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BOUNDEDNESS FOR MULTILINEAR COMMUTATORS OF INTEGRAL OPERATORS IN HARDY AND HERZ–HARDY SPACES ON HOMOGENEOUS SPACES

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Abstract. In this paper, we shall study the Hardy-boundedness for the multilinear commutators related to the singular integral operators on the space of homogeneous type. By using the Hölder's inequalities and the $L^q(1 < q < \infty)$ boundedness for the singular integral operators on the space of homogeneous type, we obtain the $(H^p_{\vec{b}}, L^p)$ and $(H\dot{K}^{\alpha,p}_{q,\vec{b}}, \dot{K}^{\alpha,p}_q)$ type boundedness for the multilinear commutators on the space of homogeneous type.

1. Introduction

As the development of singular integral operators, their commutators have been well studied. Let $b \in BMO(X)$ and T be the Calderón-Zygmund operator, the commutator $[b,T]$ generated by b and T is defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss(see [8]) proved that the commutator $[b,T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that the $[b,T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^P_{\vec{b}}(R^n)$ and $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$, then $[b,T]$ maps continuously $H^P_{\vec{b}}(R^n)$ into $L^p(R^n)$ and $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ into $\dot{K}^{\alpha,p}_q(R^n)$. The main purpose of this paper is to consider the continuity of the multilinear commutators associated with the singular integral operator and $BMO(X)$ functions in certain Hardy and Herz-Hardy spaces on spaces of homogeneous type.

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2. Definitions and Results

Let us first introduce some definitions (see [1-7]). Give a set X, a function d : $X \times X \to R^+$ is called a quasi-distance on X if the following conditions are satisfied:
(i) for every x and y in X, $d(x,y) \geqslant 0$ and $d(x,y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x,y) = d(y,x)$,
(iii) there exists a constant $k \geqslant 1$ such that

$$d(x,y) \leqslant k(d(x,z) + d(z,y)) \tag{1}$$

for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the r-balls $B(x,r) = \{y : d(x,y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant $A_1 > 1$ such that

$$0 < \mu(B(x,2r)) \leqslant A_1\mu(B(x,r)) < \infty \tag{2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X,d,μ), with d and μ as above, is called a space of homogeneous type. The constants k and A_1 in (1) and (2) will be called the constants of the space.

From (2), we can say that there exists a constant $n_0 > 1$ such that $A_1 \leqslant 2^{n_0}$, in other words, there exists a constant $n_0 > 1$ such that $\mu(B(x,2r)) \leqslant 2^{n_0}\mu(B(x,r))$. This condition is very useful in the proofs of Theorem 1 and Theorem 2.

We say that (X,d,μ) satisfies a reverse doubling condition , that is, there exists a constant $A_2 > 1$ such that

$$0 < A_2\mu(B(x,r)) \leqslant \mu(B(x,2r)) < \infty \tag{3}$$

holds for all $x \in X$ and $r > 0$.It can be proved that, under some general additional geometric assumptions on the space (X,d), (3) is actually a consequence of the doubling condition on μ (see [11]). In this paper, the homogeneous spaces which we discussing are satisfied the reverse doubling condition.

In this paper, B will denote a ball of X , and for a ball B let $f_B = \mu(B)^{-1}\int_B f(x)d\mu(x)$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y).$$

It is well-known that (see [10])

$$f^{\#}(x) \approx \sup_{B \ni x} \inf{}_{c \in C} \frac{1}{\mu(B)} \int_B |f(y) - c| d\mu(y).$$

We say that b belongs to $BMO(X)$ if $b^{\#}$ belongs to $L^\infty(X)$ and define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that(see [10])

$$||b - b_{2^kB}||_{BMO} \leqslant Ck||b||_{BMO}.$$

Given a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

DEFINITION 1. Let b_i $(i = 1, \cdots, m)$ be a locally integrable functions and $0 < p \leqslant 1$. A bounded measurable function $a(x)$ on X is called a $(p, \vec{b})$ atom, if

(1) $supp\ a \subset B = B(x_0, r)$,

(2) $||a||_{L^\infty} \leqslant \mu(B)^{-1/p}$,

(3) $\int_B a(y) d\mu(y) = \int_B a(y) \prod_{l \in \sigma} b_l(y) d\mu(y) = 0$ for any $\sigma \in C_j^m$, $1 \leqslant j \leqslant m$.

A temperate distribution (see [9][12-13]) f is said to belong to $H_{\vec{b}}^p(X)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j are $(p, \vec{b})$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$.

Moreover, $||f||_{H_{\vec{b}}^p} = \inf (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$, where the infimum are taken over all the decompositions of f as above.

DEFINITION 2. Let $\alpha \in R$, $0 < p < \infty$ and $1 \leqslant q < \infty$. For $k \in \mathbf{Z}$ and $x_0 \in X$, set $B_k = \{x \in X : d(x_0, x) \leqslant 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(X) = \left\{ f \in L_{loc}^q(X \setminus \{x_0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k=-\infty}^{\infty} \mu(B_k)^{\alpha p} ||f\chi_k||_{L^q}^p \right)^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(X) = \left\{ f \in L_{loc}^q(X) : ||f||_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} \mu(B_k)^{\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_0||_{L^q}^p \right]^{1/p}.$$

DEFINITION 3. Let b_i $(i = 1, \cdots, m)$ be locally integrable functions, $1 < q < \infty$, $\alpha \geqslant 1 - 1/q$. A function a is called a central $(\alpha, q, \vec{b})$ -atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type), if

(1) supp $a \subset B = B(x_0, r)$ (or for some $r \geqslant 1$),

(2) $||a||_{L^q} \leqslant \mu(B)^{-\alpha}$,

(3) $\int_B a(x) d\mu(x) = \int_B a(x) \prod_{i \in \sigma} b_i(x) d\mu(x) = 0$ for any $\sigma \in C_j^m$, $1 \leqslant j \leqslant m$.

A function f is said to belong to homogeneous $H\dot{K}^{\alpha,p}_{q,\vec{b}}(X)$ (or nonhomogeneous $HK^{\alpha,p}_{q,\vec{b}}(X)$), if it can be written as $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ (or $f=\sum_{j=0}^{\infty}\lambda_j a_j(x)$), where a_j is a central $(\alpha,q,\vec{b})$-atom(or a central $(\alpha,q,\vec{b})$-atom of restrict type) supported on $B(x_0,2^j)$ and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$ (or $\sum_{j=0}^{\infty}|\lambda_j|^p<\infty$). Moreover,

$$||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}=\inf(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p} \quad (\text{or } \inf(\sum_{j=0}^{\infty}|\lambda_j|^p)^{1/p}),$$

where the infimum are taken over all the decompositions of f as above.

REMARK 1. Using atoms, Coifman and Weiss (see [5]) defined the Hardy space $H^p(X)$ as a subspace of the dual of $Lip_\alpha(X)$ and they proved that $Lip_\alpha(X)$ is the dual of $H^p(X)$. In [5], $Lip_\alpha(X)$ was regarded as the space of functions modulo constants. Therefore, we denote by $(H^p(X))^*=Lip_\alpha(X)/\wp$ and $\wp$ is the space of all constant functions.

DEFINITION 4. Suppose b_j $(j=1,\cdots,m)$ are the fixed locally integrable functions on X. Let T be the singular integral operator as

$$T(f)(x)=\int_X K(x,y)f(y)d\mu(y),$$

where K is a locally integrable function on $X\times X\backslash\{(x,y)|\ x=y\}$ and satisfies the following properties:

(1) $|K(x,y)|\leqslant \dfrac{C}{\mu(B(x,d(x,y)))}$,

(2) there exists a $p_0, 1<p_0\leqslant\infty$, such that T is bounded on $L^{p_0}(X)$,

(3) $|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|\leqslant C\dfrac{d(y,y')^\delta}{\mu(B(y,d(x,y)))d(x,y)^\delta}$,

when $d(x,y)\geqslant 2d(y,y')$, with some $\delta\in(0,1]$.

The multilinear commutator of the singular integral operator is defined by

$$T_{\vec{b}}(f)(x)=\int_X\prod_{j=1}^m(b_j(x)-b_j(y))K(x,y)f(y)d\mu(y).$$

Note that when $b_1=\cdots=b_m$, $T_{\vec{b}}$ is just the m order commutators. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-4][6-7]).

Now we state our theorems as following.

THEOREM 1. *Let* $1<q<\infty$, $b_i\in BMO(X)$, $1\leqslant i\leqslant m$, $\vec{b}=(b_1,\cdots,b_m)$, $n_0/(n_0+\delta)<p\leqslant 1$, *Then the multilinear commutator* $T_{\vec{b}}$ *is bounded from* $H^p_{\vec{b}}(X)$ *to* $L^p(X)$.

THEOREM 2. *Let* $1<q<\infty$, $b_i\in BMO(X)$, $1\leqslant i\leqslant m$, $\vec{b}=(b_1,\cdots,b_m)$, $0<p<\infty$ *and* $1-1/q\leqslant\alpha<1-1/q+\delta/n_0$. *Then the multilinear commutator* $T_{\vec{b}}$ *is bounded from* $H\dot{K}^{\alpha,p}_{q,\vec{b}}(X)$ *to* $\dot{K}^{\alpha,p}_q(X)$.

3. Proof of Theorems

We begin with two preliminary lemmas.

LEMMA 1.. ([10]) *Let* $1<r<\infty$, $b_j \in BMO(X)$ *for* $j=1,\cdots,k$ *and* $k\in N$. *Then*

$$\frac{1}{\mu(B)}\int_B \prod_{j=1}^{k}|b_j(y)-(b_j)_B|d\mu(y)\leqslant C\prod_{j=1}^{k}||b_j||_{BMO}$$

and

$$\left(\frac{1}{\mu(B)}\int_B \prod_{j=1}^{k}|b_j(y)-(b_j)_B|^r d\mu(y)\right)^{1/r}\leqslant C\prod_{j=1}^{k}||b_j||_{BMO}.$$

LEMMA 2. ([10]) *Let* $1<q<p_0$, *then* T *and* $T_{\vec b}$ *are bounded on* $L^q(X)$.

Proof of Theorem 1. It suffices to show that there exist a constant $C>0$, such that for every $(p,\vec b)$ atom a,

$$||T_{\vec b}(a)||_{L^p}\leqslant C.$$

Let a be a $(p,\vec b)$ atom supported on a ball $B=B(x_0,r)$. We write

$$\int_X |T_{\vec b}(a)(x)|^p d\mu(x)=\int_{2B}|T_{\vec b}(a)(x)|^p d\mu(x)+\int_{(2B)^c}|T_{\vec b}(a)(x)|^p d\mu(x)=I+II.$$

For I, taking $q>1$, by Hölder's inequality and the L^q- boundedness of $T_{\vec b}$, we have

$$\begin{aligned}I&\leqslant\left(\int_{2B}|T_{\vec b}(a)(x)|^q d\mu(x)\right)^{p/q}\cdot\mu(2B)^{1-p/q}\\&\leqslant C||T_{\vec b}(a)||_{L^q}^p\cdot\mu(2B)^{1-p/q}\\&\leqslant C||\vec b||_{BMO}^p||a||_{L^q}^p\mu(B)^{1-p/q}\\&\leqslant C||\vec b||_{BMO}^p||a||_{L^\infty}^p\mu(B)^{p/q}\mu(B)^{1-p/q}\\&\leqslant C||\vec b||_{BMO}^p.\end{aligned}$$

For II, when $m=1$, by the Hölder's inequality and the vanishing moment of atom a, we get, for $x\in(2B)^c$,

$$\begin{aligned}|T_{b_1}(a)(x)|&\leqslant\int_B|K(x,y)-K(x,x_0)||b_1(x)-b_1(y)||a(y)|d\mu(y)\\&\leqslant C\int_B\frac{d(x_0,y)^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}(|b_1(x)-(b_1)_B|+|(b_1)_B-b_1(y)|)d\mu(y)||a||_{L^\infty}\\&\leqslant C\frac{r^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}(|b_1(x)-(b_1)_B|+||b_1||_{BMO})\mu(B)^{1-1/p},\end{aligned}$$

so

$$\begin{aligned}
II &\leqslant C\int_{(2B)^c}\left[\frac{r^{\delta}}{d(x,x_0)^{\delta}\mu(B(x_0,d(x,x_0)))}(|b_1(x)-(b_1)_B|+||b_1||_{BMO})\mu(B)^{1-1/p}\right]^p d\mu(x)\\
&\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{r^{p\delta}}{d(x,x_0)^{p\delta}\mu(B(x_0,d(x,x_0)))^p}\\
&\quad\times(|b_1(x)-(b_1)_B|+||b_1||_{BMO})^p d\mu(x)\mu(B)^{p-1}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{r^{p\delta}}{(2^kr)^{p\delta}\mu(2^kB)^p}\mu(B)^{p-1}\left[\int_{2^{k+1}B}|b_1(x)-(b_1)_B|^p d\mu(x)+||b_1||_{BMO}^p\mu(2^{k+1}B)\right]\\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\delta}\mu(2^kB)^{-p}\mu(B)^{p-1}\left[\mu(2^{k+1}B)^{1-p}\left(\int_{2^{k+1}B}|b_1(x)-(b_1)_B|d\mu(x)\right)^p\right.\\
&\quad\left.+||b_1||_{BMO}^p\mu(2^{k+1}B)\right]\\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\delta}\mu(2^kB)^{1-p}\mu(B)^{p-1}(k^p||b_1||_{BMO}^p+||b_1||_{BMO}^p)\\
&\leqslant C||b_1||_{BMO}^p\sum_{k=1}^{\infty}(k^p+1)2^{-kp\delta+n_0k(1-p)}\\
&\leqslant C||b_1||_{BMO}^p.
\end{aligned}$$

This finishes the proof of the case of $m=1$.

When $m>1$, denoting $\vec{b}_B=((b_1)_B,\cdots,(b_m)_B)$, where

$$(b_i)_B=\mu(B)^{-1}\int_{B(x_0,r)}b_i(x)d\mu(x),\qquad 1\leqslant i\leqslant m,$$

by Hölder's inequality and the vanishing moment of a, noting that $x\in 2^{k+1}B\setminus 2^kB$, we get

$$\begin{aligned}
T_{\vec{b}}(a)(x) &= \int_B\prod_{j=1}^m|b_j(x)-b_j(y)|K(x,y)a(y)d\mu(y)\\
&\leqslant C\int_B|K(x,y)-K(x,x_0)|\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|d\mu(y)\\
&\leqslant C\int_B\frac{d(y,x_0)^{\delta}}{d(x,x_0)^{\delta}\mu(B(x_0,d(x,x_0)))}\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|d\mu(y)\\
&\leqslant C\frac{r^{\delta}}{d(x,x_0)^{\delta}\mu(B(x_0,d(x,x_0)))}\int_B\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|d\mu(y),
\end{aligned}$$

so

$$
\begin{aligned}
II &\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|T_{\vec{b}}(a)(x)|^p d\mu(x)\\
&\leqslant C\sum_{k=1}^{\infty}\mu(2^{k+1}B)^{1-p}\left(\int_{2^{k+1}B\setminus 2^kB}|T_{\vec{b}}(a)(x)|d\mu(x)\right)^p\\
&\leqslant C\sum_{k=1}^{\infty}\mu(2^{k+1}B)^{1-p}\Bigg(\int_{2^{k+1}B\setminus 2^kB}\frac{r^\delta}{d(x,x_0)^\delta\mu(B(x_0,d(x,x_0)))}\\
&\quad\times\left(\int_B\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|d\mu(y)\right)d\mu(x)\Bigg)^p\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\mu(2^{k+1}B)^{1-p}r^{p\delta}}{(2^kr)^{p\delta}\mu(2^kB)^p}\Bigg(\sum_{j=0}^m\sum_{\sigma\in C_j^m}\int_{2^{k+1}B}|(\vec{b}(x)-\vec{b}_B)_\sigma|d\mu(x)\\
&\quad\times\int_B|(\vec{b}(y)-\vec{b}_B)_{\sigma^c}||a(y)|d\mu(y)\Bigg)^p\\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\delta}\frac{\mu(2^{k+1}B)^{1-p}}{\mu(2^kB)^p}\sum_{j=0}^m\sum_{\sigma\in C_j^m}\left(\int_{2^{k+1}B}|(\vec{b}(x)-\vec{b}_B)_\sigma|d\mu(x)\right)^p\\
&\quad\times\left(\int_B|(\vec{b}(y)-\vec{b}_B)_{\sigma^c}||a(y)|d\mu(y)\right)^p\\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\delta}k^p\mu(2^{k+1}B)^{1-p}\sum_{j=0}^m\sum_{\sigma\in C_j^m}||a||_{L^\infty}^p\left(\int_B|(\vec{b}(y)-\vec{b}_B)_{\sigma^c}|d\mu(y)\right)^p||\vec{b}_\sigma||_{BMO}^p\\
&\leqslant C\sum_{k=1}^{\infty}2^{-kp\delta}k^p\mu(2^{k+1}B)^{1-p}\mu(B)^{p-1}||\vec{b}||_{BMO}^p\\
&\leqslant C||\vec{b}||_{BMO}^p\sum_{j=0}^m k^p2^{-kp\delta+n_0k(1-p)}\\
&\leqslant C||\vec{b}||_{BMO}^p.
\end{aligned}
$$

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Let $f\in H\dot{K}_{q,\vec{b}}^{\alpha,p}(X)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_ja_j(x)$ be the atomic decomposition for f as in Definition 3, we write

$$
\begin{aligned}
||T_{\vec{b}}(f)(x)||_{\dot{K}_q^{\alpha,p}} &= \left(\sum_{k=-\infty}^{\infty}\mu(B)^{\alpha p}||T_{\vec{b}}(f)\chi_k||_{L^q}^p\right)^{1/p}\\
&\leqslant C\left[\sum_{k=-\infty}^{\infty}\mu(B)^{\alpha p}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\left[\sum_{k=-\infty}^{\infty}\mu(B)^{\alpha p}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q}\Big)^p\right]^{1/p}\\
&\quad+C\left[\sum_{k=-\infty}^{\infty}\mu(B)^{\alpha p}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q}\Big)^p\right]^{1/p}\\
&=J+JJ.
\end{aligned}
$$

For JJ, by the boundedness of $T_{\vec{b}}$ on $L^q(X)$ and the Hölder's inequality, we have

$$
\begin{aligned}
JJ &\leqslant C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p}\\
&\leqslant C\Big[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j\chi_k||_{L^q}\Big)^p\Big]^{1/p}\\
&\leqslant C\Big[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\Big(\sum_{j=k-2}^{\infty}|\lambda_j|\mu(B_j)^{-\alpha}\Big)^p\Big]^{1/p}\\
&\leqslant C\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big(\sum_{k=-\infty}^{j+2}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p/2}\Big)\Big(\sum_{k=-\infty}^{j+2}\left(\frac{\mu(B_k)}{\mu(B_j)}\right)^{\alpha p'/2}\Big)^{p/p'}\right]^{1/p}, & \\ \quad 1<p<\infty\end{cases}\\
&\leqslant C\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}A_2{}^{(k-j)\alpha p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=-\infty}^{j+2}A_2{}^{(k-j)\alpha p/2})(\sum_{k=-\infty}^{j+2}A_2{}^{(k-j)\alpha p'/2})^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}\\
&\leqslant C\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leqslant C||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}.
\end{aligned}
$$

For J, let $C_k=B_k\backslash B_{k-1}$, $\chi_k=\chi_{C_k}$, $b^i_j=\mu(B_j)^{-1}\int_{B_j}b_i(x)d\mu(x)$, $1\leqslant i\leqslant m, \vec{b}_{B_j}=(b^1_j,\cdots,b^m_j)$. By the Hölder's inequality, the vanishing moment of atom a, the reverse doubling condition, we get, for $x\in 2^{k+1}B\setminus 2^kB$ and $u\in B$,

$$
\begin{aligned}
&|T_{b_1}(a_j)(x)|\\
&=|\int_{B_j}K(x,y)(b_1(x)-b_1(y))a_j(y)d\mu(y)|\\
&\leqslant C\int_{B_j}|K(x,y)-K(x,u)||b_1(x)-b_1(y)||a_j(y)|d\mu(y)
\end{aligned}
$$

$$\leqslant C\int_{B_j}\frac{d(u,y)^\delta}{d(x,u)^\delta\mu(B(u,d(x,u)))}(|b_1(x)-b_1(y)|)|a_j(y)|d\mu(y)$$
$$\leqslant C\frac{{r_j}^\delta}{d(x,u)^\delta\mu(B(u,d(x,u)))}\int_{B_j}(|b_1(x)-b_1(y)|)|a_j(y)|d\mu(y)$$
$$\leqslant C\frac{{r_j}^\delta}{d(x,u)^\delta\mu(B(u,d(x,u)))}\left(\int_{B_j}|a_j(y)||b_1(x)-b_j^1|d\mu(y)+\int_{B_j}|a_j(y)||b_1(y)-b_j^1|d\mu(y)\right)$$
$$\leqslant C\frac{{r_j}^\delta}{d(x,u)^\delta\mu(B(u,d(x,u)))}\left(|b_1(x)-b_j^1|||a_j||_{L^q}\mu(B_j)^{1-1/q}+||a_j||_{L^q}\mu(B_j)^{1-1/q}||b_1||_{BMO}\right)$$
$$\leqslant C\frac{{r_j}^\delta}{d(x,u)^\delta\mu(B(u,d(x,u)))}\mu(B_j)^{1-1/q-\alpha}(|b_1(x)-b_j^1|+||b_1||_{BMO}),$$

then

$$\begin{aligned}||T_{b_1}(a_j)(x)\chi_k||_{L^q} &\leqslant C{r_j}^\delta\mu(B_j)^{1-1/q-\alpha}\left[\int_{B_k\setminus B_{k-1}}\left(|b_1(x)-b_j^1|+||b_1||_{BMO}\right)^q d(x,u)^{-q\delta}\right.\\ &\qquad\left.\times\mu(B(u,d(x,u)))^{-q}d\mu(x)\right]^{1/q}\\ &\leqslant C{r_j}^\delta\mu(B_j)^{1-1/q-\alpha}{r_{k-1}}^{-\delta}\mu(B_{k-1})^{-1}\\ &\qquad\times\left[\left(\int_{B_k}|b_1(x)-b_j^1|^q d\mu(x)\right)^{1/q}+||b_1||_{BMO}\mu(B_k)^{1/q}\right]\\ &\leqslant C2^{(j-k+1)\delta}\mu(B_j)^{1-1/q-\alpha}\mu(B_k)^{-1+1/q}||b_1||_{BMO}\\ &\leqslant C||b_1||_{BMO}2^{(j-k)\delta}\mu(B_j)^{1-1/q-\alpha}\mu(B_k)^{-1+1/q},\end{aligned}$$

thus

$$J=C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_{b_1}(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p}$$
$$\leqslant C||b_1||_{BMO}\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{(j-k)\delta}\mu(B_j)^{1-1/q-\alpha}\mu(B_k)^{-(1-1/q)}\right)^p\right]^{1/p}$$
$$\leqslant C||b_1||_{BMO}\begin{cases}\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)\delta p}\mu(B_j)^{(1-1/q-\alpha)p}\mu(B_k)^{-(1-1/q)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)\delta p/2}\mu(B_j)^{(1-1/q-\alpha)p/2}\times\mu(B_k)^{-(1-1/q)p/2}\right)\right.\\ \quad\left.\times\left(\sum_{j=-\infty}^{k-3}2^{(j-k)\delta p'/2}\mu(B_j)^{(1-1/q-\alpha)p'/2}\mu(B_k)^{-(1-1/q)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$
\leqslant C||b_1||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)\delta p}2^{(j-k)n_0(1-1/q-\alpha)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)\delta p/2}2^{(j-k)n_0(1-1/q-\alpha)p/2}\right)\right. & \\ \left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)\delta p'/2}2^{(j-k)n_0(1-1/q-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}
$$

$$
\leqslant C||b_1||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q+\delta/n_0-\alpha)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q+\delta/n_0-\alpha)p/2}\right)\right. & \\ \left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q+\delta/n_0-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}
$$

$$
\leqslant C||b_1||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}
$$

$$
\leqslant C||f||_{H\dot{K}^{\alpha,p}_{q,b_1}}.
$$

When $m>1$, let $b^i_j=\mu(B_j)^{-1}\int_{B_j}b_i(x)d\mu(x),\ 1\leqslant i\leqslant m, \vec{b}_{B_j}=(b^1_j,\cdots,b^m_j)$. We have

$$
\begin{aligned}
|T_{\vec{b}}(a_j)(x)| &= |\int_{B_j}\prod_{i=1}^{m}(b_i(x)-b_i(y))K(x,y)a_j(y)d\mu(y)|\\
&\leqslant C\int_{B_j}\prod_{i=1}^{m}|(b_i(x)-b_i(y))||K(x,y)-K(x,u)||a_j(y)|d\mu(y)\\
&\leqslant C\int_{B_j}\frac{d(u,y)^{\delta}}{d(x,u)^{\delta}\mu(B(u,d(x,u)))}\prod_{i=1}^{m}|(b_i(x)-b_i(y))||a_j(y)|d\mu(y)\\
&\leqslant C\frac{{r_j}^{\delta}}{d(x,u)^{\delta}\mu(B(u,d(x,u)))}\int_{B_j}\prod_{i=1}^{m}|(b_i(x)-b_i(y))||a_j(y)|d\mu(y)\\
&\leqslant C\frac{{r_j}^{\delta}}{d(x,u)^{\delta}\mu(B(u,d(x,u)))}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_{B_j})_{\sigma}|\\
&\quad\times\int_{B_j}|a_j(y)||(\vec{b}(y)-\vec{b}_{B_j})_{\sigma_c}|d\mu(y)\\
&\leqslant C\frac{{r_j}^{\delta}}{d(x,u)^{\delta}\mu(B(u,d(x,u)))}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_{B_j})_{\sigma}|||a_j||_{L^q}\\
&\quad\times\left(\int_{B_j}|(\vec{b}(y)-\vec{b}_{B_j})^{q'}_{\sigma_c}|d\mu(y)\right)^{1/q'}
\end{aligned}
$$

$$\leqslant C\frac{r_j{}^{\delta}}{d(x,u)^{\delta}\mu(B(u,d(x,u)))}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_{B_j})_\sigma|\mu(B_j)^{1-1/q-\alpha}||\vec{b}_{\sigma^c}||_{BMO}$$
$$\leqslant C\frac{r_j{}^{\delta}\mu(B_j)^{1-1/q-\alpha}}{d(x,u)^{\delta}\mu(B(u,d(x,u)))}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_{B_j})_\sigma|||\vec{b}_{\sigma^c}||_{BMO},$$

so

$$\begin{aligned}||T_{\vec{b}}(a_j)\chi_k||_{L^q} &\leqslant Cr_j^{\delta}\mu(B_j)^{1-1/q-\alpha}||\vec{b}_{\sigma^c}||_{BMO}\\ &\quad\times\left(\int_{B_j}\frac{1}{d(x,u)^{q\delta}\mu(B(u,d(x,u)))^q}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}_{B_j})_\sigma|^q d\mu(x)\right)^{1/q}\\ &\leqslant Cr_j^{\delta}\mu(B_j)^{1-1/q-\alpha}||\vec{b}_{\sigma^c}||_{BMO}r_k^{-\delta}\mu(B_k)^{-1+1/q}||\vec{b}_{\sigma}||_{BMO}\\ &\leqslant C||\vec{b}||_{BMO}2^{(j-k)\delta}\mu(B_j)^{1-1/q-\alpha}\mu(B_k)^{-1+1/q}\end{aligned}$$

and

$$\begin{aligned}J &= C\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q}\right)^p\right]^{1/p}\\ &\leqslant C||\vec{b}||_{BMO}\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{(j-k)\delta}\mu(B_j)^{1-1/q-\alpha}\mu(B_k)^{-(1-1/q)}\right)^p\right]^{1/p}\\ &\leqslant C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)\delta p}\mu(B_j)^{(1-1/q-\alpha)p}\mu(B_k)^{-(1-1/q)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{k=-\infty}^{\infty}\mu(B_k)^{\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)\delta p/2}\mu(B_j)^{(1-1/q-\alpha)p/2}\times\mu(B_k)^{-(1-1/q)p/2}\right)\right.\\ \quad\left.\times\left(\sum_{j=-\infty}^{k-3}2^{(j-k)\delta p'/2}\mu(B_j)^{(1-1/q-\alpha)p'/2}\mu(B_k)^{-(1-1/q)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}\\ &\leqslant C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)\delta p}2^{(j-k)n_0(1-1/q-\alpha)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)\delta p/2}2^{(j-k)n_0(1-1/q-\alpha)p/2}\right)\right.\\ \quad\left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)\delta p'/2}2^{(j-k)n_0(1-1/q-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}\end{aligned}$$

$$\leqslant C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q+\delta/n_0-\alpha)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q+\delta/n_0-\alpha)p/2}\right)\right.\\ \quad\left.\times\left(\sum_{k=j+3}^{\infty}2^{(j-k)n_0(1-1/q+\delta/n_0-\alpha)p'/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}$$

$$\leqslant C||\vec{b}||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}$$

$$\leqslant C||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}.$$

This completes the proof of Theorem 2. □

REMARK 2. Theorem 2 also holds for nonhomogeneous Herz-type spaces, we omit the details.

REFERENCES

[1] J. ALVAREZ, *Continuity properties for linear commutators of Calderón-Zygmund operators*, Collect. Math., **49** (1998), 17–31.

[2] J. ALVAREZ, R. J. BABGY, D. S. KURTZ AND C. PÉREZ, *Weighted estimates for commutators of linear operators*, Studia Math., **104** (1993), 195–209.

[3] A. BERNARDIS, S. HARTZSTEIN AND G. PRADOLINI, *Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type*, J. Math. Anal. Appl., **322** (2006), 825–846.

[4] J. J. BETANCOR, *A commutator theorem for fractional integrals in spaces of homogeneous type*, Inter. J. Math. & Math. Sci., **6**, 24 (2000), 419–421.

[5] M. BRAMANTI AND M. C. CERUTTI, *Commutators of singular integrals and fractional integrals on homogeneous spaces, Harmonic Analysis and Operator Theory*, Contemp. Math., **189**, Amer. Math. Soc., 1995, 81–94.

[6] M. BRAMANTI-M. C. CERUTTI, *Commutator of singular integrals on homogeneous spaces*, Boll. Un. Mat. Ital. B (7) **10** (1996), 843–883.

[7] W. CHEN AND E. SAWYER, *Endpoint estimates for commutators of fractional integrals on spaces of homogeneous type*, J. Math. Anal. Appl., **282** (2003), 553–566.

[8] R. COIFMAN, R. ROCHBERG AND G. WEISS, *Factorization theorem for Hardy space in several variables*, Ann. of Math., **103** (1976), 611–635.

[9] J. GARCIA-CUERVA AND M. L. HERRERO, *A Theory of Hardy spaces associated to Herz Spaces*, Proc. London Math. Soc., **69** (1994), 605–628.

[10] I. GENEBASHVILI, A. GOGATISHVILI, V. KOKILASHVILI AND M. KRBEC, *Weighted theory for integral transforms on spaces of homogeneous type*, Piman Monogr. and Surveys in Pure and Appl. Math., 92, Addison-Wesley/Longman, 1998.

[11] S. Z. LU, *Four lectures on real H^p spaces*, World Scientific, River Edge, NI, 1995.

[12] S. Z. LU AND D. C. YANG, *The decomposition of the weighted Herz spaces and its applications*, Sci. in China (ser. A), **38** (1995), 147–158.

[13] S. Z. LU AND D. C. YANG, *The weighted Herz type Hardy spaces and its applications*, Sci. in China (ser. A), **38** (1995), 662–673.

[14] C. PÉREZ, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal., **128** (1995), 163–185.

[15] C. PÉREZ AND G. PRADOLINI, *Sharp weighted endpoint estimates for commutators of singular integral operators*, Michigan Math. J., **49** (2001), 23–37.
[16] C. PÉREZ AND R. TRUJILLO-GONZALEZ, *Sharp Weighted estimates for multilinear commutators*, J. London Math. Soc., **65** (2002), 672–692.
[17] G. PRADOLINI, O. SALINAS, *Commutators of singular integrals on space of homogeneous type*, preprint, available at `http://math.unl.edu.ar/preprints/`.
[18] E. M. STEIN, *Harmonic Analysis, real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

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