Cauchy type means i.e monotonicity of these means. In section 3 we use a log-convexity criterion and obtain improvements and reverses of Slater's and related inequalities. At the end of the paper we give some determinantal inequalities.

2. Preliminary results

THEOREM 5. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$, $\phi \in C^2([a,b])$. *If* $f : \Omega \to [a,b]$ *is such that* $f, f^2, \phi(f), \phi'(f)$ *and* $\phi'(f)f$ *are all in* $L^1(\mu)$, $d \in [a,b]$ *with* $f \neq d$ *a.e. on* Ω *and* $\overline{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f)d\mu$. *Then there exists* $\xi \in [a,b]$ *such that*

$$\phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d)\phi'(f)d\mu - \overline{g} = \frac{\phi''(\xi)}{2} \int_{\Omega} (f - d)^2 d\mu. \tag{6}$$

Proof. The proof is analogous to the proof of Theorem 2.1 in [1].

COROLLARY 1. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$. *Let* $\phi, f, \overline{g}$ *be defined as in* Theorem 5 *and* $\overline{f} = \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu$ *with* $f \neq \overline{f}$ *a.e. on* Ω. *Then there exists* $\xi \in [a,b]$ *such that*

$$\phi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \overline{f})\phi'(f)d\mu - \overline{g} = \frac{\phi''(\xi)}{2} \int_{\Omega} \left(f - \overline{f}\right)^2 d\mu. \tag{7}$$

Proof. By setting $d = \overline{f}$ in Theorem 5, we get (7).

THEOREM 6. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$, $\phi, \psi \in C^2([a,b])$. *If* $f : \Omega \to [a,b]$ *is such that* $f, f^2, \phi(f), \phi'(f), \phi'(f)f, \psi(f), \psi'(f)$ *and* $\psi'(f)f$ *are all in* $L^1(\mu)$ *and* $d \in [a,b]$ *with* $f \neq d$ *a.e. on* Ω. *Then there exists* $\xi \in [a,b]$ *such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d)\phi'(f)d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f)d\mu}{\psi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d)\psi'(f)d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \psi(f)d\mu}, \tag{8}$$

provided that the denominators are non zero.

Proof. The proof is analogous to the proof of Theorem 2.3 in [1].

COROLLARY 2. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$. *Let* ϕ, ψ, f *be defined as in Theorem* 6 *and* $\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$ *with* $f \neq \overline{f}$ *a.e. on* Ω. *Then there exists* $\xi \in [a,b]$ *such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_\Omega (f - \overline{f}) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu}{\psi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_\Omega (f - \overline{f}) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \psi(f) d\mu}, \tag{9}$$

provided that the denominators are non zero.

Proof. By setting $d = \overline{f}$ in Theorem 6, we get (9).

COROLLARY 3. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$. *Let* $f : \Omega \to [a,b]$ *be such that* $f^t \in L^1(\mu)$ *for* $t \in \mathbb{R}$ *and* $d \in [a,b]$ *with* $f \neq d$ *a.e. on* Ω. *Then for* $u, v \in \mathbb{R} \setminus \{0,1\}$, $u \neq v$, *there exists* $\xi \in [a,b]$, *where* $[a,b]$ *is positive closed interval, such that*

$$\xi^{u-v} = \frac{v(v-1)[d^u + \frac{u}{\mu(\Omega)} \int_\Omega (f-d) f^{u-1} d\mu - \frac{1}{\mu(\Omega)} \int_\Omega f^u d\mu]}{u(u-1)[d^v + \frac{v}{\mu(\Omega)} \int_\Omega (f-d) f^{v-1} d\mu - \frac{1}{\mu(\Omega)} \int_\Omega f^v d\mu]}, \tag{10}$$

provided that the denominator is non zero.

Proof. To get (10), substitute $\phi(x) = x^u$ and $\psi(x) = x^v$, $x \in [a,b]$, in (8).

COROLLARY 4. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$. *Let* $f : \Omega \to [a,b]$ *be such that* $f^t \in L^1(\mu)$, *for* $t \in \mathbb{R}$ *and* $\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$ *with* $f \neq \overline{f}$ *a.e. on* Ω. *Then for* $u, v \in \mathbb{R} \setminus \{0,1\}$, $u \neq v$, *there exists* $\xi \in [a,b]$, *where* $[a,b]$ *is positive closed interval, such that*

$$\xi^{u-v} = \frac{v(v-1)[\overline{f}^u + \frac{u}{\mu(\Omega)} \int_\Omega (f-\overline{f}) f^{u-1} d\mu - \frac{1}{\mu(\Omega)} \int_\Omega f^u d\mu]}{u(u-1)[\overline{f}^v + \frac{v}{\mu(\Omega)} \int_\Omega (f-\overline{f}) f^{v-1} d\mu - \frac{1}{\mu(\Omega)} \int_\Omega f^v d\mu]}, \tag{11}$$

provided that the denominator is non zero.

Proof. To get (11), substitute $\phi(x) = x^u$ and $\psi(x) = x^v$, $x \in [a,b]$, in (9).

Now we are able to introduce generalized Cauchy means from (8) and (9). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function, then from (8) and (9) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\phi(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f-d) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu}{\psi(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f-d) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \psi(f) d\mu} \right). \tag{12}$$

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\phi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_\Omega (f-\overline{f}) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu}{\psi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_\Omega (f-\overline{f}) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \psi(f) d\mu} \right). \tag{13}$$

REMARK 2. Since the function $\xi \to \xi^{u-v}$ with $u \neq v$ is invertible, from (10) and (11) we have

$$a \leqslant \left\{ \frac{v(v-1)[d^u + \frac{u}{\mu(\Omega)}\int_\Omega (f-d) f^{u-1} d\mu - \frac{1}{\mu(\Omega)}\int_\Omega f^u d\mu]}{u(u-1)[d^v + \frac{v}{\mu(\Omega)}\int_\Omega (f-d) f^{v-1} d\mu - \frac{1}{\mu(\Omega)}\int_\Omega f^v d\mu]} \right\}^{\frac{1}{u-v}} \leqslant b. \tag{14}$$

$$a \leqslant \left\{ \frac{v(v-1)[\overline{f}^u + \frac{u}{\mu(\Omega)}\int_\Omega (f-\overline{f}) f^{u-1} d\mu - \frac{1}{\mu(\Omega)}\int_\Omega f^u d\mu]}{u(u-1)[\overline{f}^v + \frac{v}{\mu(\Omega)}\int_\Omega (f-\overline{f}) f^{v-1} d\mu - \frac{1}{\mu(\Omega)}\int_\Omega f^v d\mu]} \right\}^{\frac{1}{u-v}} \leqslant b. \tag{15}$$

We shall say that the expressions in the middle of (14) and (15) define a class of means.

To define Cauchy type means, the following family of convex functions will be useful.

LEMMA 1. ([3]) *Let us define the functions* $\varphi_t : (0,\infty) \to \mathbb{R}$

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}\,, & t \neq 0,1; \\ -\log x\,, & t = 0; \\ x \log x\,, & t = 1. \end{cases} \tag{16}$$

Then $\varphi_t''(x) = x^{t-2}$, *that is* φ_t *is convex for* $x > 0$.

DEFINITION 1. ([2]) A function $\phi : [a,b] \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{k,l=1}^{n} a_k a_l \phi(x_k + x_l) \geqslant 0,$$

for all $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $x_k \in [a,b]$, $k = 1,2,..,n$ such that $x_k + x_l \in [a,b], 1 \leqslant k,l \leqslant n$, or equivalently

$$\sum_{k,l=1}^{n} a_k a_l \phi\left(\frac{x_k + x_l}{2}\right) \geqslant 0.$$

COROLLARY 5. ([2]) *If* ϕ *is exponentially convex function, then*

$$\det\left[\phi\left(\frac{x_k + x_l}{2}\right)\right]_{k,l=1}^{n} \geqslant 0$$

for every $n \in \mathbb{N}$ $x_k \in [a,b]$, $k = 1,2,..,n$.

COROLLARY 6. ([2]) *If* $\phi : [a,b] \to (0,\infty)$ *is exponentially convex function, then* ϕ *is a* log*-convex function that is*

$$\phi(\lambda x + (1-\lambda)y) \leqslant \phi^{\lambda}(x)\phi^{1-\lambda}(y), \text{ for all } x,y \in [a,b],\ \lambda \in [0,1].$$

We define Cauchy type means for the family of functions φ_t.

THEOREM 7. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and* $f : \Omega \to \mathbb{R}^+$ *be such that* $f^t, \log f, f\log f$ *are all in* $\in L^1(\mu)$, $t \in \mathbb{R}$ *and let* $d \in \mathbb{R}^+$ *with* $f \neq d$ *a.e. on* Ω. *Consider* Υ_t *to be defined by*

$$\Upsilon_t = \varphi_t(d) + \frac{1}{\mu(\Omega)}\int_\Omega (f-d)\varphi_t'(f)d\mu - \frac{1}{\mu(\Omega)}\int_\Omega \varphi_t(f)d\mu. \tag{17}$$

Then:

(i) *for every* $n \in \mathbb{N}$ *and for every* $s_k \in \mathbb{R}$, $k \in \{1,2,3,..,n\}$, *the matrix* $[\Upsilon_{\frac{s_k+s_l}{2}}]_{k,l=1}^n$ *is a positive semi-definite matrix,*

(ii) *the function* $t \to \Upsilon_t$ *is exponentially convex,*

(iii) *the function* $t \to \Upsilon_t$ *is* $\log$*-convex.*

Proof. (i) As in [1] we can show that the function defined by

$$\mu(x) = \sum_{k,l=1}^{n} a_k a_l \varphi_{s_{kl}}(x)$$

where $s_{kl} = \frac{s_k+s_l}{2}, a_k \in \mathbb{R}$ for all $k \in \{1,2,3,..,n\}$, $x > 0$, is convex.

So by using $\mu(x)$ in (3), we get

$$\sum_{k,l=1}^{n} a_k a_l \Upsilon_{s_{kl}} \geqslant 0, \tag{18}$$

hence the matrix $[\Upsilon_{\frac{s_k+s_l}{2}}]_{k,l=1}^n$ is a positive semi-definite.

(ii) Since $lim_{t\to 0}\Upsilon_t = \Upsilon_0$ and $lim_{t\to 1}\Upsilon_t = \Upsilon_1$, we conclude that Υ_t is continuous for all $t \in \mathbb{R}, x > 0$ also by (i), $[\Upsilon_{\frac{s_k+s_l}{2}}]_{k,l=1}^n$ is positive semi-definite matrix, so using Definition 1, we have that exponentially convexity of the function $t \to \Upsilon_t$.

(iii) Since $\Upsilon_t > 0$ as $f \neq d$ $a.e.$ on Ω and φ_t is strictly convex (see Theorem 3) using Corollary 6 we obtain that Υ_t is $\log$-convex.

COROLLARY 7. *Let* (Ω, A, μ) *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $f : \Omega \to \mathbb{R}^+$ *be a non constant a.e. on* Ω *with* $f^t, \log f, f\log f$ *are all in* $L^1(\mu)$, *for* $t \in \mathbb{R}$ *and* $\overline{f} = \frac{1}{\mu(\Omega)}\int_\Omega f d\mu$. *Consider* $\tilde{\Upsilon}_t$ *to be defined by*

$$\tilde{\Upsilon}_t = \varphi_t(\overline{f}) + \frac{1}{\mu(\Omega)}\int_\Omega (f-\overline{f})\varphi_t'(f)d\mu - \frac{1}{\mu(\Omega)}\int_\Omega \varphi_t(f)d\mu. \tag{19}$$

Then:

(i) *for every* $n \in \mathbb{N}$ *and for every* $s_k \in \mathbb{R}$, $k \in \{1,2,3,..,n\}$, *the matrix* $[\tilde{\Upsilon}_{\frac{s_k+s_l}{2}}]_{k,l=1}^n$ *is a positive semi-definite matrix,*

(ii) *the function* $t \to \tilde{\Upsilon}_t$ *is exponentially convex,*

(iii) *the function* $t \to \tilde{\Upsilon}_t$ *is* $\log$*-convex.*

Proof. To get the required results set $d = \overline{f}$ in Theorem 7.

Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$. The integral power mean is defined as follows:

$$M_t(f;\mu) = \begin{cases} \left(\frac{1}{\mu(\Omega)} \int_\Omega f^t \, d\mu \right)^{\frac{1}{t}}, & \text{for } t \neq 0; \\ \exp\left(\frac{1}{\mu(\Omega)} \int_\Omega \log(f) d\mu \right), & \text{for } t = 0. \end{cases}$$

Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and f, d be defined as in Theorem 7 with $f \neq d$ a.e. on Ω. We give the following definition

$$Q_{u,v} = \left(\frac{\Upsilon_u}{\Upsilon_v} \right)^{\frac{1}{u-v}} \quad \text{for } u, v \in \mathbb{R} \text{ such that } u \neq v, \tag{20}$$

where Υ_t is defined by (17). By Remark 2 these expressions define a class of means. Moreover, we can extend these means to the other cases. Namely, for $u \neq 0, 1$ by limit we have

$$Q_{u,u} = \exp\left(\frac{\mu(\Omega) d^u \log d + (u-1) \int_\Omega f^u \log f \, d\mu + \mu(\Omega) M_u^u(f;\mu) - d(u \int_\Omega f^{u-1} \log f \, d\mu + \mu(\Omega) M_{u-1}^{u-1}(f;\mu))}{\mu(\Omega)[d^u + (u-1) M_u^u(f;\mu) - du M_{u-1}^{u-1}(f;\mu)]} \right.$$
$$\left. - \frac{2u-1}{u(u-1)} \right), \; u \neq 0, 1,$$
$$Q_{0,0} = \exp\left(\frac{\mu(\Omega) \log^2 d - \mu(\Omega) M_2^2(\log f;\mu) + 2\mu(\Omega) \log M_0(f;\mu) - 2d \int_\Omega f^{-1} \log f \, d\mu}{2\mu(\Omega)[d \log d - \log M_0(f) + 1 - d M_{-1}^{-1}(f;\mu))]} + 1 \right),$$
$$Q_{1,1} = \exp\left(\frac{\mu(\Omega) d \log^2 d + 2 \int_\Omega f \log f \, d\mu - d\mu(\Omega)(M_2^2(\log f;\mu) - 2 \log M_0(f;\mu))}{2[\mu(\Omega) d(\log d - 1) + \mu(\Omega)\overline{f} - d\mu(\Omega) \log M_0(f;\mu)]} - 1 \right).$$

THEOREM 8. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leqslant u, s \leqslant v$. Then the following inequality is valid.*

$$Q_{t,s} \leqslant Q_{u,v}. \tag{21}$$

Proof. The proof is analogous to the proof of Theorem 3.8 in [1].

Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and f be as in Theorem 7 with $f \neq \overline{f}$ a.e. on Ω. We give the following definition.

$$\tilde{Q}_{u,v} = \left(\frac{\tilde{\Upsilon}_u}{\tilde{\Upsilon}_v} \right)^{\frac{1}{u-v}} \quad \text{for } u, v \in \mathbb{R} \text{ such that } u \neq v, \tag{22}$$

where $\tilde{\Upsilon}_t$ is defined by (19). By Remark 2 these expressions define a class of means. We can extend these means to the other cases. Namely, for $u \neq 0, 1$ by limit we have

$$\tilde{Q}_{u,u} = \exp\left(\frac{\mu(\Omega)\overline{f}^u\log\overline{f}+(u-1)\int_\Omega f^u\log f d\mu+\mu(\Omega)M_u^u(f;\mu)-\overline{f}(u\int_\Omega f^{u-1}\log f d\mu+\mu(\Omega)M_{u-1}^{u-1}(f;\mu))}{\mu(\Omega)[\overline{f}^u+(u-1)M_u^u(f;\mu)-\overline{f}uM_{u-1}^{u-1}(f;\mu)]}\right.$$
$$\left. - \frac{2u-1}{u(u-1)}\right),\ u \neq 0,1,$$
$$\tilde{Q}_{0,0} = \exp\left(\frac{\mu(\Omega)\log^2\overline{f}-\mu(\Omega)M_2^2(\log f;\mu)+2\mu(\Omega)\log M_0(f;\mu)-2\overline{f}\int_\Omega f^{-1}\log f d\mu}{2\mu(\Omega)[\overline{f}\log\overline{f}-\log M_0(f)+1-\overline{f}M_{-1}^{-1}(f;\mu))]}+1\right),$$
$$\tilde{Q}_{1,1} = \exp\left(\frac{\mu(\Omega)\overline{f}\log^2\overline{f}+2\int_\Omega f\log f d\mu-\overline{f}\mu(\Omega)(M_2^2(\log f;\mu)-2\log M_0(f;\mu))}{2[\mu(\Omega)\overline{f}(\log\overline{f}-1)+\mu(\Omega)\overline{f}-\overline{f}\mu(\Omega)\log M_0(f;\mu)]}-1\right).$$

THEOREM 9. *Let $t,s,u,v\in\mathbb{R}$ such that $t\leqslant u, s\leqslant v$. Then the following inequality is valid.*

$$\tilde{Q}_{t,s} \leqslant \tilde{Q}_{u,v}. \tag{23}$$

Proof. The proof is analogous to the proof of Theorem 3.8 in [1].

3. Improvement and reversion of Slater's inequality

Let $M_t(f;\mu)$ be stated as above and define d_t as

$$d_t = \frac{\int_\Omega f\varphi_t'(f)d\mu}{\int_\Omega \varphi_t'(f)d\mu} = \begin{cases} \frac{M_t^t(f;\mu)}{M_{t-1}^{t-1}(f;\mu)}, & t\neq 0,1; \\ M_{-1}(f;\mu), & t=0; \\ \frac{\overline{f}+\int_\Omega (f\log f)d\mu}{1+\log M_0(f;\mu)}, & t=1. \end{cases} \tag{24}$$

The following improvement and reverse of Slater's inequality is valid.

THEOREM 10. *Let (Ω,A,μ) be a measure space with $0<\mu(\Omega)<\infty$. Let $f:\Omega\to\mathbb{R}^+$ be such that $f^t,\log f, f\log f\in L^1(\mu)$ and $d_t\in\mathbb{R}^+$, $t\in\mathbb{R}$. Consider Λ_t to be defined by*

$$\Lambda_t = \varphi_t(d_t) - \frac{1}{\mu(\Omega)}\int_\Omega \varphi_t(f)d\mu. \tag{25}$$

Then:

(*i*) $\Lambda_t \geqslant [H(s;t)]^{\frac{t-r}{s-r}}[H(r;t)]^{\frac{s-t}{s-r}}$ (26)

for $-\infty<r<s<t<\infty$ and $-\infty<t<r<s<\infty$.

(*ii*) $\Lambda_t \leqslant [H(s;t)]^{\frac{t-r}{s-r}}[H(r;t)]^{\frac{s-t}{s-r}}$ (27)

where $$H(s;t) = \varphi_s(d_t) + \frac{1}{\mu(\Omega)}\int_\Omega (f-d_t)\varphi_s'(f)d\mu - \frac{1}{\mu(\Omega)}\int_\Omega \varphi_s(f)d\mu. \tag{28}$$

Proof. (i) By substituting $d = d_t$ in (17) Υ_t becomes Λ_t. So by using $d = d_t$ in log-convexity criterion for Υ_t as in the proof of Theorem 3.10(i) [1] we get

$$(\Lambda_t)^{s-r} \geqslant (H(s;t))^{t-r}\,(H(r;t))^{s-t}, \quad \text{for } -\infty < r < s < t < \infty, \tag{29}$$

which is equivalent to (26).

Similarly we can prove (26) for the case $-\infty < t < r < s < \infty$.

(ii) Similar procedure can be applied here.

THEOREM 11. *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$. Let $f : \Omega \to \mathbb{R}^+$ be such that $f^t, \log f, f\log f \in L^1(\mu)$ and $d_t \in \mathbb{R}^+$, $t \in \mathbb{R}$. Then for every $n \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1,2,3,..,n\}$, the matrices $[H(\frac{s_k+s_l}{2}, s_1)]_{k,l=1}^n$, $[H(\frac{s_k+s_l}{2}, \frac{s_1+s_2}{2})]_{k,l=1}^n$ are positive semi-definite matrices. Particularly*

$$\det[H(\frac{s_k+s_l}{2}, s_1)]_{k,l=1}^n \geqslant 0, \tag{30}$$

$$\det[H(\frac{s_k+s_l}{2}, \frac{s_1+s_2}{2})]_{k,l=1}^n \geqslant 0, \tag{31}$$

where $H(s,t)$ is defined by (28).

Proof. By setting $d = d_{s_1}$ and $d = d_{\frac{s_1+s_2}{2}}$ in Theorem 7(i) we get the required results.

REMARK 3. We note that $H(t,t) = F_t$. So by setting $n = 2$ in (30) we have special case of (26) for $t = s_1, s = s_2, r = \frac{s_1+s_2}{2}$ if $s_1 < s_2$ and for $t = s_1, r = s_2, s = \frac{s_1+s_2}{2}$ if $s_2 < s_1$. Similarly by setting $n = 2$ in (31) we have special case of (27) for $r = s_1, s = s_2, t = \frac{s_1+s_2}{2}$ if $s_1 < s_2$ and for $r = s_2, s = s_1, t = \frac{s_1+s_2}{2}$ if $s_2 < s_1$.

Let $M_t(f;\mu)$ be stated as above and define $\overline{d}_t$ as

$$\overline{d}_t = M_{t-1}(f;\mu). \tag{32}$$

The following improvement and reverse of inequality in (4) is also valid.

THEOREM 12. *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and $f : \Omega \to \mathbb{R}^+$ be such that $f^t, \log f, f\log f \in L^1(\mu)$ and $\overline{d}_t \in \mathbb{R}^+, t \in \mathbb{R}$. Consider $\tilde{\Lambda}_t$ to be defined by*

$$\tilde{\Lambda}_t = \varphi_t(\overline{d}_t) + \frac{1}{\mu(\Omega)}\int_\Omega (f - \overline{d}_t)\varphi_t'(f)\,d\mu - \frac{1}{\mu(\Omega)}\int_\Omega \varphi_t(f)\,d\mu. \tag{33}$$

Then:

(i) $\tilde{\Lambda}_t \geqslant [K(s;t)]^{\frac{t-r}{s-r}}[K(r;t)]^{\frac{s-t}{s-r}}$ (34)

for $-\infty < r < s < t < \infty$ and $-\infty < t < r < s < \infty$

(ii) $\tilde{\Lambda}_t \leqslant [K(s;t)]^{\frac{t-r}{s-r}}[K(r;t)]^{\frac{s-t}{s-r}}$ (35)

where $K(s;t) = \varphi_s(\overline{d}_t) + \frac{1}{\mu(\Omega)}\int_\Omega (f - \overline{d}_t)\varphi_s'(f)\,d\mu - \frac{1}{\mu(\Omega)}\int_\Omega \varphi_s(f)\,d\mu.$ (36)

Proof. The proof is similar to the proof of Theorem 10 but use $\overline{d}_t$ instead of d_t.

THEOREM 13. *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and $f : \Omega \to \mathbb{R}^+$ be such that $f^t, \log f, f \log f \in L^1(\mu)$, $t \in \mathbb{R}$ and let $\overline{d}_t \in \mathbb{R}^+$. Then for every $n \in \mathbb{N}$ and for every $s_k \in \mathbb{R}, k \in \{1,2,3,..,n\}$, the matrices $[K(\frac{s_k+s_l}{2}, s_1)]_{k,l=1}^n$, $[K(\frac{s_k+s_l}{2}, \frac{s_1+s_2}{2})]_{k,l=1}^n$ are positive semi-definite matrices. Particularly*

$$\det\left[K\left(\frac{s_k+s_l}{2}, s_1\right)\right]_{k,l=1}^{n} \geqslant 0, \tag{37}$$

$$\det\left[K\left(\frac{s_k+s_l}{2}, \frac{s_1+s_2}{2}\right)\right]_{k,l=1}^{n} \geqslant 0, \tag{38}$$

where $K(s,t)$ is defined by (36).

Proof. By setting $d = \overline{d}_{s_1}$ and $d = \overline{d}_{\frac{s_1+s_2}{2}}$ in Theorem 7(i) we get the required results.

REMARK 4. We note that $K(t,t) = \tilde{\Lambda}_t$. So by setting $n = 2$ in (37) and in (38) we have special case of (34) and (35) respectively (see Remark 3).

REMARK 5. The results stated in [1] can be obtained from this paper if you choose the discrete measure.

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