Journal of
Mathematical
Inequalities
Volume 5, Number 2 (2011), 275–286

# BOAS–TYPE INEQUALITIES VIA SUPERQUADRATIC FUNCTIONS

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*Abstract.* We state and prove some new Boas type inequalities using the concept of superquadratic and subquadratic functions. We apply this result to refine the strengthened inequalities of the Hardy type.

## 1. Introduction

R.P. Boas, in [5] (see e.g. [18, p. 229]), proved that the inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1.1}$$

holds for all continuous convex functions $\Phi\colon [0,\infty) \to \mathbb{R}$, measurable non–negative functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and non–decreasing bounded functions $m\colon [0,\infty) \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (1.1) is the Lebesgue–Stieltjes integral with respect to $m$. In the case of a concave function $\Phi$, (1.1) holds with the sign of inequality reversed. The relation (1.1) was named the Boas inequality. The special case of the Boas inequality is the well-known Hardy inequality. It was announced by G.H. Hardy in [9] and proved in [10]. Let $p > 1$ and $f \in L^p(\mathbb{R}_+)$ be a non-negative function, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{1.2}$$

holds. Inequality (1.2) was generalized in many different ways. In [11] this result is given:
if $p > 1$, $k \neq 1$, and the function $F$ is defined on $\mathbb{R}_+$ by

$$F(x) = \begin{cases} \int_0^x f(t)dt, & k > 1; \\ \int_x^\infty f(t)dt, & k < 1. \end{cases}$$

*Mathematics subject classification* (2010): Primary 26D10, Secondary 26D15.

*Keywords and phrases*: Inequalities, Boas inequality, superquadratic function, subquadratic function, integral identities.

then

$$\int_0^\infty x^{-k} F^p(x)\,dx \leqslant \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx \tag{1.3}$$

holds for all non-negative functions $f$, such that $x^{1-\frac{k}{p}} f \in L^p(\mathbb{R}_+)$. If $0 < p < 1$, then the sign of the inequality in (1.3) is reversed, that is,

$$\int_0^\infty x^{-k} F^p(x)\,dx \geqslant \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx \tag{1.4}$$

holds. About the history of Hardy's inequality see e.g. [14].

The Boas inequality has already been generalized in many ways. Some recent results are given in papers [6], [7], [8], and [19]. Independently, S. Kaijser et al. [12] established the so-called general Hardy-Knopp-type inequality for positive functions $f\colon \mathbb{R}_+ \to \mathbb{R}$,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1.5}$$

where $\Phi$ is a real convex function on $\mathbb{R}_+$. Some new result involving general Hardy-Knopp-type inequality can be found in e.g. [13].

Another generalization gave D.Luor [15] in a setting with $\sigma$-finite Borel measures $\mu$ and $\nu$ on a topological space $X$ and a Borel probability measure on $\mathbb{R}_+$. He proved that the following inequality

$$\int_E \phi(Hf(x))\,d\mu(x) \leqslant \int_E \phi(f(x))\left(\int_0^\infty \frac{d\mu_t}{d\nu}(x)\,d\lambda(t)\right)d\nu(x),$$

holds for a $\lambda$-balanced Borel set $E$ in $X$ and measure $\mu_t$ defined by $\mu_t(D) = \mu(t^{-1}D)$ for all Borel set $D$ in $X$, $t > 0$, where $\phi$ is a non-negative convex function on an interval, $\mu_t \ll \nu$, $t \in \operatorname{supp}\lambda$ and $Hf$ is the Hardy–Littlewood average.

Our main tool in the proofs is to use the concept of superquadratic and subquadratic functions introduced by Abramovich et al. in [2] (see also [1] and [3]).

DEFINITION 1. (See [2, Definition 2.1].) A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geqslant C_x(y-x) \quad \text{for all } y \geqslant 0.$$

We say that $f$ is subquadratic if $-f$ is superquadratic.

LEMMA 1.1. (See [2, Theorem 2.3].) *Let $(\Omega,\mu)$ be a probability measure space. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leqslant \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right)d\mu(s) \tag{1.6}$$

*holds for all probability measures $\mu$ and all non-negative $\mu-$integrable functions $f$ if and only if $\varphi$ is superquadratic. Moreover,* $(1.6)$ *holds in the reversed direction if and only if $\varphi$ is subquadratic.*

*Proof.* See [2] and [3] for the details.

DEFINITION 2. A function $f:[0,\infty)\to\mathbb{R}$ is superadditive provided $f(x+y)\geqslant f(x)+f(y)$ for all $x,y\geqslant 0$. If the reverse inequality holds, then $f$ is said to be subadditive.

LEMMA 1.2. (See [2, Lemma 3.1].) *Suppose $\varphi:[0,\infty)\to\mathbb{R}$ is continuously differentiable and $\varphi(0)\leqslant 0$. If $\varphi'$ is superadditive or $\frac{\varphi'(x)}{x}$ is non-decreasing, then $\varphi$ is superquadratic.*

*Proof.* See [2] for details.

REMARK 1.1. According to Lemmas 1.1 and 1.2 it yield that if $p\geqslant 2$ in Lemma 1.1, then

$$\left(\int_\Omega f(s)d\mu(s)\right)^p\leqslant\int_\Omega (f(s))^p d\mu(s)-\int_\Omega\left|f(s)-\int_\Omega f(s)d\mu(s)\right|^p d\mu(s)$$

holds and the reversed inequality holds when $1<p\leqslant 2$ (see also [1, Example 1, p. 1448]).

This paper is organized in the following way: after the Introduction, in Section 2 we proved the Boas-type inequality in a setting with general weighted topological spaces and $\sigma$-finite measures using the concept of superquadratic and subquadratic functions. We apply this result to refine the strengthened inequalities of the Hardy type.

**Notations and Conventions.** Throughout this paper, all measures are assumed to be positive, all functions are assumed to be measurable on their respective domains and expressions of the form $0\cdot\infty$, $\frac{0}{0}$, $\frac{a}{\infty}(a\in\mathbb{R})$ and $\frac{\infty}{\infty}$ are taken to be equal to zero. As usual, by $dx$ and $d\mathbf{x}$ we denote the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n(n\in\mathbb{N},n\geqslant 2)$ respectively, while by a weight function we mean a non–negative measurable function on the actual set. An interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while Int $I$ denotes the interior of an interval $I\subseteq\mathbb{R}$.

## 2. The main results with applications

Let $\lambda$ be a finite Borel measure on $\mathbb{R}_+$. By supp $\lambda$ we denote its support, that is, the set of all $t\in\mathbb{R}_+$ such that $\lambda(N_t)>0$ holds for all open neighbourhoods $N_t$ of $t$. Hence,

$$L=\int_{\text{supp }\lambda}d\lambda(t)=\int_0^\infty d\lambda(t)=\lambda(\mathbb{R}_+)<\infty. \tag{2.1}$$

On the other hand, let $X$ be a topological space equipped with a continuous scalar multiplication $(a,\mathbf{x}) \mapsto a\mathbf{x} \in X$, for $a \in \mathbb{R}_+$, $\mathbf{x} \in X$, such that

$$1\mathbf{x} = \mathbf{x}, \;\; a(b\mathbf{x}) = (ab)\mathbf{x}, \;\; \mathbf{x} \in X, \;\; a,b \in \mathbb{R}_+.$$

Further, let Borel set $\Omega \subseteq X$ be $\lambda$-balanced, that is, $t\Omega = \{t\mathbf{x} \colon \mathbf{x} \in \Omega\} \subseteq \Omega$, for all $t \in \operatorname{supp} \lambda$. For a Borel measurable function $f\colon \Omega \to \mathbb{R}$, we define its Hardy–Littlewood average $Af$, as

$$Af(\mathbf{x}) = \frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t), \; \mathbf{x} \in \Omega. \tag{2.2}$$

Finally, suppose $\mu$ and $\nu$ are $\sigma$–finite Borel measures on $X$. For $t > 0$ and a Borel set $S \subseteq X$ we define

$$\mu_t(S) = \mu\left(\frac{1}{t}S\right). \tag{2.3}$$

Obviously $\mu_t$ is a $\sigma$-finite Borel measure on $X$. Moreover, throughout this paper we suppose that the measures $\mu_t$ are absolutely continuous with respect to the measure $\nu$, that is, $\mu_t \ll \nu$, for each $t \in \operatorname{supp} \lambda$. As usual, by $\dfrac{d\mu_t}{d\nu}$ we denote the related Radon–Nikodym derivative.

THEOREM 2.1. *Let $\lambda$ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by* (2.1)*. Let $\mu$ and $\nu$ be $\sigma$-finite Borel measures on a topological space $X$, $\mu_t$ be defined by* (2.3)*, and let $\mu_t \ll \nu$ for all $t \in \operatorname{supp}\lambda$. Further, let $\Omega \subseteq X$ be a $\lambda$-balanced set and let $u$ be a non-negative function on $X$, such that*

$$v(\mathbf{x}) = \int_0^\infty u\left(\frac{1}{t}\mathbf{x}\right)\frac{d\mu_t}{d\nu}(\mathbf{x})\,d\lambda(t) < \infty, \; \mathbf{x} \in \Omega. \tag{2.4}$$

*Suppose that $I = (0,c), c \leqslant \infty$ and let $f\colon \Omega \to \mathbb{R}$ be Borel measurable function such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$. If $\varphi\colon I \to \mathbb{R}$ is a non-negative superquadratic function and $Af$ is defined by* (2.2)*, then the inequality*

$$\begin{aligned}\int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})\varphi(|f(t\mathbf{x}) - Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x})&\\ \leqslant \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi(f(\mathbf{x}))\,d\nu(\mathbf{x})&\end{aligned} \tag{2.5}$$

*holds.*

*Proof.* For a fixed $\mathbf{x} \in \Omega$, we define $h_\mathbf{x}\colon \mathbb{R}_+ \to \mathbb{R}$ as $h_\mathbf{x}(t) = f(t\mathbf{x}) - Af(\mathbf{x})$. Then (2.2) and (2.1) imply

$$\int_0^\infty h_\mathbf{x}(t)\,d\lambda(t) = \int_0^\infty f(t\mathbf{x})\,d\lambda(t) - Af(\mathbf{x})\int_0^\infty d\lambda(t) = 0. \tag{2.6}$$

Now, let us prove that $Af(x) \in I$ for all $x \in \Omega$. Since $\Omega$ is $\lambda$-balanced and $f(\Omega) \subseteq I$, it follows that $f(t\mathbf{x}) \in I$ for all $t \in \operatorname{supp}\lambda$ and every $\mathbf{x} \in \Omega$. Suppose that there exists

$\mathbf{x}_0 \in \Omega$ such that $Af(\mathbf{x}_0) \notin I$. Then we have either $Af(\mathbf{x}_0) < f(t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$, or $Af(\mathbf{x}_0) > f(t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$, so the function $h_{\mathbf{x}_0}$ is either strictly positive or strictly negative on $\mathbb{R}_+$. Since this contradicts (2.6), we proved that $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$.

Finally, we prove (2.5). By applying the refined Jensen inequality to the first term on the left hand side of inequality (2.5) we obtain

$$\begin{aligned}\int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) &= \int_\Omega u(\mathbf{x})\varphi\left(\frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t)\right)d\mu(\mathbf{x})\\ &\leqslant \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(f(t\mathbf{x}))\,d\lambda(t)\,d\mu(\mathbf{x})\\ &\quad -\frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(|f(t\mathbf{x})-Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x})\end{aligned} \tag{2.7}$$

The inequality (2.7) can be written as

$$\begin{aligned}&\int_\Omega u(\mathbf{x})\varphi\left(\frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t)\right)d\mu(\mathbf{x})\\ &\qquad +\frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(|f(t\mathbf{x})-Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x})\\ &\qquad\qquad \leqslant \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(f(t\mathbf{x}))\,d\lambda(t)\,d\mu(\mathbf{x}).\end{aligned} \tag{2.8}$$

By applying the Fubini theorem to the right hand side of inequality (2.8), than the substitution $\mathbf{y} = t\mathbf{x}$, the fact that $\Omega$ is $\lambda$-balanced set, $\varphi$ is a non-negative function and the Radon-Nikodym theorem, we obtain

$$\begin{aligned}&\frac{1}{L}\int_0^\infty \int_\Omega u(\mathbf{x})\varphi(f(t\mathbf{x}))\,d\mu(\mathbf{x})\,d\lambda(t)\\ &\quad = \frac{1}{L}\int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\ &\quad \leqslant \frac{1}{L}\int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\ &\quad = \frac{1}{L}\int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\nu(\mathbf{y})\,d\lambda(t)\\ &\quad = \frac{1}{L}\int_\Omega\left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right)\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\lambda(t)\right)\varphi(f(\mathbf{y}))\,d\nu(\mathbf{y})\\ &\quad = \frac{1}{L}\int_\Omega v(\mathbf{y})\varphi(f(\mathbf{y}))\,d\nu(\mathbf{y}).\end{aligned} \tag{2.9}$$

This completes the proof. □

THEOREM 2.2. *Let $\lambda$, $L$, $\mu$, $\nu$, $u$ and $v$ be defined as in Theorem* 2.1. *Further, let $\Omega \subseteq X$ be such that $t\Omega = \Omega$, for all $t \in \operatorname{supp} \lambda$. Suppose that $I = (0,c), c \leqslant \infty$,*

*$\varphi\colon I \to \mathbb{R}$. If $\varphi$ is a superquadratic function on an interval $I$, then the inequality* (2.5) *holds for all Borel measurable functions $f\colon \Omega \to \mathbb{R}$, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, where $Af$ is defined by* (2.2).
*If $\varphi$ is a subquadratic function, then the inequality sign in* (2.5) *is reversed, that is*

$$\int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})\varphi(|f(t\mathbf{x}) - Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x}) \geqslant \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi(f(\mathbf{x}))\,d\nu(\mathbf{x}) \tag{2.10}$$

*holds.*

*Proof.* By analyzing (2.9), we see that if $t\Omega = \Omega$, for all $t \in \operatorname{supp}\lambda$, then

$$\frac{1}{L}\int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t) = \frac{1}{L}\int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)$$

and (2.5) holds for all superquadratic functions $\varphi\colon I \to \mathbb{R}$, that is, $\varphi$ does not need to be non-negative.
By making the same calculations with $\varphi$ subquadratic function, we see that the inequality sign in (2.5) is reversed, that is (2.10) holds. □

REMARK 2.1. Notice, that in case $\Omega = \mathbb{R}_+$ (2.5) and (2.10) hold for all superquadratic and all subquadratic functions, respectively.

Now we consider a particular superquadratic function in (2.5), namely $\varphi(x) = x^p$ which is superquadratic for $p \geqslant 2$ and subquadratic for $1 < p \leqslant 2$. We obtain the following result.

COROLLARY 2.1. *Let the assumptions in Theorem* 2.1 *be satisfied.*
*If $p \geqslant 2$, then*

$$\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p\,d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^p d\lambda(t)d\mu(\mathbf{x}) \leqslant \frac{1}{L}\int_\Omega v(\mathbf{x})f^p(\mathbf{x})\,d\nu(\mathbf{x}). \tag{2.11}$$

*If $t\Omega = \Omega$, for all $t \in \operatorname{supp}\lambda$ and $1 < p \leqslant 2$, then* (2.11) *holds in the reversed direction and for $p = 2$ we obtain the following very general identity*

$$\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2\,d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^2 d\lambda(t)d\mu(\mathbf{x}) = \frac{1}{L}\int_\Omega v(\mathbf{x})f^2(\mathbf{x})\,d\nu(\mathbf{x}).$$

COROLLARY 2.2. *Let $\lambda$ be a finite Borel measure on $\mathbb{R}_+$ and $L$ be defined by* (2.1). *Let $X=\mathbb{R}_+$ and let $\mu_t$ be defined by* (2.3). *Suppose $\Omega\subseteq\mathbb{R}_+$ is $\lambda$-balanced set and function $x\mapsto\frac{u(x)}{x}$ is a non-negative on $\mathbb{R}_+$ such that*

$$w(x)=\int_0^\infty u\left(\frac{x}{t}\right)d\lambda(t)<\infty, \tag{2.12}$$

*$x\in\Omega$. Suppose that $I=(0,c), c\leqslant\infty$, $\varphi\colon I\to\mathbb{R}$. If $\varphi$ is a non-negative superquadratic function on an interval $I$, then the inequality*

$$\int_\Omega u(x)\varphi(Af(x))\frac{dx}{x}+\frac{1}{L}\int_\Omega\int_0^\infty u(x)\varphi(|f(tx)-Af(x)|)d\lambda(t)\frac{dx}{x} \leqslant\frac{1}{L}\int_\Omega w(x)\varphi(f(x))\frac{dx}{x} \tag{2.13}$$

*holds for Borel measurable functions $f\colon\Omega\to\mathbb{R}$ such that $f(x)\in I$ for all $x\in\Omega$, where $Af$ is defined by* (2.2).

*If $t\Omega=\Omega$, for all $t\in\operatorname{supp}\lambda$, then* (2.13) *holds for all superquadratic functions $\varphi$ and the inequality sign in* (2.13) *is reversed, if $\varphi$ is a subquadratic function.*

*Proof.* It follows directly from Theorem 2.1 if we set measures $\mu$ and $\nu$ to be Lebesgue measures since $X=\mathbb{R}_+$. For such measures are $\frac{d\mu_t}{d\nu}(x)=\frac{1}{t}$, $t\in\mathbb{R}_+$. For function $w$ we take

$$w(x)=x\,v(x)=x\int_0^\infty u\left(\frac{x}{t}\right)\cdot\frac{t}{x}\cdot\frac{1}{t}\,d\lambda(t)=\int_0^\infty u\left(\frac{x}{t}\right)d\lambda(t)\ ,\ x\in\Omega,$$

where function $v$ is defined by (2.4) with the weight function $x\mapsto\frac{u(x)}{x}$ instead of $u$. $\square$

REMARK 2.2. If we apply Corollary 2.2 with $\Omega=\mathbb{R}_+$ and $u(x)\equiv 1$, then $w(x)\equiv L$ and the following inequality holds:

$$\int_0^\infty\varphi\left(\frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)\right)\frac{dx}{x}+\frac{1}{L}\int_0^\infty\int_0^\infty\varphi(|f(tx)-Af(x)|)d\lambda(t)\frac{dx}{x} \leqslant\int_0^\infty\varphi(f(x))\frac{dx}{x}. \tag{2.14}$$

In particular, for $\varphi(x)=x^p$, we obtain the following (in)equalities.
$(i)$ If $p\geqslant 2$, then

$$\int_0^\infty\left(\frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)\right)^p\frac{dx}{x}+\frac{1}{L}\int_0^\infty\int_0^\infty|f(tx)-Af(x)|^p d\lambda(t)\frac{dx}{x} \leqslant\int_0^\infty f^p(x)\frac{dx}{x}. \tag{2.15}$$

$(ii)$ If $1 < p \leqslant 2$ , then $(2.15)$ holds in the reversed direction.
$(iii)$ If $p = 2$, then the following identity holds

$$\int_0^\infty \left(\frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)\right)^2 \frac{dx}{x} + \frac{1}{L}\int_0^\infty \int_0^\infty |f(tx) - Af(x)|^2 d\lambda(t)\frac{dx}{x} = \int_0^\infty f^2(x)\frac{dx}{x}.$$

REMARK 2.3. Observe that a non-decreasing and bounded function $m\colon [0,\infty) \to \mathbb{R}$, such that $M = m(\infty) - m(0) > 0$, induces a finite Borel measure $\lambda$ on $\mathbb{R}_+$ and vice versa. For such function and measure, related Lebesgue and Lebesgue-Stieltjes integrals are equivalent. Thus, all the above results can be interpreted as for $Af(x)$ defined by

$$Af(x) = \frac{1}{M}\int_0^\infty f(tx)\,dm(t),\, x \in \mathbb{R}_+,$$

so they refine and generalize the Boas inequality (1.1).

REMARK 2.4. As a special case of inequality $(2.15)$ we obtain the refined Hardy and dual Hardy inequality. Let $\alpha > 0$ and

$$\lambda(t) = \begin{cases} \alpha^{-1}t^{\alpha}, & 0 \leqslant t \leqslant 1; \\ \alpha^{-1}, & t \geqslant 1. \end{cases}$$

Then $L = \alpha^{-1}$ and (2.15) becomes

$$\alpha^p \int_0^\infty x^{-1-\alpha p}\left(\int_0^x f(t)t^{\alpha-1}\,dt\right)^p dx + \alpha\int_0^\infty \int_0^x |f(t) - Af(x)|^p t^{\alpha-1}x^{-\alpha-1}dtdx \leqslant \int_0^\infty f^p(x)\frac{dx}{x}, \tag{2.16}$$

where

$$Af(x) = \alpha x^{-\alpha}\int_0^x t^{\alpha-1}f(t)dt.$$

If we let $f(t) = g(t)t^{1-\alpha}$ and $\alpha = \frac{k-1}{p}$ $(p \geqslant 2, k > 1)$ in (2.16) we have

$$\int_0^\infty x^{-k}\left(\int_0^x g(t)dt\right)^p dx + \frac{k-1}{p}\int_0^\infty \int_t^\infty \left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} g(t) - \frac{1}{x}\int_0^x g(s)ds\right|^p x^{p-k-\frac{k-1}{p}}dx t^{\frac{k-1}{p}-1}dt \leqslant \left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k}g^p(x)dx. \tag{2.17}$$

If $1 < p \leqslant 2$, then (2.17) holds in the reversed direction.
Now, let $\beta > 0$ and

$$\lambda(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1; \\ \beta^{-1}(1 - t^{-\beta}), & t \geqslant 1. \end{cases}$$

Then $L = \beta^{-1}$ and (2.15) becomes

$$\begin{aligned} \beta^{p} \quad & \int_0^\infty x^{\beta p - 1} \left( \int_x^\infty f(t) t^{-\beta - 1} \, dt \right)^p dx \\ & + \beta \int_0^\infty \int_x^\infty |f(t) - Af(x)|^p t^{-\beta-1} x^{\beta-1} dt dx \\ & \leqslant \int_0^\infty f^p(x) \frac{dx}{x}, \end{aligned} \tag{2.18}$$

where

$$Af(x) = \beta x^\beta \int_x^\infty t^{-\beta-1} f(t) dt.$$

If we let $f(t) = g(t)t^{1+\beta}$ and $\beta = \frac{1-k}{p}$ ($p \geqslant 2, k < 1$) in (2.18) we have

$$\begin{aligned} & \int_0^\infty x^{-k} \left( \int_x^\infty g(t) dt \right)^p dx \\ & + \frac{1-k}{p} \int_0^\infty \int_0^t \left| \frac{p}{1-k} \left( \frac{t}{x} \right)^{1 + \frac{1-k}{p}} g(t) - \frac{1}{x} \int_x^\infty g(s) ds \right|^p x^{p-k+\frac{1-k}{p}} dx t^{\frac{k-1}{p}-1} dt \\ & \leqslant \left( \frac{p}{1-k} \right)^p \int_0^\infty x^{p-k} g^p(x) dx. \end{aligned} \tag{2.19}$$

If $1 < p \leqslant 2$, then (2.19) holds in the reversed direction.

Note that for the case $p = 2$ both inequalities (2.17) and (2.19) will be equalities, Parseval type identities with the Hardy and respectfully the dual Hardy operators.

REMARK 2.5. Those results are already given in [16] (see Theorem 3.1 and 3.2). Also, some new results involving Hardy type inequalities using the concept of superquadratic and subquadratic function can be found in [4] and [17]. In [17] J. A. Oguntuase et al. proved these results in multidimensional settings. Our Theorem 2.1 can be applied in multidimensional settings to obtain those results, but here we omit the details.

We continue with two theorems that are consequences of Theorem 2.1.

THEOREM 2.3. *Let $b \in \mathbb{R}_+$ and let $x \mapsto \dfrac{u(x)}{x}$ be a non-negative function on $(0,b)$, such that the function $t \mapsto \dfrac{u(t)}{t^2}$ is locally integrable in $(0,b)$, and let*

$$w(x) = x \int_x^b u(t) \frac{dt}{t^2}, \ x \in (0,b). \tag{2.20}$$

*If $\varphi$ is non-negative superquadratic on an interval $I=(0,c)$, $c\leqslant\infty$, then the inequality*

$$\int_0^b u(x)\varphi(Hf(x))\,\frac{dx}{x}+\int_0^b\int_0^x u(x)\varphi(|f(t)-Hf(x)|)dt\,\frac{dx}{x^2}\leqslant\int_0^b w(x)\varphi(f(x))\,\frac{dx}{x} \tag{2.21}$$

*holds for all functions $f$ on $(0,b)$ with values in $I$ and for $Hf(x)$ defined by*

$$Hf(x)=\frac{1}{x}\int_0^x f(t)\,dt \tag{2.22}$$

*for $x\in(0,b)$.*

*Proof.* Rewrite Theorem 2.1 with the measures $d\lambda(t)=\chi_{(0,1)}(t)\,dt$, $\mu(x)=\chi_{(0,b)}(x)\,dx$, $d\nu(x)=dx$ and $x\mapsto\dfrac{u(x)}{x}$ instead of weight function $u$. Then we get $L=1$, $\dfrac{d\mu_t}{d\nu}(x)=\dfrac{1}{t}\chi_{(0,tb)}(x)$,

$$Af(x)=\int_0^1 f(tx)\,dt=Hf(x)$$

and

$$v(x)=\int_0^1\frac{u\left(\frac{1}{t}x\right)}{\frac{1}{t}x}\cdot\frac{1}{t}\chi_{(0,tb)}(x)\,dt=\frac{1}{x}\int_{\frac{x}{b}}^1 u\left(\frac{x}{t}\right)\,dt=\int_x^b u(y)\frac{dy}{y^2}=\frac{w(x)}{x},$$

for $x\in(0,b)$, so (2.21) holds. □

We give also a dual result to the Theorem 2.3 considering $d\lambda(t)=\chi_{[1,\infty)}(t)\dfrac{dt}{t^2}$.

THEOREM 2.4. *For $b\geqslant 0$, suppose $u\colon(b,\infty)\to\mathbb{R}$ is a non-negative function, locally integrable in $(b,\infty)$, and $w$ is defined on $(b,\infty)$ by*

$$w(x)=\frac{1}{x}\int_b^x u(t)\,dt. \tag{2.23}$$

*If $\varphi$ is non-negative superquadratic on an interval $I=(0,c)$, $c\leqslant\infty$, then the inequality*

$$\int_b^\infty u(x)\varphi(\tilde{H}f(x))\frac{dx}{x}+\int_b^\infty\int_x^\infty u(x)\varphi(|f(t)-\tilde{H}f(x)|)\frac{dt}{t^2}dx\leqslant\int_b^\infty w(x)\varphi(f(x))\frac{dx}{x} \tag{2.24}$$

*holds for all functions $f$ on $(b,\infty)$ with values in $I$ and for $\tilde{H}f(x)$ defined by*

$$\tilde{H}f(x)=x\int_x^\infty f(t)\frac{dt}{t^2} \tag{2.25}$$

*for $x\in(b,\infty)$.*

*Acknowledgements.* The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant 117-1170889-0888 (first and second author) and 082-0000000-0893 (third author).

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(Received March 2, 2011)

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