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COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS IN TVS–VALUED CONE METRIC SPACES

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Abstract. The existence of coincidence points and common fixed points for four mappings satisfying generalized contractive conditions without exploiting the notion of continuity of any map involved therein, in a TVS-valued cone metric space is proved. These results extend, unify and generalize several well known comparable results in the existing literature.

1. Introduction and preliminaries

Huang and Zhang [8] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for nonlinear mappings satisfying different contractive conditions. Subsequently, Abbas and Jungck [2] and Abbas and Rhoades [1] studied common fixed point theorems in cone metric spaces (see also, [3], [5], [9], [13], [14] and the references mentioned therein). Recently, Beg et al. [4] studied common fixed points of a pair of maps on topological vector space (TVS) valued cone metric space which is a larger class than that of introduced by Huang and Zhang [8]. Jungck [12] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In this paper, common fixed point theorems for two pairs of weakly compatible maps, which are more general than $R-$weakly commuting and compatible mappings, are obtained in the setting of cone metric spaces, without exploiting the notion of continuity. It is worth mentioning that our results do not require the assumption that the cone is normal. Our results extend and unify various comparable results in the literature ([2], [3], [6] and [9]).

The following definitions and results will be needed in the sequel.

Let E be always a topological vector space (in shortly, TVS). A subset P of E is called a *cone* if and only if

(a) P is closed, non-empty and $P \neq \{0\}$;

(b) if $a,b \in R$ with $a,b \geqslant 0$ and $x,y \in P$, then $ax+by \in P$;

(c) $P \cap (-P) = \{0\}$.

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For a given cone $P \subset E$, we define a *partial ordering* $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$, where $x \ll y$ means that $y - x \in int\, P$ (the interior of P). A cone P is said to be *normal* if there is a number $K > 0$ such that

$$0 \leqslant x \leqslant y \Longrightarrow \|x\| \leqslant K\|y\|, \quad \forall x,y \in E.$$

The least positive number satisfying the above inequality is called the *normal constant* of P.

Recently, Rezapour and Hamlbarani [15] proved that there is no normal cone with normal constant $K < 1$ and, for all $k > 1$, there are cones with normal constants $K > k$.

DEFINITION 1.1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leqslant d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \leqslant d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a *TVS-valued cone metric* on X and (X,d) is called a *TVS-valued cone metric space*.

DEFINITION 1.2. Let (X,d) be a TVS-valued cone metric space. Let $\{x_n\}$ be a sequence in X and $c \in E$ with $0 \ll c$.

(1) The sequence $\{x_n\}$ is called a *Cauchy sequence* if there is an N such that $d(x_n,x_m) \ll c$ for all $n,m > N$.

(2) The sequence $\{x_n\}$ is said to be *convergent* if there exist a positive integer N and $x \in X$ such that $d(x_n,x) \ll c$ for all $n > N$.

(3) A cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

It is known that a sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $d(x_n,x) \to 0$ as $n \to \infty$. A subset A of X is closed if every Cauchy sequence in A has its limit point in A.

DEFINITION 1.3. Let f and g be self-mappings on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a *coincidence point* of f and g, where w is called *a point of the coincidence* of f and g.

DEFINITION 1.4. Let f and g be two self-mappings defined on a set X. Then f and g are said to be *weakly compatible* if they commute at every coincidence point.

REMARK 1.1. Let E is a TVS-valued cone metric space with a cone P. Then we have the following:

(1) If $a \leqslant ha$ for all $a \in P$ and $h \in (0,1)$, then $a = 0$.

(2) If $0 \leqslant u \ll c$ for all $0 \ll c$, then $u = 0$.

(3) If $a \leqslant b + c$ for all $0 \ll c$, then $a \leqslant b$.

For more on the properties of the cone, we refer to [10].

2. Common Fixed Point Results

The following Lemma not only improves, but also extends Lemma 1 of [11] to TVS-valued cone metric spaces.

LEMMA 2.1. *Let f, g, S and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X) \subset T(X)$ and $g(X) \subset S(X)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by*

$$\begin{cases} y_{2n+1} = fx_{2n} = Tx_{2n+1}, \\ y_{2n+2} = gx_{2n+1} = Sx_{2n+2}, \quad \forall n \geqslant 0. \end{cases}$$

Suppose that there exists $\lambda \in [0,1)$ such that

$$d(y_n, y_{n+1}) \leqslant \lambda\, d(y_{n-1}, y_n), \quad \forall n \geqslant 1. \tag{2.1}$$

Then either

(1) *the pairs $\{f,S\}$, $\{g,T\}$ have coincidence points and the sequence $\{y_n\}$ converges to a point in X or*

(2) *$\{y_n\}$ is a Cauchy sequence in X.*

Moreover, if X is complete, then the sequence $\{y_n\}$ converges to a point $z \in X$ and

$$d(y_n, z) \leqslant \frac{\lambda^n}{1-\lambda}\, d(y_0, y_1), \quad \forall n \geqslant 1. \tag{2.2}$$

Proof. To prove part (1), suppose that there exists a positive integer n such that $y_{2n} = y_{2n+1}$. Then, from the definition of $\{y_n\}$, $gx_{2n-1} = Sx_{2n} = fx_{2n} = Tx_{2n+1}$ and the mappings f and S have a coincidence point x_{2n}. Moreover, from (2.1), we have

$$d(y_{2n+1}, y_{2n+2}) \leqslant \lambda\, d(y_{2n}, y_{2n+1}) = 0$$

and so $y_{2n+1} = y_{2n+2}$, i.e., $fx_{2n} = Tx_{2n+1} = gx_{2n+1} = Sx_{2n+2}$ and the mappings g and T have a coincidence point x_{2n+1}. In addition, repeating the use of (2.1) yields $y_{2n} = y_m$ for each $m > 2n$ and hence the sequence $\{y_n\}$ converges to a point in X.

The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some positive integer n.

For part (2), assume that $y_{2n} \neq y_{2n+1}$ for all $n \geqslant 1$. Then, by (2.1), we have

$$d(y_n, y_{n+1}) \leqslant \lambda^n d(y_0, y_1), \quad \forall n \geqslant 1.$$

For any $m, n \geqslant 1$ with $m > n$, it follows that

$$\begin{aligned} d(y_n, y_m) &\leqslant \sum_{i=n}^{m-1} d(y_i, y_{i+1}) \leqslant \sum_{i=n}^{m-1} \lambda^i d(y_0, y_1) \\ &= \lambda^n d(y_0, y_1) \sum_{j=0}^{m-n-1} \lambda^j \\ &\leqslant \frac{\lambda^n}{1-\lambda} d(y_0, y_1). \end{aligned} \tag{2.3}$$

Let $0 \ll c$ be given. Choose a symmetric neighborhood V of 0 such that $c+V \subseteq int P$. Also, choose a positive integer N_1 such that

$$\frac{\lambda^n}{1-\lambda} d(y_0, y_1) \in V, \quad \forall n \geqslant N_1.$$

Then $\frac{\lambda^n}{1-\lambda} d(y_0, y_1) \ll c$ for all $n \geqslant N_1$. Thus, for all $m, n \geqslant N_1$,

$$d(y_n, y_m) \leqslant \frac{\lambda^n}{1-\lambda} d(y_0, y_1) \ll c$$

and so the sequence $\{y_n\}$ is a Cauchy sequence in X. If X is complete, there exist a point $z \in X$ such that $\{y_m\}$ converges to z as $m \to \infty$. Choose a positive integer N_2 such that $d(y_m, z) \ll c$ for all $m \geqslant N_2$. Thus it follows that

$$\begin{aligned} d(y_n, z) &\leqslant d(y_n, y_m) + d(y_m, z) \\ &\leqslant \frac{\lambda^n}{1-\lambda} d(y_0, y_1) + d(y_m, z) \\ &\ll \frac{\lambda^n}{1-\lambda} d(y_0, y_1) + c, \end{aligned}$$

which yields (2.2) by using Remark 1 (3). This completes the proof. □

The following theorem extends and improves Theorem 2.1 of Ilic and Rakocevic [9].

THEOREM 2.2. *Let f, g, S and T be self-mappings of a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and there exists $h \in (0,1)$ such that*

$$d(fx, gy) \leqslant h\, u_{x,y}(f, g, S, T), \tag{2.4}$$

where

$$u_{x,y}(f,g,S,T) \in \left\{ d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(gy,Sx)}{2} \right\}, \quad \forall x, y \in X.$$

If one of $f(X) \cup g(X)$ and $S(X) \cup T(X)$ is complete, then the pairs $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then the mapping f, g, S and T have a unique common fixed point in X.

Proof. For any arbitrary point x_0 in X, construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{cases} fx_{2n} = Tx_{2n+1} = y_{2n+1}, \\ gx_{2n+1} = Sx_{2n+2} = y_{2n+2}, \quad \forall n \geqslant 0. \end{cases}$$

Then it follows from (2.4) that

$$d(y_{2n+1},y_{2n+2}) = d(fx_{2n},gx_{2n+1}) \leqslant h\,u_{x_{2n},x_{2n+1}}(f,g,S,T), \quad \forall n \geqslant 1,$$

where

$$\begin{aligned}
& u_{x_{2n},x_{2n+1}}(f,g,S,T) \\
\in & \Big\{d(Sx_{2n},Tx_{2n+1}), d(fx_{2n},Sx_{2n}), d(gx_{2n+1},Tx_{2n+1}), \\
& \frac{[d(fx_{2n},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n})]}{2}\Big\} \\
= & \Big\{d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n}), d(y_{2n+2},y_{2n+1}), \frac{[d(y_{2n+1},y_{2n+1})+d(y_{2n+2},y_{2n})]}{2}\Big\} \\
= & \Big\{d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), \frac{[d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})]}{2}\Big\}.
\end{aligned}$$

Now, if $u_{x_{2n},x_{2n+1}}(f,g,S,T) = d(y_{2n},y_{2n+1})$, then $d(y_{2n+1},y_{2n+2}) \leqslant hd(y_{2n},y_{2n+1})$. If $u_{x_{2n},x_{2n+1}}(f,g,S,T) = d(y_{2n+1},y_{2n+2})$, then $d(y_{2n+1},\ y_{2n+2}) \leqslant hd(y_{2n+1},y_{2n+2})$, which implies that $d(y_{2n+1},y_{2n+2}) = 0$ and so $y_{2n+1} = y_{2n+2}$.

If $u_{x_{2n},x_{2n+1}}(f,g,S,T) = \dfrac{[d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})]}{2}$, then we obtain

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &\leqslant \frac{h}{2}[d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})] \\
&\leqslant \frac{h}{2}d(y_{2n},y_{2n+1})+\frac{1}{2}d(y_{2n+1},y_{2n+2}),
\end{aligned}$$

which implies that

$$d(y_{2n+1},y_{2n+2}) \leqslant hd(y_{2n},y_{2n+1}), \quad \forall n \geqslant 0.$$

Hence the condition (2.1) of Lemma 2.1 is satisfied.

Now, we show that the pairs $\{f,S\}$ and $\{g,T\}$ have coincidence points in X. In fact, without loss of generality, we may assume that $y_n \neq y_{n+1}$ for any $n \geqslant 1$. If we have the equality for some n, then, from (1) of Lemma 2.1, the pairs $\{f,S\}$ and $\{g,T\}$ have coincidence points in X. Thus, from (2) of Lemma 1, the sequence $\{y_n\}$ is a Cauchy sequence.

(I) Suppose that $S(X) \cup T(X)$ is complete. Then there exists $u \in S(X) \cup T(X)$ such that $y_n \to u$ as $n \to \infty$. Further, the subsequences $\{Sx_{2n+2}\} = \{gx_{2n+1}\} = \{y_{2n+2}\}$ and $\{Tx_{2n+1}\} = \{fx_{2n}\} = \{y_{2n+1}\}$ of $\{y_n\}$ also converge to the point u. Now, since $u \in S(X) \cup T(X)$, we have $u \in S(X)$ or $u \in T(X)$.

If $u \in S(X)$, then we can find $v \in X$ such that $Sv = u$ and claim that $fv = u$. For this, consider

$$\begin{aligned}
d(fv,u) &\leqslant d(fv,gx_{2n+1})+d(gx_{2n+1},u) \\
&\leqslant h\,u_{v,x_{2n+1}}(f,g,S,T)+d(gx_{2n+1},u),
\end{aligned}$$

where

$$u_{v,x_{2n+1}}(f,g,S,T) \in \left\{d(Sv,Tx_{2n+1}),d(fv,Sv),d(gx_{2n+1},Tx_{2n+1}),\frac{d(fv,Tx_{2n+1})+d(gx_{2n+1},Sv)}{2}\right\} \tag{2.5}$$

for each $n \geqslant 1$. Then, from (2.5), we have the following four cases:

Case (a) If $u_{v,x_{2n_k+1}}(f,g,S,T) = d(Sv,Tx_{2n_k+1})$ for all $k \geqslant 1$, then we have

$$d(fv,u) \leqslant hd(Sv,Tx_{2n_k+1}) + d(gx_{2n_k+1},u)$$

and so, as $k \to \infty$, $d(fv,u) \ll c$.

Case (b) If $u_{v,x_{2n_k+1}}(f,g,S,T) = d(fv,Sv)$, then we have

$$d(fv,u) \leqslant hd(fv,Sv) + d(gx_{2n_k+1},u)$$

and so, as $k \to \infty$, $d(fv,u) \ll c$.

Case (c) If $u_{v,x_{2n_k+1}}(f,g,S,T) = d(gx_{2n_k+1},Tx_{2n_k+1})$, then we have

$$d(fv,u) \leqslant hd(gx_{2n_k+1},Tx_{2n_k+1}) + d(gx_{2n_k+1},u)$$

and so, as $k \to \infty$, $d(fv,u) \ll c$.

Case (d) If $u_{v,x_{2n_k+1}}(f,g,S,T) = \dfrac{d(fv,Tx_{2n_k+1})+d(gx_{2n_k+1},sv)}{2}$, the we have

$$\begin{aligned} d(fv,u) &\leqslant h\frac{d(fv,Tx_{2n_k+1})+d(gx_{2n_k+1},Sv)}{2} + d(gx_{2n_k+1},u) \\ &\leqslant \frac{h}{2}d(fv,Tx_{2n_k+1}) + \frac{h}{2}d(gx_{2n_k+1},Sv) + d(gx_{2n_k+1},u) \end{aligned}$$

and so, as $k \to \infty$, $d(fv,u) \ll c$.

Therefore, from the cases (a)-(d), we have $d(fv,u) \ll c$ and so $d(u,fv) \ll \dfrac{c}{m}$ and $\dfrac{c}{m} - d(u,fv) \in P$ for all $m \geqslant 1$. Since $\dfrac{c}{m} \to 0$ as $m \to \infty$ and P is closed, it follows that $-d(u,fv) \in P$. But also $d(u,fv) \in P$ and so $d(u,fv) = 0$. Consequently, we have $fv = Sv = u$ and so, since $u \in f(X) \subset T(X)$, there exists $w \in X$ such that $Tw = u$.

Now, we show that $gw = u$. In fact, consider

$$\begin{aligned} d(gw,u) &\leqslant d(gw,fx_{2n}) + d(fx_{2n},u) \\ &= d(fx_{2n},gw) + d(fx_{2n},u) \\ &\leqslant hu_{x_{2n},w}(f,g,S,T) + d(fx_{2n},u), \end{aligned}$$

where

$$u_{x_{2n},w}(f,g,S,T) \in \left\{d(Sx_{2n},Tw),d(fx_{2n},Sx_{2n}),d(gw,Tw),\frac{d(fx_{2n},Tw)+d(gw,Sx_{2n})}{2}\right\} \tag{2.6}$$

for each $n \geqslant 1$. Then, from (2.6), we have the following four cases:

Case (e) If $u_{x_{2n_k},w}(f,g,S,T) = d(Sx_{2n_k},Tw)$ for each $k \geqslant 1$, then we have

$$d(gw,u) \leqslant h\,d(Sx_{2n_k},Tw) + d(fx_{2n_k},u)$$

and so, as $k \to \infty$, $d(gw,u) \ll c$.

Case (f) If $u_{x_{2n_k},w}(f,g,S,T) = d(fx_{2n_k},Sx_{2n_k})$, then we have

$$d(gw,u) \leqslant hd(fx_{2n_k},Sx_{2n_k}) + d(fx_{2n_k},u)$$

and so, as $k \to \infty$, $d(gw,u) \ll c$.

Case (g) If $u_{x_{2n_k},w}(f,g,S,T) = d(gw,Tw)$, then we have

$$\begin{aligned} d(gw,u) &\leqslant h\,d(gw,Tw) + d(fx_{2n_k},u) \\ &= h\,d(gw,u) + d(fx_{2n_k},u) \end{aligned}$$

and so, as $k \to \infty$, $d(gw,u) \ll c$.

Case (h) If $u_{x_{2n_k},w}(f,g,S,T) = \dfrac{d(fx_{2n_k},Tw) + d(gw,Sx_{2n_k})}{2}$, then we have

$$\begin{aligned} d(gw,u) &\leqslant h\,\frac{d(fx_{2n_k},Tw) + d(gw,Sx_{2n_k})}{2} + d(fx_{2n_k},u) \\ &\leqslant \frac{h}{2}d(fx_{2n_k},u) + \frac{1}{2}d(gw,Sx_{2n_k}) + d(fx_{2n_k},u) \end{aligned}$$

and so, as $k \to \infty$, $d(gw,u) \ll c$.

Therefore, from the cases (e)-(h), $d(gw,u) \ll c$ and so, following similar arguments to those given above, we obtain $gw = Tw = u$. Thus the pairs $\{f,S\}$ and $\{g,T\}$ have a common point of coincidence in X.

Now, if the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, $fu = fSv = Sfv = Su = w_1$ (say) and $gu = gTw = Tgw = Tu = w_2$ (say). Now, we have

$$d(w_1,w_2) = d(fu,gu) \leqslant h\,u_{u,u}(f,g,S,T),$$

where

$$\begin{aligned} u_{u,u}(f,g,S,T)) &\in \left\{ d(Su,Tu), d(fu,Su), d(gu,Tu), \frac{d(fu,Tu) + d(gu,Su)}{2} \right\} \\ &= \{d(w_1,w_2)\}. \end{aligned}$$

Therefore, $d(w_1,w_2) \leqslant h\,d(w_1,w_2)$, which implies that $w_1 = w_2$ and hence $fu = gu = Su = Tu$, i.e., the point u is a coincidence point of the pairs $\{f,S\}$ and $\{g,T\}$.

Now, we show that $u = gu$. In fact, we have

$$d(u,gu) = d(fv,gu) \leqslant h\,u_{v,u}(f,g,S,T),$$

where

$$\begin{aligned} u_{v,u}(f,g,S,T) &\in \left\{ d\,(Sv,Tu)\,, d(fv,Sv), d(gu,Tu), \frac{d\,(fv,Tu) + d\,(gu,Sv)}{2} \right\} \\ &= \{d\,(u,gu)\}. \end{aligned}$$

Thus $d(u,gu) \leqslant hd(u,gu)$, which implies that $gu = u$ and hence u is a common fixed point of the mappings f, g, S and T.

Finally, for the uniqueness of the point u, suppose that u^* is also a common fixed point of f, g, S and T. From (2.4), it follows that

$$d(u,u^*) = d(fu,gu^*) \leqslant h\,u_{u,u^*}(f,g,S,T),$$

where

$$\begin{aligned} &u_{u,u^*}(f,g,S,T) \\ &\in \left\{d(Su,Tu^*), d(fu,Su), d(gu^*,Tu^*), \frac{d(fu,Tu^*)+d(gu^*,Su)}{2}\right\} \\ &= \{d(u,u^*)\}, \end{aligned}$$

which implies that $u = u^*$.

(II) Suppose that $f(X)\cup g(X)$ is complete and $u \in T(X)$. Then proof lines are similar to those of the completeness of $S(X)\cup T(X)$ and $u \in T(X)$ and so we omit here. This completes the proof. $\square$

COROLLARY 2.3. *Let f, g, S and T be self-mappings on a TVS-valued cone metric space X with cone P having non-empty interior, satisfying $f(X)\subset T(X)$, $g(X)\subset S(X)$ and, for some $m,n\geqslant 1$, there exists $h\in(0,1)$ such that*

$$d(f^m x, g^n y) \leqslant h\,u_{x,y}(f^m,g^n,S^m,T^n), \tag{2.7}$$

where

$$\begin{aligned} &u_{x,y}(f^m,g^n;S^m,T^n) \\ &\in \left\{d(S^m x,T^n y), d(f^m x,S^m x), d(g^n y,T^n y), \frac{d(f^m x,T^n y)+d(g^n y,S^m x)}{2}\right\}, \quad \forall x,y\in X. \end{aligned}$$

If one of $f(X)\cup g(X)$ and $S(X)\cup T(X)$ is complete subspace of X, then the pairs $\{f,S\}$ and $\{g,T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then the mappings f, g, S and T have a unique common fixed point in X.

Proof. It follows from Theorem 2.2 that $\{f^m,S^m\}$ and $\{g^m,T^n\}$ have a unique common fixed point $p\in X$. Now, we have

$$f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p)),$$

$$S(p) = S(S^m(p)) = S^{m+1}(p) = S^m(S(p))$$

and so $f(p)$ and $S(p)$ are also fixed points for the mappings f^m and S^m. Hence $f(p) = S(p) = p$. By using the same argument in the proof of Theorem 2.2, we obtain $g(p) = T(p) = p$. This completes the proof. $\square$

The following corollary extends Fisher's well-known result in [6] to TVS-valued cone metric spaces:

COROLLARY 2.4. *Let f, g, S and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and there exists $h \in (0,1)$ such that*

$$d(fx,gy) \leqslant h\,d(Sx,Ty), \quad \forall x,y \in X.$$

If one of $f(X) \cup g(X)$ and $S(X) \cup T(X)$ is a complete subspace of X, then the pairs $\{f,S\}$ and $\{g,T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then the mappings f, g, S and T have a unique common fixed point in X.

COROLLARY 2.5. *Let f, g and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X) \cup g(X) \subset T(X)$ and there exists $h \in (0,1)$ such that*

$$d(fx,gy) \leqslant h\,u_{x,y}(f,g,T),$$

where

$$u_{x,y}(f,g,T) \in \left\{d(Sx,Ty), d(fx,Tx), d(gy,Ty), \frac{d(fx,Ty)+d(gy,Tx)}{2}\right\}, \quad \forall x,y \in X.$$

If one of $f(X) \cup g(X)$ or $T(X)$ is a complete subspace of X, then the pairs $\{f,T\}$ and $\{g,T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f,T\}$ and $\{g,T\}$ are weakly compatible, then the mappings f, g and T have a unique common fixed point in X.

COROLLARY 2.6. *Let f and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X) \subset T(X)$ and there exists $h \in (0,1)$ such that*

$$d(fx,fy) \leqslant h\,u_{x,y}(f,T),$$

where

$$u_{x,y}(f,T) \in \left\{d(Tx,Ty), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}, \quad \forall x,y \in X.$$

If one of $f(X)$ or $T(X)$ is a complete subspace of X, then a pair $\{f,T\}$ have a unique point of coincidence in X. Moreover, if a pair $\{f,T\}$ is weakly compatible, then the mappings f and T has a unique common fixed point in X.

The following theorem extends and improves Theorem 2 of Arshad et al. [3]:

THEOREM 2.7. *Let f, g, S and T be self mappings on a TVS-valued cone metric space X with cone P having the nonempty interior satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and*

$$d(fx,gy) \leqslant pd(Sx,Ty)+qd(fx,Sx)+rd(gy,Ty)+t[d(fx,Ty)+d(gy,Sx)], \quad \forall x,y \in X, \tag{2.8}$$

where $p,q,r,t \in [0,1)$ satisfying $p+q+r+2t<1$. If one of $f(X)\cup g(X)$ and $S(X)\cup T(X)$ is a complete subspace of X, then the pairs $\{f,S\}$ and $\{g,T\}$ have a unique coincidence point in X. Moreover, if the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then the mappings f, g, S and T have a unique common fixed point in X.

Proof. For any arbitrary point x_0 in X, construct the sequences $\{x_n\}$ and $\{y_n\}$ in X as in the proof of Theorem 2.2. Then we have

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &= d(fx_{2n},gx_{2n+1})\\
&\leqslant pd(Sx_{2n},Tx_{2n+1})+qd(fx_{2n},Sx_{2n})+rd(gx_{2n+1},Tx_{2n+1})\\
&\quad +t[d(fx_{2n},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n})]\\
&= pd(y_{2n},y_{2n+1})+qd(y_{2n+1},y_{2n})+rd(y_{2n+2},y_{2n+1})\\
&\quad +t[d(y_{2n+1},y_{2n+1})+d(y_{2n+2},y_{2n})]\\
&= (p+q+r+t)d(y_{2n+1},y_{2n+2})+td(y_{2n},y_{2n+1}),
\end{aligned}$$

which implies that

$$d(y_{2n+1},y_{2n+2}) \leqslant \delta\, d(y_{2n},y_{2n+1}),$$

where $\delta=\frac{t}{1-p-q-r-t}<1$. Thus, from Lemma 2.1, it follows that $\{y_n\}$ is a Cauchy sequence.

Next, the proof lines of the theorem follow from those of Theorem 2.2. This completes the proof. □

Now, we give one example to validate Theorem 2.7:

EXAMPLE 2.1. Let $E=(C_{[0,1]},R)$, $P=\{\varphi\in E:\varphi\geqslant 0\}\subset E$, $X=[0,1]$ and $d:X\times X\to E$ defined by $d(x,y)(t)=(|x-y|)\,e^t$, where $e^t\in E$. Then (X,d) is a cone metric space. Consider four mappings $f,g,T,S:X\to X$ defined by

$$fx=\frac{x}{8},\quad gx=\frac{x}{12},\quad Tx=\frac{x}{2},\quad Sx=\frac{x}{3},\quad \forall x\in X.$$

Clearly, $f(X)\subset T(X)$ and $g(X)\subset S(X)$. For all $x,y\in X$,

$$d(fx,gy)(t)=\left(\left|\frac{x}{8}-\frac{y}{12}\right|\right)e^t=\frac{1}{8}\left(\left|x-\frac{2y}{3}\right|\right)e^t,$$

$$d(Sx,Ty)(t)=\left(\left|\frac{x}{3}-\frac{y}{2}\right|\right)e^t,$$

$$d(fx,Sx)(t)=\left(\left|\frac{x}{8}-\frac{x}{3}\right|)e^t=(\frac{5x}{24}\right)e^t,$$

$$d(gy,Ty)(t)=\left(\left|\frac{y}{12}-\frac{y}{2}\right|)e^t=(\frac{5y}{12}\right)e^t,$$

$$d(fx,Ty)+d(gy,Sx)(t)=\left(\left|\frac{x}{8}-\frac{y}{2}\right|+\left|\frac{y}{12}-\frac{x}{3}\right|\right)e^t$$

and so

$$
\begin{aligned}
& d(fx,gy)(t) \\
= & \frac{1}{8}\left(\left|x-\frac{2y}{3}\right|\right)e^t \\
\leqslant & \frac{1}{6}\left(\left|\frac{x}{3}-\frac{y}{2}\right|\right)e^t+\frac{1}{6}\left(\frac{5x}{24})e^t+\frac{1}{6}(\frac{5y}{12}\right)e^t+\frac{1}{6}\left(\left|\frac{x}{8}-\frac{y}{2}\right|+\left|\frac{y}{12}-\frac{x}{3}\right|\right)e^t \\
= & pd(Sx,Ty)(t)+qd(fx,Sx)(t)+rd(gy,Ty)(t)+t[d(fx,Ty)(t)+d(gy,Sx)(t)].
\end{aligned}
$$

Thus all the conditions of Theorem 2.7 are satisfied with $p+q+r+2t=\frac{5}{6}<1$. Note that 0 is the unique common fixed point of the mappings f, g, S and T.

COROLLARY 2.8. *Let f, g, S and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X)\subset T(X)$, $g(X)\subset S(X)$ and, for some $m,n\geqslant 1$,*

$$
\begin{aligned}
d(f^m x,g^n y)\leqslant\ & pd(S^m x,T^n y)+qd(f^m x,S^m x)+rd(g^n y,T^n y) \\
& +t[d(f^m x,T^n y)+d(g^n y,S^m x)],\quad \forall x,y\in X,
\end{aligned}
$$

where $p,q,r,t\in[0,1[$ satisfying $p+q+r+2t<1$. If one of $f(X)\cup g(X)$ and $S(X)\cup T(X)$ is complete subspace of X, then the pairs $\{f,S\}$ and $\{g,T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f,S\}$ and $\{g,T\}$ are weakly compatible, then the mappings f, g, S and T have a unique common fixed point in X.

The following corollary extends and improves results in Abbas and Rhoades [2]:

COROLLARY 2.9. *Let f, g and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X)\cup g(X)\subset T(X)$ and*

$$
\begin{aligned}
& d(fx,gy) \\
\leqslant\ & pd(Tx,Ty)+qd(fx,Tx)+rd(gy,Ty)+t[d(fx,Ty)+d(gy,Tx)],\quad \forall x,y\in X,
\end{aligned}
$$

where $p,q,r,t\in[0,1[$ satisfying $p+q+r+2t<1$. If one of $f(X)\cup g(X)$ and $T(X)$ is a complete subspace of X, then the pairs $\{f,T\}$ and $\{g,T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f,T\}$ and $\{g,T\}$ are weakly compatible, then the mappings f, g and T have a unique common fixed point in X.

COROLLARY 2.10. *Let f and T be self-mappings on a TVS-valued cone metric space X with a cone P having the non-empty interior satisfying $f(X)\subset T(X)$ and*

$$
\begin{aligned}
& d(f^n x,f^n y) \\
\leqslant\ & pd(Tx,Ty)+qd(fx,Tx)+rd(fy,Ty)+t[d(fx,Ty)+d(fy,Tx)],\quad \forall x,y\in X,
\end{aligned}
$$

where $p,q,r,t\in[0,1[$ satisfying $p+q+r+2t<1$. If one of $f(X)$ or $T(X)$ is a complete subspace of X, then a pair $\{f,T\}$ has a unique point of coincidence in X. Moreover, if a pair $\{f,T\}$ is weakly compatible, then the mappings f and T have a unique common fixed point in X.

Proof. By Theorem 2.7, we obtain $v \in X$ such that $Tv = f^n v = v$. Then the result follows from the fact that

$$\begin{aligned}
& d(fv,v) \\
&= d(ff^n v, f^n v) \\
&\leqslant pd(Tf^n v, Tv) + qd(ff^n v, Tf^n v) + rd(fv, Tv) + t[d(ff^n v, Tv) + d(fv, Tf^n v)] \\
&= qd(fv,v) + rd(fv,v) + t[d(fv,v) + d(fv,v)] \\
&= (q+r+2t)d(fv,v).
\end{aligned}$$

Thus, by using (1) of Remark 1.1, we have $fv = Tv = v$. This completes the proof. □

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