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A CLASS OF BAZILEVIC TYPE FUNCTIONS DEFINED BY CONVOLUTION OPERATOR

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Abstract. The aim of this paper is to define and study a class of analytic functions related to Bazilevic type functions in the open unit disc.This class is defined by using a convolution operator and the concept of bounded radius rotation of order ρ. A necessary condition, inclusion result, arc length and some other interesting properties of this class of functions are investigated.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Also let $S^*(\rho)$ and $C(\rho)$ denote the well known classes of starlike and convex functions of order ρ respectively.

Let $P_k(\rho)$ be the class of analytic functions $p(z)$ defined in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re}\, p(z) - \rho}{1-\rho} \right| d\theta \leqslant k\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $k \geqslant 2$ and $0 \leqslant \rho < 1$. When $\rho = 0$, we obtain the class P_k defined in [10] and for $k = 2$, $\rho = 0$, we have the class P of functions with positive real part. We can write (1.2) as

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1-2\rho) z e^{-i\theta}}{1 - z e^{-i\theta}} \, d\mu(\theta),$$

where $\mu(\theta)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(\theta) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |\, d\mu(\theta)| \leqslant k\pi.$$

Also, for $p(z) \in P_k(\rho)$, we can write from (1.2)

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in E,$$

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where $p_1(z),\ p_2(z) \in P(\rho),\ \ P(\rho)$ is the class of functions with positive real part greater than ρ.

A function $f\ (z)$ analytic in E belongs to the class $R_k(\rho),\ k \geqslant 2,\ 0 \leqslant \rho < 1$, if and only if,

$$\frac{zf'(z)}{f(z)} \in P_k(\rho), z \in E.$$

This class of functions is known as functions of bounded radius rotation of order ρ and introduced by Padmanabhan and Parvatham [8].

The class of Bazilevic functions in the open unit disc E was first introduced by Bazilevic [1] in 1955. He defined Bazilevic function by the relation

$$f(z) = \left\{ \frac{\eta}{1+\beta^2} \int_0^z (p(t) - \beta i) t^{\frac{-\eta\beta i}{2}-1} g^{\frac{\eta}{1+\beta^2}}(t)\, dt \right\}^{\frac{1+\beta^2}{\eta}},$$

where $p(z) \in P,\ g(z)\ \in S^*,\ \beta$ is real and $\eta > 0$. For $\beta = 0$, we have the class of Bazilevic functions of type η and is given by

$$f(z) = \left\{ \eta \int_0^z p(t) t^{-1} g^{\eta}(t)\, dt \right\}^{\frac{1}{\eta}}.$$

For any two analytic functions $f(z)$ and $g(z)$ with

$$f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} \text{ and } g(z) = \sum_{n=0}^{\infty} c_n z^{n+1},\ z \in E,$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n c_n z^{n+1},\ z \in E.$$

Let $K_a(z) = \frac{z}{(1-z)^a}$, $\mathrm{Re}\, a > 0$. Then

$$\begin{aligned} (K_a * f)(z) &= z + \sum_{n=2}^{\infty} \frac{(a+n-2)!}{(a-1)!\,(n-1)!} b_n z^n \\ &= z + \sum_{n=2}^{\infty} b(a,n)\, b_n z^n, \end{aligned} \tag{1.3}$$

where $b(a,n) = \binom{a+n-2}{a-1}$.

It is also noted that

$$z(K_a * f)'(z) = a(K_{a+1} * f)(z) - (a-1)(K_a * f)(z). \tag{1.4}$$

Using the concept of this convolution operator and the class of functions of bounded radius rotation of order ρ, we define the following class of Bazilevic type functions.

DEFINITION 1.1. A function $f \in A$ is said to be from the class $B_k(a,\eta,\rho,\beta)$, if and only if, there exists a function $(K_a * g)(z) \in R_k(\rho)$ such that

$$\left| \arg \frac{(K_{a+1} * f)^{\eta}(z)(K_a * f)^{1-\eta}(z)}{(K_a * g)(z)} \right| \leqslant \frac{\beta\pi}{2}, \quad z \in E. \tag{1.5}$$

where $a > 0$, $\eta \geqslant 0$, $0 < \beta \leqslant 1$. For $a = \eta = 1, \rho = 0$ and $k = 2$, $B_2(1,1,0,\beta) = B_2(\beta)$ reduces to the subclass of close-to-convex functions defined by Pommerenke [11] with $0 \leqslant \beta \leqslant 1$ and considered by Goodman [3] for $\beta \geqslant 0$. The class $B_2(1)$ coincides with the class of close-to-convex univalent functions introduced by Kaplan [5] in 1952.

2. Preliminary Lemmas

LEMMA 2.1. [7]. *Let $f(z)$ is in $R_k(\rho)$ for $k \geqslant 2$ and $0 \leqslant \rho < 1$. Then*

$$f(z) = z\left(\frac{h(z)}{z}\right)^{(1-\rho)}, \tag{2.1}$$

where $h(z) \in R_k$.

LEMMA 2.2. [7]. *Let $f(z)$ is in $R_k(\rho)$ for $k \geqslant 2$ and $0 \leqslant \rho < 1$. Then with $z = re^{i\theta}$, $r < 1$ and $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta > -\left(\frac{k}{2} - 1\right)(1-\rho)\pi. \tag{2.2}$$

LEMMA 2.3. [9]. *Let $q(z) = 1 + d_1 z + d_2 z^2 + \dots$ be analytic in E, $\delta \geqslant 1$, $\mathrm{Re}c \geqslant 0$ and $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$. If*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{q(z) + \frac{\delta z q'(z)}{c\delta + q(z)}\right\} d\theta > -\pi,$$

then

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}q(z)\, d\theta > -\pi, \quad z = re^{i\theta}. \tag{2.3}$$

3. Main Results

THEOREM 3.1. *A function $f(z) \in B_k(a,\eta,\rho,\beta)$, if and only if,*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}J(a,\eta,f(z))\, d\theta > -\left[\beta + \left(\frac{k}{2} - 1\right)(1-\rho)\right]\pi, \tag{3.1}$$

where $k \geqslant 2,\ 0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi,\ z = re^{i\theta},\ r < 1,\ \eta \geqslant 0,\ 0 < \beta \leqslant 1$ *and*

$$J(a,\eta,f(z)) = \eta(a+1)\left\{\frac{(K_{a+2}*f)(z)}{(K_{a+1}*f)(z)} - \frac{a}{a+1}\right\} + a(1-\eta)\left\{\frac{(K_{a+1}*f)(z)}{(K_a*f)(z)} - \frac{a-1}{a}\right\}, \tag{3.2}$$

with $(K_{a+1}*f)(z)/z \neq 0$ *and* $(K_a*f)(z)/z \neq 0$.

Proof. We define, for $z = re^{i\theta},\ r \in (0,1),\ \theta$ real, the following classes of functions:

$$S(r,\theta) = \arg\left[((K_{a+1}*f))^{\eta}(z)\,((K_a*f))^{1-\eta}(z)\right]. \tag{3.3}$$

$$V(r,\theta) = \arg\left[(K_a*g)(z)\right]. \tag{3.4}$$

It can be shown that the functions $S(z)$, $V(z)$ are periodic and continuous with period 2π. Since $f(z) \in B_k(a,\eta,\rho,\beta)$, therefore from (1.5), it follows that we can choose the branches of argument of $S(z)$ and $V(z)$ such that

$$|S(r,\theta) - V(r,\theta)| \leqslant \frac{\beta\pi}{2},\ \ \beta \in (0,1].$$

Since $(K_a*g)(z) \in R_k(\rho)$, therefore by using Lemma 2.2, we have

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{z((k_a*g)(z))'}{(k_a*g)(z)}d\theta > -\left(\frac{k}{2}-1\right)(1-\rho)\pi. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we have

$$\begin{aligned} S(r,\theta_2) - S(r,\theta_1) &= S(r,\theta_2) + V(r,\theta_2) - V(r,\theta_2) + V(r,\theta_1) - V(r,\theta_1) - S(r,\theta_1) \\ &= [S(r,\theta_2) - V(r,\theta_2)] - [S(r,\theta_1) - V(r,\theta_1)] + [V(r,\theta_2) - V(r,\theta_1)] \\ &> -\beta\pi - \left(\frac{k}{2}-1\right)(1-\rho)\pi \\ &= -\pi\left[\beta + \left(\frac{k}{2}-1\right)(1-\rho)\right]. \end{aligned}$$

Moreover, from (3.3)

$$\begin{aligned} \frac{d}{d\theta}S(r,\theta) = \mathrm{Re}\Bigg\{&\eta(a+1)\left(\frac{(K_{a+2}*f)(re^{i\theta})}{(K_{a+1}*f)(re^{i\theta})} - \frac{a}{a+1}\right) \\ &+ a(1-\eta)\left(\frac{(K_{a+1}*f)(re^{i\theta})}{(K_a*f)(re^{i\theta})} - \frac{a-1}{a}\right)\Bigg\}. \end{aligned}$$

Therefore

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}J(a,\eta,f(z))\,d\theta > -\left[\beta + \left(\frac{k}{2}-1\right)(1-\rho)\right]\pi. \qquad \square$$

THEOREM 3.2. *A function* $f(z) \in B_k(a,\eta,\rho,\beta)$, $\eta \geqslant 0$, $a > 0$, $k \geqslant 2$, $0 < \beta \leqslant 1$, *if and only if,*

$$(K_a * f)(z) = \left\{ \frac{a}{\eta z^{\left(\frac{a-1}{\eta}\right)}} \int_0^z t^{\frac{a+\rho-1}{\eta}-1} h^{\frac{1-\rho}{\eta}}(t)\, p^{\frac{\beta}{\eta}}(t)\, dt \right\}^{\eta}, \quad z \in E, \tag{3.6}$$

where $h(z) \in R_k$ *and* $p(z) \in P$, *for* $z \in E$.

Proof. From (1.5), we have

$$(K_{a+1} * f)^{\eta}(z)(K_a * f)^{1-\eta}(z) = H_1(z)\, p^{\beta}(z),$$

where $H_1(z) = (K_a * g)(z) \in R_k(\rho)$. By (1.4), we obtain

$$z(K_a * f)'(z)(K_a * f)^{\frac{1}{\eta}-1}(z) + (a-1)(K_a * f)^{\frac{1}{\eta}}(z) = aH_1^{\frac{1}{\eta}}(z)\, p^{\frac{\beta}{\eta}}(z).$$

Multiplying by $z^{\frac{a-1}{\eta}-1}$ and using Lemma 2.1, we have

$$\left[z^{\frac{a-1}{\eta}}(K_a * f)^{\frac{1}{\eta}}(z)\right]' = \frac{a}{\eta} z^{\frac{a+\rho-1}{\eta}-1} h^{\frac{1-\rho}{\eta}}(z)\, p^{\frac{\beta}{\eta}}(z),$$

where $h(z) \in R_k$. Integrating from 0 to z, we obtain the required result. □

THEOREM 3.3. *Let* $f(z) \in B_k(a,\eta,\rho,\beta)$ *and* $J(a,\eta,f(z))$ *is defined by* (3.2). *Then* $J(a,\eta,f(z)) \in P_k$ *for*

$$|z| < r_{\beta,\rho} = \frac{1}{(1-2\rho)}\left[(1+\beta-\rho) - \sqrt{(1+\beta-\rho)^2 - (1-2\rho)}\right]. \tag{3.7}$$

Proof. From (1.5), it follows that

$$(K_{a+1} * f)^{\eta}(z)(K_a * f)^{1-\eta}(z) = (K_a * g)(z)\, p^{\beta}(z). \tag{3.8}$$

Differentiating logrithmically (3.8), we have

$$\begin{aligned} J(a,\eta,f(z)) &= \frac{z(K_a * g)'}{(K_a * g)} + \beta \frac{zp'(z)}{p(z)} \\ &= H(z) + \beta \frac{zp'(z)}{p(z)}, \end{aligned} \tag{3.9}$$

where $H(z) \in P_k(\rho)$ and $p(z) \in P$.

Now we can write

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) H_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) H_2(z), \quad H_1(z), H_2(z) \in P(\rho) \text{ in } E.$$

Using the well-known results for the classes P and $P(\rho)$, we obtain,

$$\operatorname{Re}\left[H_i(z)+\beta\frac{zp'(z)}{p(z)}\right] \geqslant \frac{1-(1-2\rho)r}{1+r}-\beta\frac{2r}{1-r^2}$$
$$= \frac{(1-2\rho)r^2-2(1+\beta-\rho)r+1}{1-r^2}.$$

Solving for r, we obtain the required result. □

THEOREM 3.4. *Let $f(z)\in B_k(a,\eta,\rho,\beta)$, $\eta\geqslant 1$, $0<\beta<1$, $0\leqslant\rho<1$, $a\geqslant 1$ and $2<k<4$. Then*

$$B_k(a,\eta,\rho,\beta)\subset B_k(a,0,\rho,1).$$

Proof. Let

$$\frac{z(K_a*f)'}{(K_a*f)}=H(z), \tag{3.10}$$

where $H(z)$ is analytic in E and $H(0)=1$ and let $f(z)\in B_k(a,\eta,\rho,\beta)$. Differentiating logrithmically (3.10) and using (3.2), we obtain

$$\int_{\theta_1}^{\theta_2}\operatorname{Re}J(a,\eta,f(z))\,d\theta = \int_{\theta_1}^{\theta_2}\operatorname{Re}\left\{H(z)+\frac{\eta zH'(z)}{(a-1)+H(z)}\right\}d\theta$$
$$> -\left[\beta+\left(\frac{k}{2}-1\right)(1-\rho)\right]\pi,$$

for $0\leqslant\theta_1<\theta_2\leqslant 2\pi$, $z=re^{i\theta}\in E$. Now using Lemma 2.3 with $2\beta+(k-2)(1-\rho)\leqslant 2$, we obtain the required result. □

THEOREM 3.5. *Let $f(z)\in B_k(a,\eta,\rho,\beta)$, $\eta>0$, $M(r)=\max_{|z|=r}|(K_a*f)(z)|$, $0\leqslant\theta\leqslant 2\pi$. Then*

$$M^{\frac{1}{\eta}}(r)\leqslant 2^{\frac{(k-2)(1-\rho)+2\beta}{2\eta}}r^{\frac{1}{\eta}}\,{}_2F_1\left(\frac{a}{\eta};\frac{(k-2)(1-\rho)+2\beta}{2\eta};\frac{a}{\eta}+1;r\right), \tag{3.11}$$

where ${}_2F_1$ is thehypergeometric function.

Proof. From the Theorem 3.2, we have

$$(K_a*f)^{\frac{1}{\eta}}(z)=\left\{\frac{a}{\eta z^{\left(\frac{a-1}{\eta}\right)}}\int_0^z t^{\frac{a+\rho-1}{\eta}-1}h^{\frac{1-\rho}{\eta}}(t)\,p^{\frac{\beta}{\eta}}(t)\,dt\right\},\ z\in E.$$

Now for $h(z)\in R_k$, it is well known [2] that for $s_1(z)$, $s_2(z)\in S^*$

$$h(z)=\frac{(s_1(z))^{\frac{k}{4}+\frac{1}{2}}}{(s_2(z))^{\frac{k}{4}-\frac{1}{2}}}. \tag{3.12}$$

Using the well-known distortion results for the claases S^* and P, we have

$$|(K_a * f)(z)|^{\frac{1}{\eta}} \leqslant \frac{a}{\eta r^{\frac{a-1}{\eta}}} 2^{\frac{(k-2)(1-\rho)+2\beta}{2\eta}} \int_0^r t^{\frac{a}{\eta}-1}(1-t)^{\frac{(k-2)(1-\rho)+2\beta}{2\eta}} dt.$$

Now for $t = ru$, we obtain

$$\begin{aligned} |(K_a * f)(z)|^{\frac{1}{\eta}} &\leqslant \frac{a}{\eta} 2^{\frac{(k-2)(1-\rho)+2\beta}{2\eta}} r^{\frac{1}{\eta}} \int_0^1 u^{\frac{a}{\eta}-1}(1-ru)^{\frac{(k-2)(1-\rho)+2\beta}{2\eta}} du \\ &= 2^{\frac{(k-2)(1-\rho)+2\beta}{2\eta}} r^{\frac{1}{\eta}} {}_2F_1\left(\frac{a}{\eta}; \frac{(k-2)(1-\rho)+2\beta}{2\eta}; \frac{a}{\eta}+1; r\right), \end{aligned}$$

Hence the proof. □

THEOREM 3.6. *Let $f(z) \in B_k(a,\eta,\rho,\beta)$, $1 \leqslant \eta < \left(\frac{k}{2}+1\right)(1-\rho)+\beta$, $0<\beta \leqslant 1$, $0 \leqslant \rho < 1$ and $k \geqslant 2$. Then, for $F(z) = (K_a * f)(z)$ and $M(r) = \max_{|z|=r} |(K_a * f)(z)|$*

$$L_r F(z) \leqslant C_1(a) M(r) + C_2(a,\eta,\beta,\rho,k) M^{1-\frac{1}{\eta}}(r) \left(\frac{1}{1-r}\right)^{\frac{(k+2)(1-\rho)+2\beta}{2\eta}-1}, \quad (r \to 1),$$

where $C_1(a)$ and $C_2(a,\eta,\beta,\rho,k)$ are constants depending upon a and a, η, β, ρ, k respectively.

Proof. We know that

$$L_r F(z) = \int_0^{2\pi} \left|z(K_a * f)'(z)\right| d\theta, \quad z = re^{i\theta},\ 0<r<1,\ 0 \leqslant \theta \leqslant 2\pi.$$

Now from Theorem 3.2, we have

$$z(K_a * f)'(z) = (1-a)(K_a * f)(z) + az^{\frac{\rho}{\eta}}(K_a * f)^{1-\frac{1}{\eta}}(z) h^{\frac{1-\rho}{\eta}}(z) p^{\frac{\beta}{\eta}}(z),$$

where $h(z) \in R_k$ and $p(z) \in P$. From (3.12), we have

$$\begin{aligned} L_r F(z) \leqslant\ & 2\pi|1-a| M(r) + aM^{1-\frac{1}{\eta}}(r) 2^{\left(\frac{1-\rho}{\eta}\right)\left(\frac{k}{2}-1\right)} r^{\frac{\rho}{\eta}-\left(\frac{1-\rho}{\eta}\right)\left(\frac{k}{4}-\frac{1}{2}\right)} \\ & \times \int_0^{2\pi} |s_1(z)|^{\left(\frac{1-\rho}{\eta}\right)\left(\frac{k}{4}+\frac{1}{2}\right)} |p(z)|^{\frac{\beta}{\eta}} d\theta. \end{aligned}$$

Using the Schwarz inequality, subordination for starlike functions and a result for class P due to Hayman [4], we have

$$\begin{aligned} L_r F(z) &\leqslant 2\pi|1-a| M(r) + aC(\eta,\beta) M^{1-\frac{1}{\eta}}(r) 2^{\left(\frac{1-\rho}{\eta}\right)\left(\frac{k}{2}-1\right)} r^{\frac{1}{\eta}} \left(\frac{1}{1-r}\right)^{\frac{(k+2)(1-\rho)+2\beta}{2\eta}-1} \\ &= C_1(a) M(r) + C_2(a,\eta,\beta,\rho,k) M^{1-\frac{1}{\eta}}(r) \left(\frac{1}{1-r}\right)^{\frac{(k+2)(1-\rho)+2\beta}{2\eta}-1}, \quad (r \to 1). \end{aligned}$$

This completes the proof. □

COROLLARY 3.7. *Let $f(z) \in B_2(1,1,0,\beta)$ Then $f(z)$ is strongly close-to-convex function and*

$$L_r F(z) = O(1)\left(\frac{1}{1-r}\right)^{\beta+1},$$

where $O(1)$ is the constant.

COROLLARY 3.8. *Let $f(z) \in B_k(1,1,0,1) = T_k$. Then*

$$L_r F(z) = O(1)\left(\frac{1}{1-r}\right)^{\frac{k}{2}+1}.$$

THEOREM 3.9. *Let $f(z) \in B_k(a,\eta,\rho,\beta)$ and given by (1.1). Then, for $n \geqslant 2$, $1 \leqslant \eta < \left(\frac{k}{2}+1\right)(1-\rho)+\beta$*

$$|b_n| \leqslant C_3(a,\eta,\beta,\rho,k) M^{1-\frac{1}{\eta}}(n) n^{\frac{(k+2)(1-\rho)+2\beta}{2\eta}-a-1},$$

where $C_3(a,\eta,\beta,\rho,k)$ is constant depending upon a, η, β, ρ, k respectively.

Proof. Since with $z = re^{i\theta}$, Cauchy theorem gives

$$nb(a,n) b_n = \frac{1}{2\pi r^n} L_r F(z).$$

Using Theorem 3.6, we obtain

$$nb(a,n)|b_n| \leqslant \frac{1}{2\pi r^n}\left(C_1(a) M(r) + C_2(a,\eta,\beta,\rho,k) M^{1-\frac{1}{\eta}}(r)\left(\frac{1}{1-r}\right)^{\frac{(k+2)(1-\rho)+2\beta}{2\eta}-1}\right).$$

Taking $r = 1-\frac{1}{n}$, we have

$$|b_n| \leqslant C_3(a,\eta,\beta,\rho,k) M^{1-\frac{1}{\eta}}(n) n^{\frac{(k+2)(1-\rho)+2\beta}{2\eta}-a-1}, \quad (n \to \infty).$$

which is the required result. □

COROLLARY 3.10. *Let $f(z) \in B_k(1,1,0,1) = T_k$. Then*

$$b_n = O(1) n^{\frac{k}{2}}.$$

This result is proved in [6].

COROLLARY 3.11. *Let $f(z) \in B_2(1,1,0,1)$ be close-to-convex functions. Then*

$$b_n = O(1) n.$$

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REFERENCES

[1] I. E. BAZILEVIC, *On a class of integrability in quadratures of the Löewner-Kufarev equation*, Math. Sb. **37** (1955), 471–476.
[2] D. A. BRANNAN, *On functions of bounded boundary rotation*, Proc. Edin. Math. Soc. **2** (1968-1969), 339–347.
[3] A. W. GOODMAN, *On close-to-convex functions of higher order*, Ann. univ. Sci. Budapest. Eötvös. Sect. Math. **15** (1972), 17–30.
[4] W. K. HAYMAN, *On functions with positive real part*, J. London Math. Soc. **36** (1961), 34–48.
[5] W. KAPLAN, *Close-to-convex Schlicht functions*, Mich. Math. J. **1** (1952), 169–185.
[6] K. I NOOR, *Hankel determinant problem for functions of bounded boundary rotation*, Rev. Roum. Math. Pures Appl. **28** (1983), 731–739.
[7] K. I. NOOR, *On subclasses of close-to-convex functions of higher order*, Int. J. Math. and Math. Sci. **6**, 2 (1983), 327–334.
[8] K. S. PADMANABHAN AND R. PARVATHAM, *Properties of a class of functions with bounded boundary rotation*, Ann. Polon. Math. **31** (1975), 311–323.
[9] R. PARVATHAM AND S. RADHA, *On certain classes of analytic functions*, Ann. Polon. Math. **49** (1988), 31–34.
[10] B. PINCHUK, *Functions of bounded boundary rotation*, Isr. J. Math. **10** (1971), 6–16.
[11] CH. POMMERENKE, *On starlike and close-to-convex analytic functions*, Proc. Lond. Math. Soc. **13** (1963), 290–304.

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