ON EXPONENTIAL CONVEXITY FOR POWER SUMS AND RELATED RESULTS

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Abstract. In this paper, we use parameterized class of increasing functions to give exponential convexity of the non-negative difference of certain inequality as a function of parameter in connection with power sums. We define new means of Cauchy type and give its relation to the means defined in [5] and [6]. Also we give related mean value theorems of Cauchy type.

1. Introduction and Preliminaries

Bernstein [3] introduced the important sub-class of convex functions in a given interval \((a, b)\). Akhiezer [1, page 209] denoted this sub-class by \(W_{a,b}\). Independently of Bernstein, but somewhat later, Widder [7] also introduced the class \(W_{a,b}\) and studied it. Bernstein called functions \(f \in W_{a,b}\) exponentially convex.

**Definition 1.** A function \(f: (a,b) \to \mathbb{R}\) is exponentially convex if it is continuous and
\[
\sum_{i,j=1}^{n} v_i v_j f(x_i + x_j) \geq 0
\]
for all \(n \in \mathbb{N}\) and all choices \(v_i \in \mathbb{R}, i = 1, ..., n\) such that \(x_i + x_j \in (a, b)\), \(1 \leq i, j \leq n\).

**Proposition 1.1.** Let \(f: (a,b) \to \mathbb{R}\). The following propositions are equivalent

(i) \(f\) is exponentially convex

(ii) \(f\) is continuous and
\[
\sum_{i,j=1}^{n} v_i v_j f\left(\frac{x_i + x_j}{2}\right) \geq 0
\]
for every \(v_i \in \mathbb{R}\) and for every \(x_i \in (a,b)\), \(1 \leq i \leq n\).

**Corollary 1.2.** If \(f : (a,b) \to \mathbb{R}^+\) is exponentially convex function then \(f\) is a log-convex function.

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In [5], we defined the following function:

\[
\Delta_t = \Delta_t(x; p) = \begin{cases} 
\frac{1}{t-1} \left( \left( \sum_{i=1}^{n} p_i x_i^t \right)^{-1} - \sum_{i=1}^{n} p_i x_i^t \right), & t \neq 1; \\
\sum_{i=1}^{n} p_i x_i \log \sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i x_i \log x_i, & t = 1,
\end{cases}
\]

where \( x = (x_1, \ldots, x_n) \) and \( p = (p_1, \ldots, p_n) \) are positive \( n \)-tuples such that \( \sum_{i=1}^{n} p_i x_i \geq x_j \) for \( j = 1, \ldots, n \).

In [2], we proved that \( t \mapsto \Delta_t \) is an exponentially convex function on \( \mathbb{R} \). Also in [5], we introduced the Cauchy means by considering an increasing function of the type \( f(x)/x \) related to power sums, that is, the following means were defined.

**Definition 2.** Let \( x = (x_1, \ldots, x_n) \) and \( p = (p_1, \ldots, p_n) \) be two positive \( n \)-tuples \((n \geq 2)\) such that \( p_i \geq 1 \) \((i = 1, \ldots, n)\). Then for \( t, r, s \in \mathbb{R}^+ \),

\[
A_{t,r}^s(x; p) = \begin{cases} 
\frac{r-s}{t-s} \left( \frac{\sum_{i=1}^{n} p_i x_i^t}{t} - \frac{\sum_{i=1}^{n} p_i x_i^s}{s} \right) & t \neq r, r \neq s, t \neq s, \\
\sum_{i=1}^{n} p_i x_i \log \left( \frac{\sum_{i=1}^{n} p_i x_i^t}{t} - \frac{\sum_{i=1}^{n} p_i x_i^s}{s} \right) & s \neq r.
\end{cases}
\]

In [6] we introduced the Cauchy means by considering convex function, that is, the following means were defined.

**Definition 3.** Let \( x = (x_1, \ldots, x_n) \) and \( p = (p_1, \ldots, p_n) \) be two positive \( n \)-tuples such that \( p_i \geq 1 \) \((i = 1, \ldots, n)\). Then for \( t, r, s \in \mathbb{R}^+ \),

\[
B_{t,r}^s(x; p) = \begin{cases} 
\frac{r(r-s)}{t(t-s)} \left( \frac{\sum_{i=1}^{n} p_i x_i^t}{t} - \frac{\sum_{i=1}^{n} p_i x_i^s}{s} \right) & t \neq r, r \neq s, t \neq s, \\
\sum_{i=1}^{n} p_i x_i \log \left( \frac{\sum_{i=1}^{n} p_i x_i^t}{t} - \frac{\sum_{i=1}^{n} p_i x_i^s}{s} \right) & s \neq r.
\end{cases}
\]
One can find the following relation between $A^s_{t,r}(x,p)$ and $B^s_{t,r}(x,p)$ [6].

\[
B^s_{t,r}(x,p) = \left( \frac{r}{t} \right)^{\frac{1}{r-1}} A^s_{t,r}(x,p),
\]

\[
B^s_{t,s}(x,p) = B^s_{s,r}(x,p) = \left( \frac{r}{s} \right)^{\frac{1}{r-1}} A^s_{s,r}(x,p) = \left( \frac{r}{s} \right)^{\frac{1}{s-1}} A^s_{r,s}(x,p),
\]

In this paper, we use the class of increasing functions to give some results related to power sums as shown in [5] and [6]; we use the following theorem [4, page 151].

**Theorem 1.3.** Let $(x_1,\ldots,x_n) \in I^n$, where $I$ is an interval, $(p_1,\ldots,p_n)$ and $(q_1,\ldots,q_n)$ be non-negative $n$-tuples such that

\[
\sum_{i=1}^n p_i x_i \geq x_j, \text{ for } j = 1,\ldots,n \text{ and } \sum_{i=1}^n p_i x_i \in I. \tag{1}
\]

If $f : I \rightarrow \mathbb{R}$ is an increasing function, then

\[
\sum_{i=1}^n q_i f \left( \sum_{i=1}^n p_i x_i \right) \geq \sum_{i=1}^n q_i f(x_i). \tag{2}
\]

**Remark 1.4.** If $f$ is strictly increasing on $I$ and all $x_i$’s are not equal, then

\[
\sum_{i=1}^n p_i x_i > x_j,
\]

implies

\[
f \left( \sum_{i=1}^n p_i x_i \right) > f(x_j).
\]

Thus we obtain strict inequality in (2).

In this paper we use parameterized class of an increasing functions to give exponential convexity of non-negative difference of (2) as a function of parameter. We introduce means of Cauchy type and use logarithmic convexity of the difference to prove a monotonicity property of newly defined means. We also prove related mean value theorem of Cauchy type.

**2. Main results**

Let $t \in \mathbb{R}$ and $h_t : (0,\infty) \rightarrow \mathbb{R}$ be the function defined as

\[
h_t(x) = \begin{cases} 
\frac{d}{t}, & t \neq 0; \\
\log x, & t = 0.
\end{cases}
\tag{3}
\]

It is easy to check that $h_t$ is strictly increasing on $(0,\infty)$ for each $t \in \mathbb{R}$. 

\[
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\]
THEOREM 2.1. Let \( x = (x_1, \ldots, x_n) \), \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) be positive n-tuples \( (n \geq 2) \) such that \( \sum_{i=1}^{n} p_ix_i \geq x_j \) for \( j = 1, \ldots, n \). Also let \{\( h_t : t \in \mathbb{R} \)\} be the family of functions define in (3) and

\[
\tilde{\mathcal{U}}_t := \tilde{\mathcal{U}}_t(x; p; q) = \sum_{i=1}^{n} q_i h_t \left( \sum_{i=1}^{n} p_i x_i \right) - \sum_{i=1}^{n} q_i h_t (x_i). \tag{4}
\]

(a) For \( m \in \mathbb{N} \), let \( r_1, \ldots, r_m \) be arbitrary real numbers. Then the matrix

\[
\begin{bmatrix}
\tilde{\mathcal{U}}_{r_i + r_j} - 1/2
\end{bmatrix}, \quad \text{where} \quad 1 \leq i, j \leq m,
\]

is a positive semi-definite matrix. Particularly

\[
\det \left[ \tilde{\mathcal{U}}_{r_i + r_j} - 1/2 \right]_{i,j=1}^k \geq 0 \quad \text{for all} \quad k = 1, \ldots, m.
\]

(b) The function \( t \mapsto \tilde{\mathcal{U}}_t \), where \( t \in \mathbb{R} \), is an exponentially convex.

(c) If all \( x_i \)'s are not equal, then \( t \mapsto \tilde{\mathcal{U}}_t \) is log-convex function.

Proof. (a) Define a \( m \times m \) matrix \( M = \begin{bmatrix} h_{r_i + r_j} - 1/2 \end{bmatrix} \), where \( i, j = 1, \ldots, m \), and let \( v \) be a nonzero arbitrary vector from \( \mathbb{R}^m \).

Consider the function

\[
\zeta(x) = v M v^T = \sum_{i,j=1}^{m} v_i v_j h_{r_i + r_j} (x).
\]

Now we have

\[
\zeta'(x) = \sum_{i,j=1}^{m} v_i v_j x^{r_i + r_j - 1} = \left( \sum_{i=1}^{m} v_i x^{r_i - 1} \right)^2 \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^+, \]

concluding \( \zeta \) is an increasing on \( \mathbb{R}^+ \). Now by Theorem 1.3 with \( f = \zeta \), we have

\[
\sum_{k=1}^{n} q_k \zeta \left( \sum_{k=1}^{n} p_k x_k \right) - \sum_{k=1}^{n} q_k \zeta (x_k) \geq 0,
\]

this implies

\[
\sum_{i,j=1}^{m} v_i v_j \left( \sum_{k=1}^{n} q_k h_{r_i + r_j} - \sum_{k=1}^{n} q_k h_{r_i + r_j} (x_k) \right) \geq 0,
\]

and finally we have

\[
\sum_{i,j=1}^{m} v_i v_j \tilde{\mathcal{U}}_{r_i + r_j} \geq 0.
\]
Therefore the given matrix is positive semi-definite.

Specially, we get

\[
\begin{vmatrix}
\mathcal{V}_{r_1} & \cdots & \mathcal{V}_{r_1+n_t} \\
\vdots & \ddots & \vdots \\
\mathcal{V}_{r_t+n_t} & \cdots & \mathcal{V}_{r_k+n_t}
\end{vmatrix} \geq 0
\]  

(5)

for all \( k = 1, \ldots, m \).

(b) Since \( \lim_{t \to 0} \mathcal{V}_t = \mathcal{V}_0 \), it follows that \( t \mapsto \mathcal{V}_t \) is continuous on \( \mathbb{R} \). Now using Proposition 1.1 we have exponential convexity of the function \( t \mapsto \mathcal{V}_t \).

(c) Since all \( x_i \)'s are not equal and \( x \mapsto h_t(x) \) is strictly increasing for any \( t \in \mathbb{R} \) therefore from Remark 1.4 we have \( \mathcal{V}_t > 0 \). Now logarithmic convexity of \( t \mapsto \mathcal{V}_t \) is follows from the Corollary 1.2.

Let us introduce the following:

**Definition 4.** Let \( x = (x_1, \ldots, x_n) \), \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) be positive \( n \)-tuples \((n \geq 2)\) such that \( \sum_{i=1}^n p_i x_j \geq x_j \) for \( j = 1, \ldots, n \). Then for \( t, r, r', \in \mathbb{R} \), we define

\[
H_{t,r}(x;p;q) = \left( \frac{r \sum_{i=1}^n q_i (\sum_{i=1}^n p_i x_i)^r - \sum_{i=1}^n q_i x_i^r}{t \sum_{i=1}^n q_i (\sum_{i=1}^n p_i x_i)^r - \sum_{i=1}^n q_i x_i^r} \right)^{\frac{1}{r}}, \quad r \neq t, r, t \neq 0.
\]

\[
H_{r,r}(x;p;q) = \exp \left( -\frac{1}{r} + \frac{\sum_{i=1}^n q_i (\sum_{i=1}^n p_i x_i)^r - \sum_{i=1}^n q_i x_i^r}{\sum_{i=1}^n q_i (\sum_{i=1}^n p_i x_i)^r - \sum_{i=1}^n q_i x_i^r} \right) \log \frac{\sum_{i=1}^n q_i (\sum_{i=1}^n p_i x_i)^r}{\sum_{i=1}^n q_i x_i^r}, \quad r \neq 0.
\]

\[
H_{r,0}(x;p;q) = H_{0,r}(x;p;q) = \left( \frac{\sum_{i=1}^n q_i (\sum_{i=1}^n p_i x_i)^r - \sum_{i=1}^n q_i x_i^r}{r \{ \sum_{i=1}^n q_i \log (\sum_{i=1}^n p_i x_i) - \sum_{i=1}^n q_i \log x_i \}} \right)^{\frac{1}{r}}, \quad r \neq 0.
\]

\[
H_{0,0}(x;p;q) = \exp \left( \frac{\sum_{i=1}^n q_i \{ \log (\sum_{i=1}^n p_i x_i) \}^2 - \sum_{i=1}^n q_i (\log x_i)^2}{2 \{ \sum_{i=1}^n q_i \log (\sum_{i=1}^n p_i x_i) - \sum_{i=1}^n q_i \log x_i \}} \right).
\]

**Remark 2.2.** Note that \( \lim_{r \to 1} H_{t,r}(x;p;q) = H_{t,r}(x;p;q) \), \( \lim_{r \to 0} H_{t,r}(x;p;q) = \lim_{r \to 0} H_{r,t}(x;p;q) = H_{0,r}(x;p;q) = H_{r,0}(x;p;q) \) and \( \lim_{r \to 0} H_{r,r}(x;p;q) = H_{0,0}(x;p;q) \).

We shall use a following lemma [5] to prove the monotonicity of the means defined above.

**Lemma 2.3.** Let \( f \) be a log-convex function and assume that if \( x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2 \). Then the following inequality is valid:

\[
\left( \frac{f(y_2)}{f(y_1)} \right)^{\frac{1}{x_2-y_2}} \leq \left( \frac{f(y_2)}{f(y_1)} \right)^{\frac{1}{x_1-y_1}}.
\]

\[
(6)
\]

**Theorem 2.4.** Let \( x = (x_1, \ldots, x_n) \), \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) be positive \( n \)-tuples \((n \geq 2)\) such that \( \sum_{i=1}^n p_i x_j \geq x_j \) for \( j = 1, \ldots, n \). Also let \( r, t, u, v \in \mathbb{R} \) such that \( r \leq u, t \leq v \). Then we have

\[
H_{t,r}(x;p;q) \leq H_{v,u}(x;p;q).
\]

(7)
Proof. Let $\mathcal{U}_r$ be defined by (4). Taking $x_1 = r$, $x_2 = t$, $y_1 = u$, $y_2 = v$, where $r \neq t$, $u \neq v$, and $f(t) = \mathcal{U}_r$ in Lemma 2.3, we have

$$\left(\frac{r \sum_{i=1}^{n} q_i (\sum_{i=1}^{n} p_i x_i^r) - \sum_{i=1}^{n} q_i x_i^t}{t \sum_{i=1}^{n} q_i (\sum_{i=1}^{n} p_i x_i^t) - \sum_{i=1}^{n} q_i x_i^r}\right)^{\frac{1}{t-r}} \leq \left(\frac{u \sum_{i=1}^{n} q_i (\sum_{i=1}^{n} p_i x_i^u) - \sum_{i=1}^{n} q_i x_i^v}{v \sum_{i=1}^{n} q_i (\sum_{i=1}^{n} p_i x_i^v) - \sum_{i=1}^{n} q_i x_i^u}\right)^{\frac{1}{v-u}}.$$

This is equivalent to (7) for $t \neq r$, $u \neq v$. From Remark 2.2, we get (7) is also valid for $t = r$, $u = v$.

REMARK 2.5. If we put $r \rightarrow r - 1$, $t \rightarrow t - 1$ and $q_i \rightarrow p_i x_i$ in $H_{t,r}(x;p,q)$, we have

$$\tilde{H}_{t,r}(x;p) = \left(\frac{r-1}{t-1} \left(\frac{\sum_{i=1}^{n} p_i x_i^r - \sum_{i=1}^{n} p_i x_i^t}{\sum_{i=1}^{n} p_i x_i^t - \sum_{i=1}^{n} p_i x_i^r}\right)\right)^{\frac{1}{r-t}}, \quad r \neq t, t \neq 1. $$

$$\tilde{H}_{r,r}(x;p) = \exp\left(\frac{1}{1-r} + \frac{\sum_{i=1}^{n} p_i x_i^r \log \sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i x_i^r \log x_i}{\sum_{i=1}^{n} p_i x_i^r - \sum_{i=1}^{n} p_i x_i^t}\right), \quad r \neq 1. $$

$$\tilde{H}_{r,0}(x;p) = \tilde{H}_{0,r}(x;p)$$

where

$$\tilde{H}_{t,s}(x^s) = (A_{t,s}(x^s;p))^s,$$

$$\tilde{H}_{s,s}(x^s; p) = (A_{s,s}(x^s;p))^s,$$

$$\tilde{H}_{s,0}(x^s; p) = \tilde{H}_{0,s}(x^s; p) = (A_{s,s}(x^s;p))^s,$$

$$\tilde{H}_{0,0}(x^s; p) = (A_{s,s}(x^s;p))^s.$$

Also note that

$$B_{t,r}(x;p) = \left(\frac{r}{t}\right)^{\frac{1}{t-r}} \left(\tilde{H}_{t,s}(x^s; p)\right)^{\frac{1}{s}},$$

$$B_{r,s}(x;p) = B_{s,r}(x;p) = \left(\frac{r}{s}\right)^{\frac{1}{r-s}} \left(\tilde{H}_{0,s}(x^s; p)\right)^{\frac{1}{s}} = \left(\frac{r}{s}\right)^{\frac{1}{s-r}} \left(\tilde{H}_{0,0}(x^s; p)\right)^{\frac{1}{s}},$$

$$B_{t,t}(x;p) = \exp\left(-\frac{1}{r}\right) \left(\tilde{H}_{t,s}(x^s; p)\right)^{\frac{1}{s}},$$

$$B_{s,s}(x;p) = \exp\left(-\frac{1}{s}\right) \left(\tilde{H}_{0,0}(x^s; p)\right)^{\frac{1}{s}}.$$

The following result has been proved in [5].
COROLLARY 2.6. Let $x = (x_1, ..., x_n)$ and $p = (p_1, ..., p_n)$ be positive $n$-tuples $(n \geq 2)$ such that $\sum_{i=1}^{n} p_i x_i \geq x_j$ for $j = 1, ..., n$. Also let $r, t, u, v \in \mathbb{R}^+$ such that $r \leq u$, $t \leq v$. Then we have

$$A_{t,r}^s(x;p) \leq A_{v,u}^s(x;p).$$

Proof. Taking $r \rightarrow r - 1$, $t \rightarrow t - 1$, $u \rightarrow u - 1$, $v \rightarrow v - 1$ and $q_i \rightarrow p_i x_i$ in (7), we have

$$\tilde{H}_{t,r}(x;p) \leq \tilde{H}_{v,u}(x;p).$$

Now taking $x_i \rightarrow x_i^s$, $r \rightarrow \frac{r}{s}$, $t \rightarrow \frac{t}{s}$, $u \rightarrow \frac{u}{s}$, $v \rightarrow \frac{v}{s}$ where $r, t, u, v \neq s$ and $s \neq 0$, we have

$$(A_{t,r}^s(x;p))^s \leq (A_{v,u}^s(x;p))^s.$$ 

This follows (8).

REMARK 2.7. Similarly, we can prove the monotonicity of $B_{t,r}^s(x;p)$ which we have given in [6], that is, for $t, r, u, v \in \mathbb{R}^+$ such that $r \leq u$, $t \leq v$, we have

$$B_{t,r}^s(x;p) \leq B_{v,u}^s(x;p).$$

In fact we have shown in [6] that such results can be obtained from the results given in [5].

3. Mean value theorems

In this section, we prove mean value theorems of Cauchy type by using Theorem 1.3 with the help of functions defined in a following lemma.

LEMMA 3.1. Let $f \in C^1(I)$, such that

$$m \leq f'(x) \leq M, \ x \in I.$$ (10)

Consider the functions $\phi_1, \phi_2$ defined as,

$$\phi_1(x) = Mx - f(x)$$

$$\phi_2(x) = f(x) - mx.$$ 

Then $\phi_i$ for $i = 1, 2$ are monotonically increasing.

Proof. We have that

$$\phi_1'(x) = M - f'(x) \geq 0,$$

$$\phi_2'(x) = f'(x) - m \geq 0.$$ 

i.e. $\phi_i$ for $i = 1, 2$ are monotonically increasing.
THEOREM 3.2. Let \((x_1, \ldots, x_n) \in I^n\), where \(I\) is a compact interval, \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) be non-negative \(n\)-tuples such that all \(x_i\)'s are not equal and condition (1) is satisfied. If \(f \in C^1(I)\), then there exists \(\xi \in I\) such that

\[
\sum_{i=1}^{n} q_i f \left( \sum_{i=1}^{n} p_ix_i \right) - \sum_{i=1}^{n} q_i f (x_i) = f' (\xi) \sum_{j=1}^{n} q_j \left( \sum_{i=1}^{n} p_ix_i - x_j \right). \tag{11}
\]

Proof. Since \(I\) is compact and \(f \in C^1(I)\), therefore let \(m = \min f'\) and \(M = \max f'\). In Theorem 1.3, setting \(f = \phi_1\) and \(f = \phi_2\) respectively as defined in Lemma 3.1, we get the following inequalities

\[
\sum_{i=1}^{n} q_i f \left( \sum_{i=1}^{n} p_ix_i \right) - \sum_{i=1}^{n} q_i f (x_i) \leq M \sum_{j=1}^{n} q_j \left( \sum_{i=1}^{n} p_ix_i - x_j \right), \tag{12}
\]

\[
\sum_{i=1}^{n} q_i f \left( \sum_{i=1}^{n} p_ix_i \right) - \sum_{i=1}^{n} q_i f (x_i) \geq m \sum_{j=1}^{n} q_j \left( \sum_{i=1}^{n} p_ix_i - x_j \right). \tag{13}
\]

Taking \(f(x) = x\) in Theorem 1.3 with all \(x_i\)'s are not equal, we get

\[
\sum_{j=1}^{n} q_j \left( \sum_{i=1}^{n} p_ix_i - x_j \right) > 0, \]

therefore combining (12) and (13), we have

\[
m \leq \frac{\sum_{i=1}^{n} q_i f \left( \sum_{i=1}^{n} p_ix_i \right) - \sum_{i=1}^{n} q_i f (x_i)}{\sum_{j=1}^{n} q_j \left( \sum_{i=1}^{n} p_ix_i - x_j \right)} \leq M. \tag{14}
\]

Hence, there exists \(\xi \in I\) such that

\[
\frac{\sum_{i=1}^{n} q_i f \left( \sum_{i=1}^{n} p_ix_i \right) - \sum_{i=1}^{n} q_i f (x_i)}{\sum_{j=1}^{n} q_j \left( \sum_{i=1}^{n} p_ix_i - x_j \right)} = f' (\xi).
\]

Which implies (11).

From above Theorem we can deduce the results which we have proved in [5].

COROLLARY 3.3. Let \((x_1, \ldots, x_n) \in I^n\), where \(I \subseteq (0, \infty)\) is a compact interval, \((p_1, \ldots, p_n)\) be non-negative \(n\)-tuple such that all \(x_i\)'s are not equal and condition (1) is satisfied. If \(f \in C^1(I)\), then there exists \(\xi \in I\) such that

\[
f \left( \sum_{i=1}^{n} p_ix_i \right) - \sum_{i=1}^{n} p_if (x_i) = \frac{\xi f' (\xi) - f(\xi)}{\xi^2} \left\{ \left( \sum_{i=1}^{n} p_ix_i \right)^2 - \sum_{i=1}^{n} p_ix_i^2 \right\}. \tag{15}
\]

Proof. Taking \(q_i \to p_ix_i\), \(f(x) \to f(x)/\xi\) in (11), we get (15).
THEOREM 3.4. Let \((x_1, \ldots, x_n) \in I^n\), where \(I\) is a compact interval, \((p_1, \ldots, p_n)\) and \((q_1, \ldots, q_n)\) be non-negative \(n\)-tuples such that all \(x_i\)'s are not equal and condition (1) is satisfied. If \(f_1, f_2 \in C^1(I)\), then there exists \(\xi \in I\) such that

\[
\frac{\sum_{i=1}^n q_i f_1 \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n q_i f_1 (x_i)}{\sum_{i=1}^n q_i f_2 \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n q_i f_2 (x_i)} = \frac{f_1' (\xi)}{f_2' (\xi)}. \tag{16}
\]

Provided that the denominators are non-zero.

Proof. Let a function \(k \in C^1(I)\) be defined as

\[k = c_1 f_1 - c_2 f_2,\]

where \(c_1\) and \(c_2\) are defined as

\[c_1 = \sum_{i=1}^n q_i f_2 \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n q_i f_2 (x_i),\]

\[c_2 = \sum_{i=1}^n q_i f_1 \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n q_i f_1 (x_i).\]

Then, using Theorem 3.2 with \(f = k\), we have

\[0 = (c_1 f_1' (\xi) - c_2 f_2' (\xi)) \sum_{j=1}^n q_j \left( \sum_{i=1}^n p_i x_i - x_j \right). \tag{17}\]

\(\sum_{j=1}^n q_j \left( \sum_{i=1}^n p_i x_i - x_j \right)\) is non-zero, so we have

\[
\frac{c_2}{c_1} = \frac{f_1' (\xi)}{f_2' (\xi)}.
\]

After putting the values of \(c_1\) and \(c_2\), we get (16).

COROLLARY 3.5. [5] Let \((x_1, \ldots, x_n) \in I^n\), where \(I \subseteq (0, \infty)\) is a compact interval, \((p_1, \ldots, p_n)\) be non-negative \(n\)-tuple such that all \(x_i\)'s are not equal and condition (1) is satisfied. If \(f_1, f_2 \in C^1(I)\), then there exists \(\xi \in I\) such that

\[
\frac{f_1 \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n p_i f_1 (x_i)}{f_2 \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n p_i f_2 (x_i)} = \frac{\xi f_1' (\xi) - f_1 (\xi)}{\xi f_2' (\xi) - f_2 (\xi)}. \tag{18}
\]

Provided that the denominators are non-zero.

Proof. Taking \(q_i \to p_i x_i\), \(f(x) \to f(x)/x\) in (16), we get (18).
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