Journal of
Mathematical
Inequalities
Volume 5, Number 2 (2011), 265–274

ON EXPONENTIAL CONVEXITY FOR POWER SUMS AND RELATED RESULTS

J. PEČARIĆ AND ATIQ UR REHMAN

Abstract. In this paper, we use parameterized class of increasing functions to give exponential convexity of the non-negative difference of certain inequality as a function of parameter in connection with power sums. We define new means of Cauchy type and give its relation to the means defined in [5] and [6]. Also we give related mean value theorems of Cauchy type.

1. Introduction and Preliminaries

Bernstein [3] introduced the important sub-class of convex functions in a given interval (a,b). Akhiezer [1, page 209] denoted this sub-class by $\mathrm{W}_{a,b}$. Independently of Bernstein, but somewhat later, Widder [7] also introduced the class $W_{a,b}$ and studied it. Bernstein called functions $f\in \mathrm{W}_{a,b}$ exponentially convex.

DEFINITION 1. A function $f:(a,b)\to\mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} v_i v_j f(x_i+x_j)\geqslant 0$$

for all $n\in\mathbb{N}$ and all choices $v_i\in\mathbb{R}$, $i=1,\dots,n$ such that $x_i+x_j\in(a,b)$, $1\leqslant i,j\leqslant n$.

PROPOSITION 1.1. *Let $f:(a,b)\to\mathbb{R}$. The following propositions are equivalent*

(i) f is exponentially convex

(ii) f is continuous and

$$\sum_{i,j=1}^{n} v_i v_j f\left(\frac{x_i+x_j}{2}\right)\geqslant 0$$

for every $v_i\in\mathbb{R}$ and for every $x_i\in(a,b)$, $1\leqslant i\leqslant n$.

COROLLARY 1.2. *If $f:(a,b)\to\mathbb{R}^+$ is exponentially convex function then f is a log-convex function.*

Mathematics subject classification (2010): 26D15, 26D20, 26D99.
Keywords and phrases: convex functions, log-convex functions, power sums, mean value theorems.
This research was partially funded by Higher Education Commission of Pakistan. The research of the first author was supported by the Croatian Ministry of Science, Education and Sports under the Research Grant 117-1170889-0888.

In [5], we defined the following function:

$$\Delta_t = \Delta_t(\mathbf{x};\mathbf{p}) = \begin{cases} \frac{1}{t-1}\left(\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n p_i x_i^t\right), & t \neq 1; \\ \sum_{i=1}^n p_i x_i \log \sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i \log x_i, & t = 1, \end{cases}$$

where $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ are positive n-tuples such that, $\sum_{i=1}^n p_i x_i \geqslant x_j$ for $j = 1,\ldots,n$.

In [2], we proved that $t \mapsto \Delta_t$ is an exponentially convex function on $\mathbb{R}$. Also in [5], we introduced the Cauchy means by considering an increasing function of the type $f(x)/x$ related to power sums, that is, the following means were defined.

DEFINITION 2. Let $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ be two positive n-tuples $(n \geqslant 2)$ such that $p_i \geqslant 1$ $(i = 1,\ldots,n)$. Then for $t,r,s \in \mathbb{R}^+$,

$$A_{t,r}^s(\mathbf{x};\mathbf{p}) = \left\{ \frac{r-s}{t-s} \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r} \right\}^{\frac{1}{t-r}}, \quad t \neq r, r \neq s, t \neq s.$$

$$A_{s,r}^s(\mathbf{x};\mathbf{p}) = A_{r,s}^s(\mathbf{x};\mathbf{p}) = \left\{ \frac{r-s}{s} \frac{\left(\sum_{i=1}^n p_i x_i^s\right)\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r} \right\}^{\frac{1}{s-r}}, \; s \neq r.$$

$$A_{r,r}^s(\mathbf{x};\mathbf{p}) = \exp\left(\frac{1}{s-r} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} \log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r\right\}} \right), \; s \neq r.$$

$$A_{s,s}^s(\mathbf{x};\mathbf{p}) = \exp\left(\frac{\left(\sum_{i=1}^n p_i x_i^s\right)\left(\log\sum_{i=1}^n p_i x_i^s\right)^2 - s^2\sum_{i=1}^n p_i x_i^s \left(\log x_i\right)^2}{2s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)\log\left(\sum_{i=1}^n p_i x_i^s\right) - s\sum_{i=1}^n p_i x_i^s \log x_i\right\}} \right).$$

In [6] we introduced the Cauchy means by considering convex function, that is, the following means were defined.

DEFINITION 3. Let $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ be two positive n-tuples such that $p_i \geqslant 1$ $(i = 1,\ldots,n)$. Then for $t,r,s \in \mathbb{R}^+$,

$$B_{t,r}^s(\mathbf{x};\mathbf{p}) = \left\{ \frac{r(r-s)}{t(t-s)} \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r} \right\}^{\frac{1}{t-r}}, \quad t \neq r, r \neq s, t \neq s,$$

$$B_{s,r}^s(\mathbf{x};\mathbf{p}) = B_{r,s}^s(\mathbf{x};\mathbf{p}) = \left\{ \frac{r(r-s)}{s^2} \frac{\left(\sum_{i=1}^n p_i x_i^s\right)\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r} \right\}^{\frac{1}{s-r}}, \; s \neq r,$$

$$B_{r,r}^s(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{2r-s}{r(r-s)} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} \log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r\right\}} \right), \; s \neq r,$$

$$B_{s,s}^s(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{1}{s} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)\left(\log\sum_{i=1}^n p_i x_i^s\right)^2 - s^2\sum_{i=1}^n p_i x_i^s \left(\log x_i\right)^2}{2s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i\right\}} \right).$$

One can found the following relation between $A^s_{t,r}(\mathbf{x},\mathbf{p})$ and $B^s_{t,r}(\mathbf{x},\mathbf{p})$ [6].

$$B^s_{t,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{t}\right)^{\frac{1}{t-r}} A^s_{t,r}(\mathbf{x};\mathbf{p}),$$

$$B^s_{r,s}(\mathbf{x};\mathbf{p}) = B^s_{s,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{s}\right)^{\frac{1}{s-r}} A^s_{s,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{s}\right)^{\frac{1}{s-r}} A^s_{r,s}(\mathbf{x};\mathbf{p}),$$

$$B^s_{r,r}(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{1}{r}\right) A^s_{r,r}(\mathbf{x};\mathbf{p}),$$

$$B^s_{s,s}(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{1}{s}\right) A^s_{s,s}(\mathbf{x};\mathbf{p}).$$

In this paper, we use the class of increasing functions to give some results related to power sums as shown in [5] and [6]; we use the following theorem [4, page 151].

THEOREM 1.3. *Let* $(x_1,...,x_n) \in I^n$*, where* I *is an interval,* $(p_1,...,p_n)$ *and* $(q_1,...,q_n)$ *be non-negative n-tuples such that*

$$\textstyle\sum_{i=1}^n p_i x_i \geqslant x_j, \ \ for\ j = 1,...,n \ and\ \sum_{i=1}^n p_i x_i \in I. \tag{1}$$

If $f : I \to \mathbb{R}$ *is an increasing function, then*

$$\sum_{i=1}^n q_i f\left(\sum_{i=1}^n p_i x_i\right) \geqslant \sum_{i=1}^n q_i f(x_i). \tag{2}$$

REMARK 1.4. If f is strictly increasing on I and all x_i's are not equal, then

$$\textstyle\sum_{i=1}^n p_i x_i > x_j,$$

implies

$$f\left(\textstyle\sum_{i=1}^n p_i x_i\right) > f(x_j).$$

Thus we obtain strict inequality in (2).

In this paper we use parameterized class of an increasing functions to give exponential convexity of non-negative difference of (2) as a function of parameter. We introduce means of Cauchy type and use logarithmic convexity of the difference to prove a monotonicity property of newly defined means. We also prove related mean value theorem of Cauchy type.

2. Main results

Let $t \in \mathbb{R}$ and $h_t : (0,\infty) \to \mathbb{R}$ be the function defined as

$$h_t(x) = \begin{cases} \frac{x^t}{t}, & t \neq 0; \\ \log x, & t = 0. \end{cases} \tag{3}$$

It is easy to check that h_t is strictly increasing on $(0,\infty)$ for each $t \in \mathbb{R}$.

THEOREM 2.1. *Let* $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{p}=(p_1,\dots,p_n)$ *and* $\mathbf{q}=(q_1,\dots,q_n)$ *be positive n-tuples* $(n \geqslant 2)$ *such that* $\sum_{i=1}^{n} p_i x_i \geqslant x_j$ *for* $j=1,\dots,n$. *Also let* $\{h_t : t \in \mathbb{R}\}$ *be the family of functions define in* (3) *and*

$$\mho_t := \mho_t(\mathbf{x};\mathbf{p};\mathbf{q}) = \sum_{i=1}^{n} q_i h_t\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i h_t(x_i). \tag{4}$$

(a) For $m \in \mathbb{N}$, *let* $r_1,\dots,r_m$ *be arbitrary real numbers. Then the matrix*

$$\left[\mho_{\frac{r_i+r_j}{2}}\right], \quad \text{where} \quad 1 \leqslant i,j \leqslant m,$$

is a positive semi-definite matrix. Particularly

$$\det\left[\mho_{\frac{r_i+r_j}{2}}\right]_{i,j=1}^{k} \geqslant 0 \text{ for all } k=1,\dots,m.$$

(b) The function $t \mapsto \mho_t$, *where* $t \in \mathbb{R}$, *is an exponentially convex.*

(c) If all x_i*'s are not equal, then* $t \mapsto \mho_t$ *is log-convex function.*

Proof. (a) Define a $m \times m$ matrix $M = \left[h_{\frac{r_i+r_j}{2}}\right]$, where $i,j=1,\dots,m$, and let $\mathbf{v}=(v_1,\dots,v_m)$ be a nonzero arbitrary vector from $\mathbb{R}^m$.

Consider the function

$$\zeta(x) = \mathbf{v}M\mathbf{v}^{\tau} = \sum_{i,j=1}^{m} v_i v_j h_{\frac{r_i+r_j}{2}}(x).$$

Now we have

$$\zeta'(x) = \sum_{i,j=1}^{m} v_i v_j x^{\frac{r_i+r_j}{2}-1} = \left(\sum_{i=1}^{m} v_i x^{\frac{r_i-1}{2}}\right)^2 \geqslant 0 \text{ for all } x \in \mathbb{R}^+,$$

concluding ζ is an increasing on $\mathbb{R}^+$. Now by Theorem 1.3 with $f=\zeta$, we have

$$\sum_{k=1}^{n} q_k \zeta\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} q_k \zeta(x_k) \geqslant 0,$$

this implies

$$\sum_{i,j=1}^{m} v_i v_j \left(\sum_{k=1}^{n} q_k h_{\frac{r_i+r_j}{2}}\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} q_k h_{\frac{r_i+r_j}{2}}(x_k)\right) \geqslant 0,$$

and finally we have

$$\sum_{i,j=1}^{m} v_i v_j \mho_{\frac{r_i+r_j}{2}} \geqslant 0.$$

Therefore the given matrix is positive semi-definite.

Specially, we get

$$\begin{vmatrix} \mho_{r_1} & \cdots & \mho_{\frac{r_1+r_k}{2}} \\ \vdots & \ddots & \vdots \\ \mho_{\frac{r_k+r_1}{2}} & \cdots & \mho_{r_k} \end{vmatrix} \geqslant 0 \tag{5}$$

for all $k = 1,...,m$.

(b) Since $\lim_{t\to 0}\mho_t = \mho_0$, it follows that $t \mapsto \mho_t$ is continuous on $\mathbb{R}$. Now using Proposition 1.1 we have exponential convexity of the function $t \mapsto \mho_t$.

(c) Since all x_i's are not equal and $x \mapsto h_t(x)$ is strictly increasing for any $t \in \mathbb{R}$ therefore from Remark 1.4 we have $\mho_t > 0$. Now logarithmic convexity of $t \mapsto \mho_t$ is follows from the Corollary 1.2.

Let us introduce the following:

DEFINITION 4. Let $\mathbf{x} = (x_1,...,x_n)$, $\mathbf{p} = (p_1,...,p_n)$ and $\mathbf{q} = (q_1,...,q_n)$ be positive n-tuples $(n \geqslant 2)$ such that $\sum_{i=1}^n p_i x_i \geqslant x_j$ for $j = 1,...,n$. Then for $t,r, \in \mathbb{R}$, we define

$$H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \left(\frac{r}{t}\frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n q_i x_i^t}{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r}\right)^{\frac{1}{t-r}}, \ r \neq t,\ r,t \neq 0.$$

$$H_{r,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \exp\left(-\frac{1}{r} + \frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^r \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i x_i^r \log x_i}{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r}\right), \ r \neq 0.$$

$$H_{r,0}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{0,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \left(\frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r}{r\left\{\sum_{i=1}^n q_i \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i \log x_i\right\}}\right)^{\frac{1}{r}}, \ r \neq 0.$$

$$H_{0,0}(\mathbf{x};\mathbf{p};\mathbf{q}) = \exp\left(\frac{\sum_{i=1}^n q_i\left\{\log\left(\sum_{i=1}^n p_i x_i\right)\right\}^2 - \sum_{i=1}^n q_i\left(\log x_i\right)^2}{2\left\{\sum_{i=1}^n q_i \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i \log x_i\right\}}\right).$$

REMARK 2.2. Note that $\lim_{t\to r} H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{r,r}(\mathbf{x};\mathbf{p};\mathbf{q})$, $\lim_{t\to 0} H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \lim_{t\to 0} H_{r,t}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{0,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{r,0}(\mathbf{x};\mathbf{p};\mathbf{q})$ and $\lim_{r\to 0} H_{r,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{0,0}(\mathbf{x};\mathbf{p};\mathbf{q})$.

We shall use a following lemma [5] to prove the monotonicity of the means defined above.

LEMMA 2.3. *Let f be a log-convex function and assume that if $x_1 \leqslant y_1, x_2 \leqslant y_2, x_1 \neq x_2, y_1 \neq y_2$. Then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \leqslant \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{6}$$

THEOREM 2.4. *Let $\mathbf{x} = (x_1,...,x_n)$, $\mathbf{p} = (p_1,...,p_n)$ and $\mathbf{q} = (q_1,...,q_n)$ be positive n-tuples $(n \geqslant 2)$ such that $\sum_{i=1}^n p_i x_i \geqslant x_j$ for $j = 1,...,n$. Also let $r,t,u,v \in \mathbb{R}$ such that $r \leqslant u$, $t \leqslant v$. Then we have*

$$H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) \leqslant H_{v,u}(\mathbf{x};\mathbf{p};\mathbf{q}). \tag{7}$$

Proof. Let $\mho_t$ be defined by (4). Taking $x_1 = r,\ x_2 = t,\ y_1 = u,\ y_2 = v$, where $r \neq t,\ u \neq v$, and $f(t) = \mho_t$ in Lemma 2.3, we have

$$\left(\frac{r}{t}\frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n q_i x_i^t}{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r}\right)^{\frac{1}{t-r}} \leqslant \left(\frac{u}{v}\frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^v - \sum_{i=1}^n q_i x_i^v}{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^u - \sum_{i=1}^n q_i x_i^u}\right)^{\frac{1}{v-u}}.$$

This is equivalent to (7) for $t \neq r,\ u \neq v$. From Remark 2.2, we get (7) is also valid for $t = r,\ u = v$.

REMARK 2.5. If we put $r \to r-1,\ t \to t-1$ and $q_i \to p_i x_i$ in $H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q})$, we have

$$\widetilde{H}_{t,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r-1}{t-1}\frac{\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n p_i x_i^r}\right)^{\frac{1}{t-r}},\ r \neq t,\ r,t \neq 1.$$

$$\widetilde{H}_{r,r}(\mathbf{x};\mathbf{p}) = \exp\left(\frac{1}{1-r} + \frac{\left(\sum_{i=1}^n p_i x_i\right)^r \log\sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i^r \log x_i}{\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n p_i x_i^r}\right),\ r \neq 1.$$

$$\begin{aligned}\widetilde{H}_{r,0}(\mathbf{x};\mathbf{p}) &= \widetilde{H}_{0,r}(\mathbf{x};\mathbf{p}) \\ &= \left(\frac{\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n p_i x_i^r}{(r-1)\left\{\left(\sum_{i=1}^n p_i x_i\right)\log\sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i \log x_i\right\}}\right)^{\frac{1}{r-1}},\ r \neq 1.\end{aligned}$$

$$\widetilde{H}_{0,0}(\mathbf{x};\mathbf{p}) = \exp\left(\frac{\sum_{i=1}^n p_i x_i\left(\log\sum_{i=1}^n p_i x_i\right)^2 - \sum_{i=1}^n p_i x_i(\log x_i)^2}{2\left\{\left(\sum_{i=1}^n p_i x_i\right)\log\sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i \log x_i\right\}}\right).$$

Now if $x_i \to x_i^s$, $r \to \frac{r}{s}$ and $t \to \frac{t}{s}$ where $r,t \neq s$ and $s \neq 0$, we have

$$\begin{aligned}\widetilde{H}_{\frac{t}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p}) &= \left(A_{t,r}^s(\mathbf{x};\mathbf{p})\right)^s, \\ \widetilde{H}_{\frac{r}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p}) &= \left(A_{r,r}^s(\mathbf{x};\mathbf{p})\right)^s, \\ \widetilde{H}_{\frac{r}{s},0}(\mathbf{x}^s;\mathbf{p}) &= \widetilde{H}_{0,\frac{r}{s}}(\mathbf{x}^s;\mathbf{p}) = \left(A_{r,s}^s(\mathbf{x};\mathbf{p})\right)^s = \left(A_{s,r}^s(\mathbf{x};\mathbf{p})\right)^s, \\ \widetilde{H}_{0,0}(\mathbf{x}^s;\mathbf{p}) &= \left(A_{s,s}^s(\mathbf{x};\mathbf{p})\right)^s.\end{aligned}$$

Also note that

$$\begin{aligned}B_{t,r}^s(\mathbf{x};\mathbf{p}) &= \left(\frac{r}{t}\right)^{\frac{1}{t-r}}\left(\widetilde{H}_{\frac{t}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}, \\ B_{r,s}^s(\mathbf{x};\mathbf{p}) = B_{s,r}^s(\mathbf{x};\mathbf{p}) &= \left(\frac{r}{s}\right)^{\frac{1}{s-r}}\left(\widetilde{H}_{0,\frac{r}{s}}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}} = \left(\frac{r}{s}\right)^{\frac{1}{s-r}}\left(\widetilde{H}_{\frac{r}{s},0}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}, \\ B_{r,r}^s(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{r}\right)\left(\widetilde{H}_{\frac{r}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}, \\ B_{s,s}^s(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{s}\right)\left(\widetilde{H}_{0,0}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}.\end{aligned}$$

The following result has been proved in [5].

COROLLARY 2.6. *Let* $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{p} = (p_1, ..., p_n)$ *be positive n-tuples* $(n \geqslant 2)$ *such that* $\sum_{i=1}^{n} p_i x_i \geqslant x_j$ *for* $j = 1, ..., n$. *Also let* $t, r, u, v \in \mathbb{R}^+$ *such that* $r \leqslant u$, $t \leqslant v$. *Then we have*

$$A^s_{t,r}(\mathbf{x};\mathbf{p}) \leqslant A^s_{v,u}(\mathbf{x};\mathbf{p}). \tag{8}$$

Proof. Taking $r \to r-1$, $t \to t-1$, $u \to u-1$, $v \to v-1$ and $q_i \to p_i x_i$ in (7), we have

$$\widetilde{H}_{t,r}(\mathbf{x};\mathbf{p}) \leqslant \widetilde{H}_{v,u}(\mathbf{x};\mathbf{p}).$$

Now taking $x_i \to x_i^s$, $r \to \frac{r}{s}$, $t \to \frac{t}{s}$, $u \to \frac{u}{s}$, $v \to \frac{v}{s}$ where $r, t, u, v \neq s$ and $s \neq 0$, we have

$$\left(A^s_{t,r}(\mathbf{x};\mathbf{p})\right)^s \leqslant \left(A^s_{v,u}(\mathbf{x};\mathbf{p})\right)^s.$$

This follows (8).

REMARK 2.7. Similarly, we can prove the monotonicity of $B^s_{t,r}(\mathbf{x};\mathbf{p})$ which we have given in [6], that is, for $t, r, u, v \in \mathbb{R}^+$ such that $r \leqslant u$, $t \leqslant v$, we have

$$B^s_{t,r}(\mathbf{x};\mathbf{p}) \leqslant B^s_{v,u}(\mathbf{x};\mathbf{p}). \tag{9}$$

In fact we have shown in [6] that such results can be obtained from the results given in [5].

3. Mean value theorems

In this section, we prove mean value theorems of Cauchy type by using Theorem 1.3 with the help of functions defined in a following lemma.

LEMMA 3.1. *Let* $f \in C^1(I)$, *such that*

$$m \leqslant f'(x) \leqslant M,\ x \in I. \tag{10}$$

Consider the functions ϕ_1, ϕ_2 *defined as,*

$$\phi_1(x) = Mx - f(x)$$

$$\phi_2(x) = f(x) - mx.$$

Then ϕ_i *for* $i = 1, 2$ *are monotonically increasing.*

Proof. We have that

$$\phi_1'(x) = M - f'(x) \geqslant 0,$$

$$\phi_2'(x) = f'(x) - m \geqslant 0.$$

i.e. ϕ_i for $i = 1, 2$ are monotonically increasing.

THEOREM 3.2. *Let* $(x_1,...,x_n) \in I^n$*, where* I *is a compact interval,* $(p_1,...,p_n)$ *and* $(q_1,...,q_n)$ *be non-negative n-tuples such that all* x_i*'s are not equal and condition* (1) *is satisfied. If* $f \in C^1(I)$*, then there exists* $\xi \in I$ *such that*

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) = f'(\xi) \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right). \tag{11}$$

Proof. Since I is compact and $f \in C^1(I)$, therefore let $m = \min f'$ and $M = \max f'$.
In Theorem 1.3, setting $f = \phi_1$ and $f = \phi_2$ respectively as defined in Lemma 3.1, we get the following inequalities

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) \leqslant M \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right), \tag{12}$$

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) \geqslant m \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right). \tag{13}$$

Taking $f(x) = x$ in Theorem 1.3 with all x_i's are not equal, we get

$$\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right) > 0,$$

therefore combining (12) and (13), we have

$$m \leqslant \frac{\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i)}{\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right)} \leqslant M. \tag{14}$$

Hence, there exists $\xi \in I$ such that

$$\frac{\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i)}{\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right)} = f'(\xi).$$

Which implies (11).
From above Theorem we can deduce the results which we have proved in [5].

COROLLARY 3.3. *Let* $(x_1,...,x_n) \in I^n$*, where* $I \subseteq (0,\infty)$ *is a compact interval,* $(p_1,...,p_n)$ *be non-negative n-tuple such that all* x_i*'s are not equal and condition* (1) *is satisfied. If* $f \in C^1(I)$*, then there exists* $\xi \in I$ *such that*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) = \frac{\xi f'(\xi) - f(\xi)}{\xi^2} \left\{ \left(\sum_{i=1}^{n} p_i x_i\right)^2 - \sum_{i=1}^{n} p_i x_i^2 \right\}. \tag{15}$$

Proof. Taking $q_i \to p_i x_i$, $f(x) \to f(x)/x$ in (11), we get (15).

THEOREM 3.4. *Let* $(x_1,...,x_n) \in I^n$*, where* I *is a compact interval,* $(p_1,...,p_n)$ *and* $(q_1,...,q_n)$ *be non-negative n-tuples such that all* x_i*'s are not equal and condition* (1) *is satisfied. If* $f_1, f_2 \in C^1(I)$*, then there exists* $\xi \in I$ *such that*

$$\frac{\sum_{i=1}^{n} q_i f_1\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f_1\left(x_i\right)}{\sum_{i=1}^{n} q_i f_2\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f_2\left(x_i\right)} = \frac{f_1'(\xi)}{f_2'(\xi)}. \tag{16}$$

Provided that the denominators are non-zero.

Proof. Let a function $k \in C^1(I)$ be defined as

$$k = c_1 f_1 - c_2 f_2,$$

where c_1 and c_2 are defined as

$$c_1 = \sum_{i=1}^{n} q_i f_2\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f_2\left(x_i\right),$$

$$c_2 = \sum_{i=1}^{n} q_i f_1\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f_1\left(x_i\right).$$

Then, using Theorem 3.2 with $f = k$, we have

$$0 = \left(c_1 f_1'(\xi) - c_2 f_2'(\xi)\right) \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right). \tag{17}$$

$\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right)$ is non-zero, so we have

$$\frac{c_2}{c_1} = \frac{f_1'(\xi)}{f_2'(\xi)}.$$

After putting the values of c_1 and c_2, we get (16).

COROLLARY 3.5. [5] *Let* $(x_1,...,x_n) \in I^n$*, where* $I \subseteq (0,\infty)$ *is a compact interval,* $(p_1,...,p_n)$ *be non-negative n-tuple such that all* x_i*'s are not equal and condition* (1) *is satisfied. If* $f_1, f_2 \in C^1(I)$*, then there exists* $\xi \in I$ *such that*

$$\frac{f_1\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f_1\left(x_i\right)}{f_2\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f_2\left(x_i\right)} = \frac{\xi f_1'(\xi) - f_1(\xi)}{\xi f_2'(\xi) - f_2(\xi)}. \tag{18}$$

Provided that the denominators are non-zero.

Proof. Taking $q_i \to p_i x_i$, $f(x) \to f(x)/x$ in (16), we get (18).

REFERENCES

[1] N. I. AKHIEZER, *The classical moment problem and some related questions in analyisis*, Oliver and Boyd Ltd. The University Press, Glasgow 1965.

[2] M. ANWAR, J. JAKŠETIĆ, J. PEČARIĆ AND ATIQ UR REHMAN, *Exponential convexity, positive semi-definite matrices and fundamental inequalities*, J. Math. Inequal. **4**, 2 (2010), 171–189.

[3] S. N. BERNSTEIN, *Sur les fonctions absolument monotones*, Acta Math. **52** (1929), 1–66.

[4] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, Partial Orderings and Statistical Applications*, Vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[5] J. PEČARIĆ AND ATIQ UR REHMAN, *On Logarithmic convexity for power sums and related results*, J. Inequal. Appl., 2008, Article ID 389410, (2008), 9 pp.

[6] J. PEČARIĆ AND ATIQ UR REHMAN, *On Logarithmic convexity for power sums and related results II*, J. Inequal. Appl., 2008, Article ID 305623, (2008), 12 pp.

[7] D. V. WIDDER, *The laplace transform*, Princeton 1941, 1946.

(Received February 24, 2011)

J. Pečarić
Abdus Salam School of Mathematical Sciences
GC University
68-B, New Muslim Town
Lahore 54600, Pakistan
and
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr

Atiq ur Rehman
Abdus Salam School of Mathematical Sciences
GC University
68-B, New Muslim Town
Lahore 54600, Pakistan
e-mail: atiq@mathcity.org