ON SLATER’S INTEGRAL INEQUALITY

M. ADIL KHAN AND J. PEČARIĆ

Abstract. In this paper we give a generalization of results given by Pečarić and Adil (2010). We use a log-convexity criterion and establish improvements and reverses of Slater’s and related inequalities.

1. Introduction

In paper [10] M. L. Slater has proved an interesting companion inequality to Jensen’s inequality. Here we cite his result.

THEOREM 1. Let \((\Omega, A, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\) and let \(\phi : (a,b) \to \mathbb{R}\) be increasing convex function defined on open interval \((a,b)\). If \(f : \Omega \to (a,b)\) is such that \(\phi(f), \phi'_+(f)\) and \(\phi'_-(f)f\) are all in \(L^1(\mu)\), then

\[
\frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu \leq \phi \left( \frac{\int_\Omega f \phi'_+(f) d\mu}{\int_\Omega \phi'_+(f) d\mu} \right)
\]

(1)

holds whenever \(\int_\Omega \phi'_+(f) d\mu > 0\). In the case when \(\phi\) is strictly convex, we have equality in (1) if and only if \(f\) is constant almost everywhere on \(\Omega\).

Here \(\phi'_+\) denotes the right derivative of \(\phi\) and similarly \(\phi'_-\) denotes the left derivative of \(\phi\). The inequality in (1) remains valid if any occurrence of \(\phi'_+(x)\) is replaced by any value from the interval \([\phi'_-(x), \phi'_+(x)]\). In [7] Pečarić noted that (1) remains true if we drop the assumption about monotonicity of \(\phi\), provided that

\[
\int_\Omega \phi'_+(f) d\mu \neq 0, \quad \frac{\int_\Omega f \phi'_+(f) d\mu}{\int_\Omega \phi'_+(f) d\mu} \in (a,b).
\]

In [8] the interested reader can find the multidimensional case of Slater’s inequality. For different types of converses of Jensen’s inequality see [5]. The following converse of Jensen’s inequality is the integral analogue of theorem given in [4].


Keywords and phrases: Slater’s integral inequality, convex function, exponential convexity, Cauchy means.

The research of the first and second author were funded by Higher Education Commission Pakistan. The research of the second author was supported by the Croatian Ministry of Science, Education and Sports under the Research Grants 117-1170889-0888.
THEOREM 2. Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\) and let \(\phi : (a, b) \to \mathbb{R}\) be differentiable convex function defined on interval \((a, b)\). If \(f : \Omega \to (a, b)\) is such that \(f, \phi(f), \phi'(f)\) and \(\phi'(f)f\) are all in \(L^1(\mu)\), and if

\[
\bar{f} = \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu, \quad \overline{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu.
\]

Then the inequalities

\[
0 \leq \overline{g} - \phi(\bar{f}) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f)(f - \bar{f}) d\mu
\]  

(2)

hold. In the case when \(\phi\) is strictly convex, we have equalities in (2) if and only if \(f\) is constant almost everywhere on \(\Omega\).

The following inequality is due to M. Matič and J. Pečarić in [6] which implies inequalities (1) and (2).

THEOREM 3. Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\) and let \(\phi : (a, b) \to \mathbb{R}\) be differentiable convex function defined on interval \((a, b)\). If \(f : \Omega \to (a, b)\) is such that \(\phi(f), \phi'(f)\) and \(\phi'(f)f\) are in \(L^1(\mu)\), then for any \(d \in (a, b)\) one has

\[
\overline{g} \leq \phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d) \phi'(f) d\mu,
\]

(3)

where \(\overline{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu\). Also, when \(\phi\) is strictly convex we have equality in (3) if and only if \(f = d\) almost everywhere on \(\Omega\).

REMARK 1. Let \(\phi, f\) and \(\bar{f}\) be stated as in Theorem 2, \(\int_{\Omega} \phi'(f) d\mu \neq 0\) and let

\[
\overline{f} = \frac{\int_{\Omega} \phi'(f) f d\mu}{\int_{\Omega} \phi'(f) d\mu} \in (a, b).
\]

If we put \(d = \overline{f}\) in (3) we immediately obtain Slater’s inequality (1). On the other hand, if we put \(d = \overline{f}\) in (3) we immediately obtain (2).

The following refinement of (2) is also valid (see [6]).

THEOREM 4. Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\) and let \(\phi : (a, b) \to \mathbb{R}\) be a differentiable, strictly convex function on interval \((a, b)\). If \(f : \Omega \to (a, b)\) is such that \(f, \phi(f), \phi'(f)\) and \(\phi'(f)f\) are all in \(L^1(\mu)\), and if \(\overline{f}, \overline{g}\) are defined as in Theorem 2, then there is exactly one \(\overline{d} \in (a, b)\) such that

\[
\phi'(\overline{d}) = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f) d\mu
\]

holds, and

\[
\overline{g} \leq \phi(\overline{d}) + \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f)(f - \overline{d}) d\mu,
\]

(4)
0 \leq \overline{g} - \phi(\overline{f}) \leq \phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f-f-d) d\mu - \phi(\overline{f}) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} \phi'(f-\overline{f}) d\mu. 

(5)

The third inequality in (5) is strict unless \( \overline{f} = \overline{d} \).

This paper is organized in the following manner: in section 2 we prove mean value theorems and introduce generalized Cauchy type means. We give some important applications of these generalized Cauchy type means i.e. monotonicity of these means. In section 3 we use a log-convexity criterion and obtain improvements and reverses of Slater’s and related inequalities. At the end of the paper we give some determinantal inequalities.

2. Preliminary results

THEOREM 5. Let \((\Omega,A,\mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\), \(\phi \in C^2([a,b])\). If \(f : \Omega \to [a,b]\) is such that \(f, f^2, \phi(f), \phi'(f)\) and \(\phi'(f)f\) are all in \(L^1(\mu)\), \(d \in [a,b]\) with \(f \neq d\) a.e. on \(\Omega\) and \(\overline{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu\). Then there exists \(\xi \in [a,b]\) such that

\[
\phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f-d) \phi'(f) d\mu - \overline{g} = \frac{\phi''(\xi)}{2} \int_{\Omega} (f-d)^2 d\mu.
\]

Proof. The proof is analogous to the proof of Theorem 2.1 in [1].

COROLLARY 1. Let \((\Omega,A,\mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\). Let \(\phi, f, \overline{g}\) be defined as in Theorem 5 and \(\overline{f} = \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu\) with \(f \neq \overline{f}\) a.e. on \(\Omega\). Then there exists \(\xi \in [a,b]\) such that

\[
\phi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_{\Omega} (\overline{f}-f) \phi'(f) d\mu - \overline{g} = \frac{\phi''(\xi)}{2} \int_{\Omega} (\overline{f}-f)^2 d\mu.
\]

Proof. By setting \(d = \overline{f}\) in Theorem 5, we get (7).

THEOREM 6. Let \((\Omega,A,\mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\), \(\phi, \psi \in C^2([a,b])\). If \(f : \Omega \to [a,b]\) is such that \(f, f^2, \phi(f), \phi'(f), \phi'(f)f, \psi(f), \psi'(f)\) and \(\psi'(f)f\) are all in \(L^1(\mu)\) and \(d \in [a,b]\) with \(f \neq d\) a.e. on \(\Omega\). Then there exists \(\xi \in [a,b]\) such that

\[
\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f-d) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu}{\psi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f-d) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \psi(f) d\mu},
\]

provided that the denominators are non-zero.

Proof. The proof is analogous to the proof of Theorem 2.3 in [1].
COROLLARY 2. Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\). Let \(\phi, \psi, f\) be defined as in Theorem 6 and \(\overline{f} = \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu\) with \(f \neq \overline{f}\) a.e. on \(\Omega\). Then there exists \(\xi \in [a, b]\) such that

\[
\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\phi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \overline{f}) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu}{\psi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \overline{f}) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \psi(f) d\mu},
\]

provided that the denominators are non zero.

Proof. By setting \(d = \overline{f}\) in Theorem 6, we get (9).

COROLLARY 3. Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\). Let \(f : \Omega \to [a, b]\) be such that \(f' \in L^1(\mu)\) for \(t \in \mathbb{R}\) and \(d \in [a, b]\) with \(f \neq d\) a.e. on \(\Omega\). Then for \(u, v \in \mathbb{R} \smallsetminus \{0, 1\}\), \(u \neq v\), there exists \(\xi \in [a, b]\), where \([a, b]\) is positive closed interval, such that

\[
\xi^{u-v} = \frac{v(v-1)[d^u + \frac{u}{\mu(\Omega)} \int_{\Omega} (f - d) f'^{u-1} d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} f^u d\mu]}{u(u-1)[d^v + \frac{v}{\mu(\Omega)} \int_{\Omega} (f - d) f'^{v-1} d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} f^v d\mu]},
\]

provided that the denominator is non zero.

Proof. To get (10), substitute \(\phi(x) = x^u\) and \(\psi(x) = x^v\), \(x \in [a, b]\), in (8).

COROLLARY 4. Let \((\Omega, \Lambda, \mu)\) be a measure space with \(0 < \mu(\Omega) < \infty\). Let \(f : \Omega \to [a, b]\) be such that \(f' \in L^1(\mu)\), for \(t \in \mathbb{R}\) and \(\overline{f} = \frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu\) with \(f \neq \overline{f}\) a.e. on \(\Omega\). Then for \(u, v \in \mathbb{R} \smallsetminus \{0, 1\}\), \(u \neq v\), there exists \(\xi \in [a, b]\), where \([a, b]\) is positive closed interval, such that

\[
\xi^{u-v} = \frac{v(v-1)[\overline{f}^u + \frac{u}{\mu(\Omega)} \int_{\Omega} (f - \overline{f}) f'^{u-1} d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} f^u d\mu]}{u(u-1)[\overline{f}^v + \frac{v}{\mu(\Omega)} \int_{\Omega} (f - \overline{f}) f'^{v-1} d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} f^v d\mu]},
\]

provided that the denominator is non zero.

Proof. To get (11), substitute \(\phi(x) = x^u\) and \(\psi(x) = x^v\), \(x \in [a, b]\), in (9).

Now we are able to introduce generalized Cauchy means from (8) and (9). Namely, suppose that \(\frac{\phi''}{\psi''}\) has inverse function, then from (8) and (9) we have

\[
\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\phi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu}{\psi(d) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \psi(f) d\mu}\right),
\]

\[
\xi = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{\phi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \overline{f}) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \phi(f) d\mu}{\psi(\overline{f}) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \overline{f}) \psi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \psi(f) d\mu}\right).
\]
Remark 2. Since the function \( \xi \rightarrow \xi^{u-v} \) with \( u \neq v \) is invertible, from (10) and (11) we have

\[
a \leq \left\{ \frac{v(v-1)[d^u + \frac{u}{\mu(\Omega)} \int_{\Omega} (f - d) f^{u-1} d\mu] - \frac{1}{\mu(\Omega)} \int_{\Omega} f^u d\mu}{u(u-1)[d^v + \frac{v}{\mu(\Omega)} \int_{\Omega} (f - d) f^{v-1} d\mu] - \frac{1}{\mu(\Omega)} \int_{\Omega} f^v d\mu} \right\}^{\frac{1}{u-v}} \leq b.
\]

We shall say that the expressions in the middle of (14) and (15) define a class of means.

To define Cauchy type means, the following family of convex functions will be useful.

**Lemma 1.** ([3]) Let us define the functions \( \phi_t : (0, \infty) \rightarrow \mathbb{R} \)

\[
\phi_t(x) = \begin{cases} 
\frac{x^t}{t(t-1)}, & t \neq 0, 1; \\
-x \log x, & t = 0; \\
x \log x, & t = 1.
\end{cases}
\]  

Then \( \phi''_t(x) = x^{t-2} \), that is \( \phi_t \) is convex for \( x > 0 \).

**Definition 1.** ([2]) A function \( \phi : [a, b] \rightarrow \mathbb{R} \) is exponentially convex if it is continuous and

\[
\sum_{k,l=1}^n a_k a_l \phi(x_k + x_l) \geq 0,
\]

for all \( n \in \mathbb{N} \), \( a_k \in \mathbb{R} \) and \( x_k \in [a, b] \), \( k = 1, 2, \ldots, n \) such that \( x_k + x_l \in [a, b] \), \( 1 \leq k, l \leq n \), or equivalently

\[
\sum_{k,l=1}^n a_k a_l \phi\left(\frac{x_k + x_l}{2}\right) \geq 0.
\]

**Corollary 5.** ([2]) If \( \phi \) is exponentially convex function, then

\[
\det \left[ \phi\left(\frac{x_k + x_l}{2}\right) \right]_{k,l=1}^n \geq 0
\]

for every \( n \in \mathbb{N} \) \( x_k \in [a, b] \), \( k = 1, 2, \ldots, n \).

**Corollary 6.** ([2]) If \( \phi : [a, b] \rightarrow (0, \infty) \) is exponentially convex function, then \( \phi \) is a log-convex function that is

\[
\phi(\lambda x + (1-\lambda)y) \leq \phi^\lambda(x)\phi^{1-\lambda}(y), \text{ for all } x, y \in [a, b], \lambda \in [0, 1].
\]

We define Cauchy type means for the family of functions \( \phi_t \).
\textbf{Theorem 7.} Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and $f : \Omega \to \mathbb{R}^+$ be such that $f^t, \log f, f \log f$ are all in $L^1(\mu)$, $t \in \mathbb{R}$ and let $d \in \mathbb{R}^+$ with $f \neq d$ a.e. on $\Omega$. Consider $\Upsilon_t$ to be defined by

$$\Upsilon_t = \phi_t(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f - d) \varphi_t'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \varphi_t(f) d\mu. \quad (17)$$

Then:

(i) for every $n \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \ldots, n\}$, the matrix $[\Upsilon_{s_k + x}]_{k,l=1}^n$ is a positive semi-definite matrix,

(ii) the function $t \to \Upsilon_t$ is exponentially convex,

(iii) the function $t \to \Upsilon_t$ is log-convex.

\textit{Proof.} (i) As in [1] we can show that the function defined by

$$\mu(x) = \sum_{k,l=1}^n a_k a_l \varphi_{s_{kl}}(x)$$

where $s_{kl} = \frac{s_k + s_l}{2}, a_k \in \mathbb{R}$ for all $k \in \{1, 2, 3, \ldots, n\}, x > 0$, is convex. So by using $\mu(x)$ in (3), we get

$$\sum_{k,l=1}^n a_k a_l \Upsilon_{s_{kl}} \geq 0, \quad (18)$$

hence the matrix $[\Upsilon_{s_k + x}]_{k,l=1}^n$ is a positive semi-definite.

(ii) Since $\lim_{t \to 0} \Upsilon_t = \Upsilon_0$ and $\lim_{t \to 1} \Upsilon_t = \Upsilon_1$, we conclude that $\Upsilon_t$ is continuous for all $t \in \mathbb{R}, x > 0$ also by (i), $[\Upsilon_{s_k + x}]_{k,l=1}^n$ is positive semi-definite matrix, so using Definition 1, we have that exponentially convexity of the function $t \to \Upsilon_t$.

(iii) Since $\Upsilon_t > 0$ as $f \neq d$ a.e. on $\Omega$ and $\varphi_t$ is strictly convex (see Theorem 3) using Corollary 6 we obtain that $\Upsilon_t$ is log-convex.

\textbf{Corollary 7.} Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $f : \Omega \to \mathbb{R}^+$ be a non constant a.e. on $\Omega$ with $f^t, \log f, f \log f$ are all in $L^1(\mu)$, for $t \in \mathbb{R}$ and $\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$. Consider $\tilde{\Upsilon}_t$ to be defined by

$$\tilde{\Upsilon}_t = \phi_t(\overline{f}) + \frac{1}{\mu(\Omega)} \int_\Omega (f - \overline{f}) \varphi_t'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \varphi_t(f) d\mu. \quad (19)$$

Then:

(i) for every $n \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \ldots, n\}$, the matrix $[\tilde{\Upsilon}_{s_k + x}]_{k,l=1}^n$ is a positive semi-definite matrix,

(ii) the function $t \to \tilde{\Upsilon}_t$ is exponentially convex,

(iii) the function $t \to \tilde{\Upsilon}_t$ is log-convex.
\textbf{Proof.} To get the required results set $d = \overline{f}$ in Theorem 7.

Let $(\Omega, A, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$. The integral power mean is defined as follows:

$$M_t(f; \mu) = \begin{cases} \left( \frac{1}{\mu(\Omega)} \int_\Omega f^t \mu \right)^{\frac{1}{t}}, & \text{for } t \neq 0; \\ \exp\left( \frac{1}{\mu(\Omega)} \int_\Omega \log(f) \mu \right), & \text{for } t = 0. \end{cases}$$

Let $(\Omega, A, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and $f, d$ be defined as in Theorem 7 with $f \neq d$ a.e. on $\Omega$. We give the following definition

$$Q_{u,v} = \left( \frac{\Upsilon_u}{\Upsilon_v} \right)^{\frac{1}{u-v}} \text{ for } u, v \in \mathbb{R} \text{ such that } u \neq v,$$

where $\Upsilon_t$ is defined by (17). By Remark 2 these expressions define a class of means. Moreover, we can extend these means to the other cases. Namely, for $u \neq 0, 1$ by limit we have

$$Q_{u,u} = \exp\left( \frac{\mu(\Omega)d^u \log_2 (\log_2 d + (u-1) \int_\Omega f^u \log f \mu + \mu(\Omega)M_1^u(f; \mu)) - (u-1) \int_\Omega f^u \log_2 f \mu - \mu(\Omega)M_1^u(f; \mu)}{2u(\Omega)d^u \log_2 (\log_2 d + (u-1) \int_\Omega f^u \log f \mu + \mu(\Omega)M_1^u(f; \mu))} \right), \quad Q_{0,0} = \exp\left( \frac{\mu(\Omega)d^2 \log_2 d - \mu(\Omega)M_2^2(f; \mu) + 2\mu(\Omega) \log M_0(f; \mu) - 2 \int_\Omega f^2 \log f \mu}{\mu(\Omega)d^2 \log_2 d - \mu(\Omega)M_2^2(f; \mu) + 2 \log M_0(f; \mu)} \right) + 1, \quad Q_{1,1} = \exp\left( \frac{\mu(\Omega)d \log_2 d + 2 \int_\Omega f \log_2 f \mu - \mu(\Omega)M_2(f; \mu) - 2 \log M_0(f; \mu)}{2\mu(\Omega)d \log_2 d + \mu(\Omega)d \log_2 M_0(f; \mu)} \right) - 1).$$

\textbf{Theorem 8.} \textit{Let }$t, s, u, v \in \mathbb{R}$\textit{ such that }$t \leq u, s \leq v$. \textit{Then the following inequality is valid.}

$$Q_{t,s} \leq Q_{u,v}. \tag{21}$$

\textbf{Proof.} The proof is analogous to the proof of Theorem 3.8 in [1].

Let $(\Omega, A, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and $f$ be as in Theorem 7 with $f \neq \overline{f}$ a.e. on $\Omega$. We give the following definition

$$\tilde{Q}_{u,v} = \left( \frac{\mathcal{Y}_u}{\mathcal{Y}_v} \right)^{\frac{1}{u-v}} \text{ for } u, v \in \mathbb{R} \text{ such that } u \neq v,$$

where $\mathcal{Y}_t$ is defined by (19). By Remark 2 these expressions define a class of means. We can extend these means to the other cases. Namely, for $u \neq 0, 1$ by limit we have
\[ \tilde{Q}_{u,0} = \exp \left( \frac{\mu(\Omega) f_m \log^- (u-1) \int_{\Omega} f^m \log f \, d\mu + \mu(\Omega) M_{u-1}^m (f; \mu)}{\mu(\Omega) (f_m \log^- (u-1) M_{u-1}^m (f; \mu) - f_m M_{u-1}^m (f; \mu))} - \frac{2u-1}{u(u-1)} \right), \quad u \neq 0, 1, \]

\[ \tilde{Q}_{0,0} = \exp \left( \frac{\mu(\Omega) \log^- f - \mu(\Omega) M_2^m (f; \mu) + 2\mu(\Omega) \log M_0 (f; \mu) - 2\mu(\Omega) f_{\mu} (f; \mu)}{2\mu(\Omega) (\log f - 1) + \mu(\Omega) (1 - f_{\mu} (f; \mu))} + 1 \right), \]

\[ \tilde{Q}_{1,1} = \exp \left( \frac{\mu(\Omega) \log^- f + 2 \int_{\Omega} f \log f \, d\mu - M_0^2 (f; \mu) + 2 \log M_0 (f; \mu)}{2 \mu(\Omega) (\log f - 1) + \mu(\Omega) (1 - f_{\mu} (f; \mu))} - 1 \right). \]

**Theorem 9.** Let \( t, s, u, v \in \mathbb{R} \) such that \( t \leq u, s \leq v \). Then the following inequality is valid.

\[ \tilde{Q}_{t,s} \leq \tilde{Q}_{u,v}. \quad (23) \]

**Proof.** The proof is analogous to the proof of Theorem 3.8 in [1].

3. Improvement and reversion of Slater’s inequality

Let \( M_t (f; \mu) \) be stated as above and define \( d_t \) as

\[ d_t = \frac{\int_{\Omega} f \varphi_t'(f) \, d\mu}{\int_{\Omega} \varphi_t'(f) \, d\mu} = \begin{cases} \frac{M_t^m (f; \mu)}{M_{t-1}^m (f; \mu)}, & t \neq 0, 1; \\ M_{t-1} (f; \mu), & t = 0; \\ \frac{\log^- f + \int_{\Omega} f \log f \, d\mu}{1 + \log M_0 (f; \mu)}, & t = 1. \end{cases} \quad (24) \]

The following improvement and reverse of Slater’s inequality is valid.

**Theorem 10.** Let \( (\Omega, \Lambda, \mu) \) be a measure space with \( 0 < \mu(\Omega) < \infty \). Let \( f : \Omega \to \mathbb{R}^+ \) be such that \( f', \log f, f \log f \in L^1 (\mu) \) and \( d_t \in \mathbb{R}^+ \), \( t \in \mathbb{R} \). Consider \( \Lambda_t \) to be defined by

\[ \Lambda_t = \varphi_t (d_t) - \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi_t (f) \, d\mu. \quad (25) \]

Then:

(i) \( \Lambda_t \geq [H(s; t)]^{\frac{t-s}{s}} [H(r; t)]^{\frac{t-r}{s}} \)

for \( -\infty < r < s < t < \infty \) and \( -\infty < t < r < s < \infty \).

(ii) \( \Lambda_t \leq [H(s; t)]^{\frac{t-s}{s}} [H(r; t)]^{\frac{t-r}{s}} \)

where \( H(s; t) = \varphi_s (d_t) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - d_t) \varphi_t'(f) \, d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi_t (f) \, d\mu. \quad (28) \)
Proof. (i) By substituting $d = d_r$ in (17) $\Psi_r$ becomes $\Lambda_r$. So by using $d = d_r$ in log-convexity criterion for $\Psi_r$ as in the proof of Theorem 3.10(i) [1] we get

$$(\Lambda_r)^{s-r} \geq (H(s;t))^{r-t} (H(r;t))^{s-t}, \quad \text{for } -\infty < r < s < t < \infty,$$

which is equivalent to (26).

Similarly we can prove (26) for the case $-\infty < t < r < s < \infty$.

(ii) Similar procedure can be applied here.

**Theorem 11.** Let $(\Omega, A, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$. Let $f: \Omega \rightarrow \mathbb{R}^+$ be such that $f^*, \log f, f \log f \in L^1(\mu)$ and $d_r \in \mathbb{R}^+$, $t \in \mathbb{R}$. Then for every $n \in \mathbb{N}$ and for every $s_k \in \mathbb{R}$, $k \in \{1, 2, 3, \ldots, n\}$, the matrices $[H(\frac{s_k + s_l}{2}, s_1)]_{k,l=1}^n$, $[H(\frac{s_k + s_l}{2}, \frac{s_1 + s_2}{2})]_{k,l=1}^n$ are positive semi-definite matrices. Particularly

$$\det[H(\frac{s_k + s_l}{2}, s_1)]_{k,l=1}^n \geq 0,$$

$$\det[H(\frac{s_k + s_l}{2}, \frac{s_1 + s_2}{2})]_{k,l=1}^n \geq 0,$$

where $H(s,t)$ is defined by (28).

Proof. By setting $d = d(s_1)$ and $d = d_s$ in Theorem 7 (i) we get the required results.

**Remark 3.** We note that $H(t,t) = F_t$. So by setting $n = 2$ in (30) we have special case of (26) for $t = s_1, s = s_2, r = \frac{s_1 + s_2}{2}$ if $s_1 < s_2$ and for $t = s_1, r = s_2, s = \frac{s_1 + s_2}{2}$ if $s_2 < s_1$. Similarly by setting $n = 2$ in (31) we have special case of (27) for $r = s_1, s = s_2, t = \frac{s_1 + s_2}{2}$ if $s_1 < s_2$ and for $r = s_2, s = s_1, t = \frac{s_1 + s_2}{2}$ if $s_2 < s_1$.

Let $M_r(f; \mu)$ be stated as above and define $\hat{a}_r$ as

$$\hat{a}_r = M_{r-1}(f; \mu).$$

The following improvement and reverse of inequality in (4) is also valid.

**Theorem 12.** Let $(\Omega, A, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and $f: \Omega \rightarrow \mathbb{R}^+$ be such that $f^*, \log f, f \log f \in L^1(\mu)$ and $d_r \in \mathbb{R}^+, t \in \mathbb{R}$. Consider $\tilde{\Lambda}_r$ to be defined by

$$\tilde{\Lambda}_r = \varphi_r(\hat{a}_r) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \hat{a}_r) \varphi_r'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi_r(f) d\mu.$$

Then:

(i) $\tilde{\Lambda}_r \geq [K(s;t)]^{\frac{r-t}{s-r}} [K(r;t)]^{\frac{s-t}{s-r}}$

for $-\infty < r < s < t < \infty$ and $\tilde{\Lambda}_r \leq [K(s;t)]^{\frac{r-t}{s-r}} [K(r;t)]^{\frac{s-t}{s-r}}$

(ii) $\tilde{\Lambda}_r \leq [K(s;t)]^{\frac{r-t}{s-r}} [K(r;t)]^{\frac{s-t}{s-r}}$

where $K(s;t) = \varphi_3(\hat{a}_r) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \hat{a}_r) \varphi'_3(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi_3(f) d\mu.$
Proof. The proof is similar to the proof of Theorem 10 but use $\overline{d}_t$ instead of $d_t$.

**Theorem 13.** Let $(\Omega,A,\mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and $f : \Omega \to \mathbb{R}^+$ be such that $f', \log f, f \log f \in L^1(\mu)$, $t \in \mathbb{R}$ and let $\overline{d}_t \in \mathbb{R}^+$. Then for every $n \in \mathbb{N}$ and for every $s_k \in \mathbb{R}, k \in \{1,2,3,...,n\}$, the matrices $[K\left(\frac{s_k+s_l}{2}, s_1\right)]_{k,l=1}^n$, $[K\left(\frac{s_k+s_l}{2}, \frac{s_1+s_2}{2}\right)]_{k,l=1}^n$ are positive semi-definite matrices. Particularly

$$\det \left[ K\left(\frac{s_k+s_l}{2}, s_1\right) \right]_{k,l=1}^n \geq 0,$$

(37)

$$\det \left[ K\left(\frac{s_k+s_l}{2}, \frac{s_1+s_2}{2}\right) \right]_{k,l=1}^n \geq 0,$$

(38)

where $K(s,t)$ is defined by (36).

Proof. By setting $d = \overline{d}_{s_1}$ and $d = \overline{d}_{\frac{s_1+s_2}{2}}$ in Theorem 7 (i) we get the required results.

**Remark 4.** We note that $K(t,t) = \tilde{\Lambda}_t$. So by setting $n = 2$ in (37) and in (38) we have special case of (34) and (35) respectively (see Remark 3).

**Remark 5.** The results stated in [1] can be obtained from this paper if you choose the discrete measure.

**References**


(Received February 24, 2011)

M. Adil Khan
Abdus Salam School of Mathematical Sciences
GC University
Lahore, Pakistan
e-mail: adilbandai@yahoo.com

J. Pečarić
University of Zagreb, Faculty of Textile Technology
Zagreb, Croatia
and
Abdus Salam School of Mathematical Sciences
GC University
Lahore, Pakistan
e-mail: pecaric@mahazu.hazu.hr