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WEIGHTED ENDPOINT ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD–PALEY OPERATOR

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Abstract. In this paper, we prove the weighted endpoint estimates for multilinear commutator of Littlewood-Paley operator.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b,T]$ generated by b and T is defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [4]) proved that the commutator $[b,T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [3], [6], the boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we will introduce the multilinear commutator of Littlewood-Paley operator and prove the weighted boundedness properties of the operator for the extreme cases.

First let us introduce some notations (see [2], [5], [9], [10]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f(Q) = \int_Q f(x)dx$, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy.$$

It is well-known that (see [5])

$$f^{\#}(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y) - c|dy.$$

Moreover, for a weight function ω, f is said to belongs to $BMO(\omega)$ if $f^{\#} \in L^{\infty}(\omega)$ and define $||f||_{BMO} = ||f^{\#}||_{L^{\infty}(\omega)}$, if $\omega = 1$, we denote $BMO(\omega) = BMO(R^n)$. It has been known that (see[11])

$$||b - b_{2^kQ}||_{BMO} \leqslant Ck||b||_{BMO}.$$

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DEFINITION 1. A function a is called a $H^1(\omega)$-atom, if there exists a cube Q, such that

1) $supp a \subset Q = Q(x_0, r)$,
2) $||a||_{L^\infty(\omega)} \leqslant \omega(Q)^{-1}$,
3) $\int_{R^n} a(x)dx = 0$.

It is well known that the weighted Hardy space $H^1(\omega)$ has the atomic decomposition characterization (see [2], [5]).

DEFINITION 2.
Let $\varepsilon > 0$ and ψ be a fixed function satisfies the following properties:

1) $\int \psi(x)dx = 0$,
2) $|\psi(x)| \leqslant C(1+|x|)^{-(n+1)}$,
3) $|\psi(x+y) - \psi(x)| \leqslant C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

Let $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \leqslant j \leqslant m$. Then the multilinear commutator of Littlewood-Paley operator is defined by

$$S_\psi^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(z))\psi_t(y-z)f(z)dz;$$

When $m = 1$, set

$$S_\psi^{b}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{b}(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^b(f)(x,y) = \int_{R^n} (b(x) - b(z))\psi_t(y-z)f(z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define that

$$S_\psi(f)(x) = \left[\int\int_{\Gamma(x)} |f * \psi_t(x)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley operator (see [1][7-8][11]).

Let H be the Hilbert space $H = \{h : ||h|| = (\int\int_{R_+^{n+1}} |h(y,t)|^2 dydt/t^{n+1})^{1/2} < \infty\}$. Then for each fixed $x \in R^n$, $F_t(f)(x,y)$ may be viewed as a mapping from $[0,+\infty)$ to H. It is clear that

$$S_\psi(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)|| \text{ and } S_\psi^{\vec{b}}(f)(x) = ||\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(y)||.$$

Given a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

DEFINITION 3. For a non-negative weighted function ω, we define the weighted central BMO space by $CMO(\omega)$, which is the space of those functions $f \in L_{loc}(R^n)$, such that

$$||f||_{CMO(\omega)} = \sup_{r>1} \omega(Q(0,r))^{-1} \int_Q |f(x) - f_Q|\omega(x)dx < \infty.$$

It is well known that (see [5][11])

$$||f||_{CMO(\omega)} \approx \sup_{r>1} \inf_{c\in C} \omega(Q(0,r))^{-1} \int_Q |f(x) - c|\omega(x)dx.$$

DEFINITION 4. Let $1 < p < \infty$ and ω be a non-negative weighted functions on R^n. We shall call $B_p(\omega)$ the space of those functions f on R^n, such that

$$||f||_{B_p(\omega)} = \sup_{r>1} [\omega(Q(0,r))]^{-1/p} ||f\chi_{Q(0,r)}||_{L^p(\omega)} < \infty.$$

2. Theorems and Proofs

We begin with two preliminaries lemmas.

LEMMA 1. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1,\cdots,k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|dy \leqslant C\prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \leqslant C\prod_{j=1}^k ||b_j||_{BMO}.$$

LEMMA 2. *Let $w \in A_1$ and $1 < p < \infty$. Then S_ψ is boundedness on $L^p(w)$.*

LEMMA 3. *Let $\omega \in A_p$, $1 < p < \infty$, then $\omega\chi_Q \in A_p$ for any cube Q.*

THEOREM 1. *Let $\omega \in A_1$ and $\vec{b} = (b_1,\cdots,b_m)$ with $b_j \in BMO(R^n)$ for $1 \leqslant j \leqslant m$. Then $S_\psi^{\vec{b}}$ is bounded from $L^\infty(\omega)$ to $BMO(\omega)$.*

Proof. It is only to prove that there exist a constant C_Q such that

$$\frac{1}{\omega(Q)}\int_Q |S_\psi^{\vec{b}}(f)(x) - C_Q|\omega(x)dx \leqslant C||f||_{L^\infty(\omega)}.$$

Let $\omega \in A_1$, for a cube Q, $Q = Q(x_0,r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q$, $f_2 = f\chi_{(R^n\setminus Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1}\int_Q b_1(y)dy$, we have

$$F_t^{b_1}(f)(x,y) = (b_1(x) - (b_1)_Q)F_t(f)(y) - F_t((b_1 - (b_1)_Q)f_1)(y) - F_t((b_1 - (b_1)_Q)f_2)(y),$$

so

$$\begin{aligned}
&|S_\psi^{b_1}(f)(x) - s_\psi(((b_1)_Q - b_1)f_2)(x_0)| \\
&= \Big| ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x,y)|| - ||\chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1)f_2)(y)|| \Big| \\
&\leqslant ||\chi_{\Gamma(x)} F_t^{b_1}(f)(x,y) - \chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1)f_2)(y)|| \\
&\leqslant ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q) F_t(f)(y)|| + ||\chi_{\Gamma(x)} F_t(((b_1)_Q - b_1)f_1)(y)|| \\
&\quad + ||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q)f_2)(y) - \chi_{\Gamma(x_0)} F_t((b_1 - (b_1)_Q)f_2)(y)|| \\
&= A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, since $\omega \in A_1$, where ω satisfies the reverse of Hölder's inequality (see[10]):

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{1/q} \leqslant \frac{C}{|Q|}\int_Q \omega(x)dx$$

for some $1 < q < \infty$. Let $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, by the Hölder's and reverse of Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q |A(x)|\omega(x)dx \\
&= \frac{1}{\omega(Q)}\int_Q |b_1(x) - (b_1)_Q||S_\psi(f)(x)|\omega(x)dx \\
&\leqslant \frac{C}{\omega(Q)}\left(\int_Q |b_1(x) - (b_1)_Q|^{p'}\omega(x)dx\right)^{1/p'}\left(\int_Q |S_\psi(f)(x)|^p\omega(x)dx\right)^{1/p} \\
&\leqslant \frac{C}{\omega(Q)}\left(\int_Q |b_1(x) - (b_1)_Q|^{p'}\omega(x)dx\right)^{1/p'}\left(\int_Q |f(x)|^p\omega(x)dx\right)^{1/p} \\
&\leqslant \frac{C}{\omega(Q)}\left(\int_Q |b_1(x) - (b_1)_Q|^{p'}\omega(x)dx\right)^{1/p'} ||f||_{L^\infty(\omega)}\left(\int_Q \omega(x)dx\right)^{1/p} \\
&\leqslant \frac{C}{\omega(Q)}\left[\left(\int_Q |b_1(x) - (b_1)_Q|^{p'q'}dx\right)^{1/q'}\left(\int_Q \omega(x)^q dx\right)^{1/q}\right]^{1/p'} ||f||_{L^\infty(\omega)}\omega(Q)^{1/p} \\
&\leqslant C\omega(Q)^{1/p-1}|Q|^{1/p'}||b_1||_{BMO}\left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{1/qp'} ||f||_{L^\infty(\omega)} \\
&\leqslant C||b_1||_{BMO}||f||_{L^\infty(\omega)}.
\end{aligned}$$

For $B(x)$, taking $p > 1$, by the $L^p(\omega)$-boundedness of S_ψ and the Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q |B(x)|\omega(x)dx \\
&= \frac{1}{\omega(Q)}\int_Q |S_\psi((b_1 - (b_1)_Q)f_1)(y)|\omega(x)dx \\
&\leqslant \left(\frac{1}{\omega(Q)}\int_Q |S_\psi((b_1 - (b_1)_Q)f_1)(y)|^p\omega(x)dx\right)^{1/p}
\end{aligned}$$

$$\leqslant C\omega(Q)^{-1/p}\left(\int_{R^n}|(b_1(x)-(b_1)_Q)f_1(y)|^p\omega(x)dx\right)^{1/p}$$

$$\leqslant C\omega(Q)^{-1/p}\left[\left(\int_Q|b_1(x)-(b_1)_Q|^{pq'}dx\right)^{1/q'}\left(\int_Q|f(y)|^{pq}\omega(x)^q dx\right)^{1/q}\right]^{1/p}$$

$$\leqslant C\omega(Q)^{-1/p}\left(\int_Q|b_1(x)-(b_1)_Q|^{pq'}dx\right)^{1/pq'}\left(\int_Q|f(y)|^{pq}\omega(x)^q dx\right)^{1/pq}$$

$$\leqslant C\omega(Q)^{-1/p}\left(\int_Q|b_1(x)-(b_1)_Q|^{pq'}dx\right)^{1/pq'}\left(\int_Q\omega(x)^q dx\right)^{1/pq}||f||_{L^\infty(\omega)}$$

$$\leqslant C\omega(Q)^{-1/p}|Q|^{1/pq'}||b_1||_{BMO}|Q|^{1/pq}\left(\frac{\omega(Q)}{|Q|}\right)^{1/p}||f||_{L^\infty(\omega)}$$

$$\leqslant C||b_1||_{BMO}||f||_{L^\infty(\omega)}.$$

For $C(x)$, we have

$$C(x)=||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_Q)f_2)(y)||$$

$$\leqslant\left[\int\int_{R^{n+1}_+}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}||b_1(z)-(b_1)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

$$\leqslant C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\left|\int\int_{|x-y|\leqslant t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2}}\right.$$

$$\left.-\int\int_{|x_0-y|\leqslant t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2}}\right|^{1/2}dz$$

$$\leqslant C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\left(\int\int_{|y|\leqslant t,|x+y-z|\leqslant t}\left|\frac{1}{(t+|x+y-z|)^(2n+2)}\right.\right.$$

$$\left.\left.-\frac{1}{(t+|x_0+y-z|)^(2n+2)}\right|\frac{dydt}{t^{n-1}}\right)^{1/2}dz$$

$$\leqslant\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\left(\int\int_{|y|\leqslant t,|x+y-z|\leqslant t}\frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3}}dydt\right)^{1/2}dz,$$

note that $2t+|x+y-z|\geqslant 2t+|x-z|-|y|\geqslant t+|x-z|$ when $|y|\leqslant t$ and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+3}}=C|x-z|^{-2n-1},$$

then, for $x\in Q$,

$$C(x)\leqslant\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\left(\int\int_{|y|\leqslant t}\frac{2^{2n+3}|x_0-x|t^{1-n}dydt}{(2t+2|x+y-z|)^{2n+3}}\right)^{1/2}dz$$

$$\leqslant C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\int\int_{|y|\leqslant t}\frac{t^{1-n}dydt}{(t+|x-z|)^{2n+3}}\right)^{1/2}dz$$

$$
\begin{aligned}
&\leqslant C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+3}}\right)^{1/2}dz\\
&\leqslant C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}}dz\\
&\leqslant C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(z)-(b_1)_Q||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}}dz\\
&\leqslant C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|b_1(z)-(b_1)_Q||f(z)|dz\\
&\leqslant C\sum_{k=1}^{\infty}k\cdot 2^{-k/2}||b_1||_{BMO}||f||_{L^\infty(\omega)}\\
&\leqslant C||b_1||_{BMO}||f||_{L^\infty(\omega)},
\end{aligned}
$$

thus

$$\frac{1}{\omega(Q)}\int_Q|C(x)|\omega(x)dx\leqslant C||b_1||_{BMO}||f||_{L^\infty(\omega)}.$$

When $m>1$, for any $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where

$$(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy,\quad 1\leqslant j\leqslant m,$$

we have

$$
\begin{aligned}
F_t^{\vec{b}}(f)(x,y)&=\int_{R^n}\left[\prod_{j=1}^m(b_1(x)-b_1(z))\right]\psi_t(y-z)f(z)dz\\
&=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\\
&\quad+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{R^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz\\
&=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\\
&\quad+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)\\
&\quad+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y),
\end{aligned}
$$

thus

$$
\begin{aligned}
&|S_\psi^{\vec{b}}(f)(x)-S_\psi(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
&\leqslant||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(y)||\\
&\leqslant||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)||
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)||\\
&+||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)\\
&-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)||\\
=\;& I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x),$ same as $m=1$. For some $1<q<\infty,$ let $1/q_1+1/q_2+\cdots+1/q_m+1/q=1,\ 1/p+1/p'=1,$ by the Hölder's and reverse of Hölder's inequality, we get

$$
\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q|I_1(x)|\omega(x)dx\\
=\;&\frac{1}{\omega(Q)}\int_Q|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)||S_\psi(f)(x)|\omega(x)dx\\
\leqslant\;&\frac{C}{\omega(Q)}\left(\int_Q|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^{p'}\omega(x)dx\right)^{1/p'}\\
&\times\left(\int_Q|S_\psi(f)(x)|^p\omega(x)dx\right)^{1/p}\\
\leqslant\;&\frac{C}{\omega(Q)}\left(\int_Q|b_1(x)-(b_1)_Q|^{p'}\cdots|b_m(x)-(b_m)_Q|^{p'}\omega(x)dx\right)^{1/p'}||f||_{L^\infty(\omega)}\\
&\times\left(\int_Q\omega(x)dx\right)^{1/p}\\
\leqslant\;&\frac{C}{\omega(Q)}||f||_{L^\infty(\omega)}\omega(Q)^{1/p}\Bigg[\left(\int_Q|b_1(x)-(b_1)_Q|^{p'q_1}dx\right)^{1/q_1}\cdots\\
&\left(\int_Q|b_m(x)-(b_m)_Q|^{p'q_m}dx\right)^{1/q_m}\left(\int_Q\omega(x)^qdx\right)^{1/q}\Bigg]^{1/p'}\\
\leqslant\;&\frac{C}{\omega(Q)}||f||_{L^\infty(\omega)}\omega(Q)^{1/p}|Q|^{1/p'q_1+\cdots+1/p'q_m}||\vec{b}||_{BMO}|Q|^{1/p'q}\left(\frac{1}{|Q|}\int_Q\omega(x)^qdx\right)^{1/p'q}\\
\leqslant\;&C||\vec{b}||_{BMO}||f||_{L^\infty(\omega)}\omega(Q)^{1/p'+1/p-1}|Q|^{1/p'[1/q_1+\cdots+1/q_m+1/q-1]}\\
\leqslant\;&C||\vec{b}||_{BMO}||f||_{L^\infty(\omega)}.
\end{aligned}
$$

For $I_2(x),$ by the Hölder's and reverse of Hölder's inequality, we have

$$
\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q|I_2(x)|\omega(x)dx\\
\leqslant\;&\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\omega(Q)}\int_Q|(b(\vec{x})-\vec{b}_Q)_\sigma||S_\psi((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|\omega(x)dx
\end{aligned}
$$

$$\leqslant \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{C}{\omega(Q)} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{p'} \omega(x) dx \right)^{1/p'}$$

$$\times \left(\int_Q |S_\psi((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|^p \omega(x) dx \right)^{1/p}$$

$$\leqslant C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{\omega(Q)} \int_Q |\vec{b}(x) - \vec{b}_Q)_\sigma|^{p'} \omega(x) dx \right)^{1/p'}$$

$$\times \left(\frac{1}{\omega(Q)} \int_Q |S_\psi((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|^p \omega(x) dx \right)^{1/p}$$

$$= C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} K_1 K_2.$$

For K_1, by the Hölder's inequality, we have

$$K_1 \leqslant C\omega(Q)^{-1/p'} \left[\left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{p'q'} dx \right)^{1/q'} \left(\int_Q \omega(x)^q dx \right)^{1/q} \right]^{1/p'}$$

$$\leqslant C\omega(Q)^{-1/p'} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)|^{p'q'} dx \right)^{1/p'q'} \left[\left(\int_Q \omega(x)^q dx \right)^{1/q} \right]^{1/p'}$$

$$\leqslant C\omega(Q)^{-1/p'} |Q|^{1/p'q'} ||\vec{b}_\sigma||_{BMO} |Q|^{1/p'q} \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/p'q}$$

$$\leqslant C\omega(Q)^{-1/p'} |Q|^{1/p'q' + 1/p'q - 1/p'} \omega(Q)^{1/p'} ||\vec{b}_\sigma||_{BMO}$$

$$\leqslant C ||\vec{b}_\sigma||_{BMO}.$$

For K_2, we have

$$K_2 \leqslant C\omega(Q)^{-1/p} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c} f(x)|^p \omega(x) dx \right)^{1/p}$$

$$\leqslant C\omega(Q)^{-1/p} \left(\int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|^{pq'} dx \right)^{1/pq'} \left(\int_Q |f(x)|^{pq} \omega(x)^q dx \right)^{1/pq}$$

$$\leqslant C\omega(Q)^{-1/p} |Q|^{1/pq'} ||\vec{b}_{\sigma^c}||_{BMO} |Q|^{1/pq} \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/pq} ||f||_{L^\infty(\omega)}$$

$$\leqslant C\omega(Q)^{-1/p} |Q|^{1/p} ||\vec{b}_{\sigma^c}||_{BMO} \left(\frac{\omega(Q)}{|Q|} \right)^{1/p} ||f||_{L^\infty(\omega)}$$

$$\leqslant C ||\vec{b}_{\sigma^c}||_{BMO} ||f||_{L^\infty(\omega)},$$

so

$$\frac{1}{\omega(Q)} \int_Q |I_2(x)| \omega(x) dx \leqslant C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ||\vec{b}_\sigma||_{BMO} ||\vec{b}_{\sigma^c}||_{BMO} ||f||_{L^\infty(\omega)} \leqslant C ||\vec{b}||_{BMO} ||f||_{L^\infty(\omega)}.$$

For $I_3(x)$, taking $p>1$, by the $L^p(\omega)$-boundedness of S_ψ and the Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q |I_3(x)|\omega(x)dx\\
&=\frac{1}{\omega(Q)}\int_Q |S_\psi((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|\omega(x)dx\\
&\leqslant C\left(\frac{1}{\omega(Q)}\int_{R^n}|S_\psi((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^p\omega(x)dx\right)^{1/p}\\
&\leqslant C\omega(Q)^{-1/p}(\int_Q |(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f(x)|^p\omega(x)dx)^{1/p}\\
&\leqslant C\omega(Q)^{-1/p}\left(\int_Q |(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^{pq'}dx\right)^{1/pq'}\\
&\quad\times\left(\int_Q |f(x)|^{pq}\omega(x)^q dx\right)^{1/pq}\\
&\leqslant C\omega(Q)^{-1/p}|Q|^{1/pq'}||\vec{b}||_{BMO}|Q|^{1/pq}\left(\frac{1}{|Q|}\int_Q\omega^q dx\right)^{1/pq}||f||_{L^\infty(\omega)}\\
&\leqslant C||\vec{b}||_{BMO}\left(\frac{|Q|}{\omega(Q)}\right)^{1/p}\left(\frac{\omega(Q)}{|Q|}\right)^{1/p}||f||_{L^\infty(\omega)}\\
&\leqslant C||\vec{b}||_{BMO}||f||_{L^\infty(\omega)}.
\end{aligned}$$

For $I_4(x)$, we have

$$\begin{aligned}
I_4(x)&=||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)\\
&\quad-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)||\\
&\leqslant\left[\int\int_{R^{n+1}_+}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}|\left[\prod_{j=1}^m|b_j(z)-(b_j)_Q|\right]|\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2},
\end{aligned}$$

similar to the proof of $C(x)$ in Case $m=1$, we have

$$\begin{aligned}
I_4(x)&\leqslant C\int_{Q^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}\left|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\right||f(z)|dz\\
&\leqslant C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}\left|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\right||f(z)|dz\\
&\leqslant C\sum_{k=1}^\infty 2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\right||f(z)|dz\\
&\leqslant C||\vec{b}||_{BMO}\sum_{k=1}^\infty k^m\cdot 2^{-km/2}||f||_{L^\infty(\omega)}\\
&\leqslant C||\vec{b}||_{BMO}||f||_{L^\infty(\omega)},
\end{aligned}$$

so

$$\frac{1}{\omega(Q)}\int_Q |I_4(x)|\omega(x)dx \leqslant C||\vec{b}||_{BMO}||f||_{L^\infty(\omega)}.$$

This completes the proof of Theorem 1. $\square$

THEOREM 2. *Let $\omega\in A_1$, and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\leqslant j\leqslant m$. If for any $H^1(\omega)$-atom a supported on certain cube Q and $u\in Q$, there is*

$$\sum_{j=1}^m\sum_{\sigma\in C_j^m}\int_{(2Q)^c}|(b(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\left|\int_Q(b(z)-b_Q)_\sigma a(z)dz\right|^2|\psi_t(y-u)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$
$$\times\omega(x)dx\leqslant C.$$

then $S_\psi^{\vec{b}}$ is bounded from $H^1(\omega)$ to $L^1(\omega)$.

Proof. Let a be an atom supported in some cube Q, since a is bounded and with compact support, when $m=1$, let $u\in Q$, we write

$$\int_{R^n}|S_\psi^{b_1}(a)(x)|\omega(x)dx=\int_{2Q}|S_\psi^{b_1}(a)(x)|\omega(x)dx+\int_{(2Q)^c}|S_\psi^{b_1}(a)(x)|\omega(x)dx.$$

We have

$$\begin{aligned}&\int_{2Q}|S_\psi^{b_1}(a)(x)|\omega(x)dx\\&\leqslant C||S_\psi^{b_1}(a)(x)||_{L^\infty(\omega)}\cdot\omega(2Q)\\&\leqslant C||b_1||_{BMO}||a||_{L^\infty(\omega)}\cdot\omega(Q)\\&\leqslant C||b_1||_{BMO}.\end{aligned}$$

For $F_t^{b_1}(a)(x,y)$, we have

$$\begin{aligned}|F_t^{b_1}(a)(x,y)| &= \left|\int_Q\psi_t(y-z)a(z)b_1(x)dz-\int_Q\psi_t(y-z)a(z)b_1(z)dz\right|\\&\leqslant\left|\int_Q\psi_t(y-z)a(z)(b_1(x)-(b_1)_Q)dz\right|\\&\quad+\left|\int_Q(\psi_t(y-z)-\psi_t(y-u))a(z)(b_1(z)-(b_1)_Q)dz\right|\\&\quad+\left|\int_Q\psi_t(y-u)(b_1(z)-(b_1)_Q)a(z)dz\right|\\&\quad=v_1'+v_2'+v_3',\end{aligned}$$

so

$$\begin{aligned}S_\psi^{b_1}(a)(x) &= ||\chi_{\Gamma(x)}F_t^{b_1}(a)(x,y)||\\&\leqslant\left(\int\int_{\Gamma(x)}|v_1'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|v_2'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|v_3'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\&=A'(x)+B'(x)+C'(x).\end{aligned}$$

For $A'(x)$, we have

$$\begin{aligned}A'(x) &= \left(\int\int_{\Gamma(x)} \left|\int_Q \psi_t(y-z)a(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} |b_1(x)-(b_1)_Q| \\ &= S_\psi(a)(x)|b_1(x)-(b_1)_Q|.\end{aligned}$$

For $B'(x)$, we have

$$\begin{aligned}B'(x) &= \left(\int\int_{\Gamma(x)} \left|\int_Q (\psi_t(y-z)-\psi_t(y-u))a(z)(b_1(z)-(b_1)_Q)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\leqslant C\left(\int\int_{\Gamma(x)} \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon}}|a(z)||b(z)-(b_1)_Q|dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\leqslant C\left(\int\int_{\Gamma(x)} \frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+1+\varepsilon)}}\right)^{1/2} \left(\int_Q |a(z)||u-z|^\varepsilon|b_1(z)-(b_1)_Q|dz\right) \\ &\leqslant C\left(\int\int_{\Gamma(x)} \frac{t^{1-n}2^{2(n+1+\varepsilon)}}{(2t+2|y-u|)^{2(n+1+\varepsilon)}}dydt\right)^{1/2} \int_Q |u-z|^\varepsilon|a(z)||b_1(z)-(b_1)_Q|dz.\end{aligned}$$

Notice that $2t+|y-u|>2t+|u-x|-|x-y|>t+|u-x|$ when $|x-y|<t$, and it is easy to calculate that

$$\int_0^\infty \frac{tdt}{(t+|x-u|)^{2(n+1+\varepsilon)}} = C|x-u|^{-2(n+\varepsilon)}.$$

then, we deduce

$$\begin{aligned}B'(x) &\leqslant C\left(\int\int_{\Gamma(x)} \frac{t^{1-n}dydt}{(t+|x-u|)^{2(n+1+\varepsilon)}}\right)^{1/2} \int_Q |u-z|^\varepsilon|a(z)||b_(z)-(b_1)_Q|dz \\ &\leqslant C\left(\int_0^\infty \frac{tdt}{(t+|x-u|)^{2(n+1+\varepsilon)}}\right)^{1/2} \int_Q |u-z|^\varepsilon|a(z)||b_1(z)-(b_1)_Q|dz \\ &\leqslant C|x-u|^{-(n+\varepsilon)}|Q|^{\varepsilon/n}\int_Q |a(z)||b_1(z)-(b_1)_Q|dz \\ &\leqslant C||b_1||_{BMO}|x-u|^{-(n+\varepsilon)}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}.\end{aligned}$$

For $A'(x)$, taking some $1<p<\infty$, and $1/p+1/p'=1$, and noting $\omega\in A_1$, we get $\frac{\omega(B_2)}{|B_2|}\frac{|B_1|}{\omega(B_1)} \leqslant C$ for all cubes B_1,B_2 with $B_1\subset B_2$. Thus by the Hölder's and reverse of Hölder' inequality, we obtain

$$\begin{aligned}&\int_{(2Q)^c} A'(x)\omega(x)dx \\ &= \int_{(2Q)^c} |b_1(x)-(b_1)_Q| \left(\int\int_{\Gamma(x)} \left|\int_Q (\psi_t(y-z)-\psi_t(y-u))a(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \omega(x)dx\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int_{(2Q)^c}|b_1(x)-(b_1)_Q|\left(\int_0^\infty \frac{tdt}{(t+|x-u|)^{2(n+1+\varepsilon)}}\right)^{1/2}\left(\int_Q|u-z|^\varepsilon|a(z)|dz\right)\omega(x)dx\\
&\leqslant C\int_{(2Q)^c}|b_1(x)-(b_1)_Q||x-u|^{-(n+\varepsilon)}\omega(x)dx\int_Q|u-z|^\varepsilon|a(z)|dz\\
&\leqslant C\int_{(2Q)^c}|b_1(x)-(b_1)_Q||x-u|^{-(n+\varepsilon)}\omega(x)dx\cdot|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\\
&\leqslant C\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}|b_1(x)-(b_1)_Q||x-u|^{-(n+\varepsilon)}\omega(x)dx\cdot|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\\
&\leqslant C\sum_{k=2}^\infty|2^kQ|^{-(1+\varepsilon/n)}|Q|^{1+\varepsilon/n}\int_{2^kQ}|b_1(x)-(b_1)_Q|\omega(x)dx\cdot||a||_{L^\infty(\omega)}\\
&\leqslant C\sum_{k=2}^\infty 2^{-k\varepsilon}|Q|\omega(Q)^{-1}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|b_1(x)-(b_1)_Q|^{p'}dx\right)^{1/p'}\left(\frac{1}{|2^kQ|}\int_{2^kQ}\omega(x)^pdx\right)^{1/p}\\
&\leqslant C\sum_{k=2}^\infty k2^{-k\varepsilon}||b_1||_{BMO}\left(\frac{\omega(2^kQ)}{|2^kQ|}\frac{|Q|}{\omega(Q)}\right)\\
&\leqslant C||b_1||_{BMO},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{(2Q)^c}B'(x)\omega(x)dx\\
&\leqslant C||b_1||_{BMO}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\int_{(2Q)^c}|x-u|^{-(n+\varepsilon)}\omega(x)dx\\
&\leqslant C||b_1||_{BMO}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\sum_{k=1}^\infty\int_{2^{k+1}Q\backslash 2^kQ}|x-u|^{-(n+\varepsilon)}\omega(x)dx\\
&\leqslant C||b_1||_{BMO}.
\end{aligned}
$$

From that we know, if

$$
\begin{aligned}
\int_{(2Q)^c}C'(x)\omega(x)dx &= \int_{(2Q)^c}\left(\int\int_{\Gamma(x)}\left|\int_Q(b_1(z)-(b_1)_Q)a(z)dz\right|^2|\psi_t(y-u)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\quad\times\omega(x)dx\\
&\leqslant C.
\end{aligned}
$$

then

$$
\int_{R^n}|S_\psi^{b_1}(a)(x)|\omega(x)dx\leqslant C.
$$

When $m>1$, we have

$$
\begin{aligned}
\int_{R^n}|S_\psi^{\vec{b}}(a)(x)|\omega(x)dx &= \int_{2Q}|S_\psi^{\vec{b}}(a)(x)|\omega(x)dx+\int_{(2Q)^c}|S_\psi^{\vec{b}}(a)(x)|\omega(x)dx\\
&= I+II.
\end{aligned}
$$

For I, by the boundedness of $S_\psi^{\vec{b}}$ and the Hölder's inequality, we have

$$\begin{aligned} I &\leqslant C\|S_\psi^{\vec{b}}(a)(x)\|_{L^\infty(\omega)}\cdot\omega(2Q) \\ &\leqslant C\|\vec{b}\|_{BMO}\|a\|_{L^\infty(\omega)}\cdot\omega(Q) \\ &\leqslant C\|\vec{b}\|_{BMO}. \end{aligned}$$

For II, we first calculate $F_t^{\vec{b}}(a)(x,y)$,

$$\begin{aligned} |F_t^{\vec{b}}(a)(x,y)| &= \left|\int_Q \prod_{j=1}^m (b_j(x)-b_j(z))\psi_t(y-z)a(z)dz\right| \\ &\leqslant \left|\prod_{j=1}^m (b_j(x)-(b_j)_Q)\int_Q \psi_t(y-z)a(z)dz\right| \\ &\quad+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q(\psi_t(y-z)-\psi_t(y-u))(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right| \\ &\quad+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q\psi_t(y-u)(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right| \\ &= \nu_1+\nu_2+\nu_3, \end{aligned}$$

so

$$\begin{aligned} S_\psi^{\vec{b}}(a)(x) &= \|\chi_{\Gamma(x)}F_t^{\vec{b}}(a)(x,y)\| \\ &\leqslant \left(\int\int_{\Gamma(x)}|\nu_1|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|\nu_2|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|\nu_3|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &= A(x)+B(x)+C(x). \end{aligned}$$

For $A(x)$, we have

$$\begin{aligned} A(x) &= \left(\int\int_{\Gamma(x)}\prod_{j=1}^m|b_j(x)-(b_j)_Q|^2\left|\int_Q\psi_t(y-z)a(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &= \prod_{j=1}^m|b_j(x)-(b_j)_Q|S_\psi(a)(x). \end{aligned}$$

For $B(x)$, by the Hölder's inequality, we have

$$\begin{aligned} B(x) = \Bigg(\int\int_{\Gamma(x)}\Bigg|&\sum_{j=1}^m\sum_{\sigma\in C_j^m}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_Q(\psi_t(y-z)-\psi_t(y-u)) \\ &\times(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\Bigg|^2\frac{dydt}{t^{n+1}}\Bigg)^{1/2} \end{aligned}$$

$$\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\left(\int_Q\frac{t|u-z|^{\varepsilon}}{(t+|y-u|)^{n+1+\varepsilon}}\right.\right.$$

$$\left.\left.\times|(\vec{b}(z)-\vec{b}_Q)_{\sigma}||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+1+\varepsilon)}}\right)^{1/2}$$

$$\times\int_Q|u-z|^{\varepsilon}|a(z)||(\vec{b}(z)-\vec{b}_Q)_{\sigma}|dz$$

$$\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-b_Q)_{\sigma^c}|\left(\int\int_{\Gamma(x)}\frac{t^{1-n}2^{2(n+1+\varepsilon)}}{(2t+2|y-u|)^{2(n+1+\varepsilon)}}dydt\right)^{1/2}$$

$$\times\int_Q|u-z|^{\varepsilon}|a(z)||(\vec{b}(z)-\vec{b}_Q)_{\sigma}|dz.$$

similar to the proof of $B'(x)$ in Case $m=1$, we obtain

$$B(x)\leqslant C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}||x-u|^{-(n+\varepsilon)}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}||\vec{b}_{\sigma}||_{BMO}.$$

so

$$\int_{(2Q)^c}A(x)\omega(x)dx$$

$$\leqslant C\int_{(2Q)^c}\prod_{j=1}^{m}|b(x)-(b_j)_Q||S_{\psi}(a)(x)|\omega(x)dx$$

$$\leqslant C\int_{(2Q)^c}\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\left(\int\int_{\Gamma(x)}\left(\int_Q|\psi_t(y-z)-\psi_t(y-u)||a(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\times\omega(x)dx$$

$$\leqslant C\int_{(2Q)^c}\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\left(\int_0^{\infty}\frac{tdt}{(t+|x-u|)^{2(n+1+\varepsilon)}}\right)^{1/2}$$

$$\times\left(\int_Q|u-z|^{\varepsilon}|a(z)|dz\right)\omega(x)dx$$

$$\leqslant C\int_{(2Q)^c}\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\cdot|x-u|^{-(n+\varepsilon)}\omega(x)dx\cdot|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}$$

$$\leqslant C\sum_{k=2}^{\infty}|2^kQ|^{-(1+\varepsilon/n)}|Q|^{1+\varepsilon/n}\int_{2^kQ}\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\omega(x)dx\cdot||a||_{L^\infty(\omega)}$$

$$\leqslant C||\vec{b}||_{BMO}\frac{\omega(2^kQ)}{|2^kQ|}\frac{|Q|}{\omega(Q)}\sum_{k=2}^{\infty}k\cdot 2^{-k\varepsilon}$$

$$\leqslant C||\vec{b}||_{BMO},$$

and

$$\begin{aligned}
&\int_{(2Q)^c} B(x)\omega(x)dx \\
&\leqslant C||\vec{b}_\sigma||_{BMO}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\int_{(2Q)^c}|x-u|^{-(n+\varepsilon)}\sum_{j=1}^m\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\omega(x)dx \\
&\leqslant C||\vec{b}_\sigma||_{BMO}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\sum_{j=1}^m\sum_{\sigma\in C_j^m}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x-u|^{-(n+\varepsilon)}| \\
&\quad\times(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\omega(x)dx \\
&\leqslant C\sum_{j=1}^m\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}|Q|^{1+\varepsilon/n}||a||_{L^\infty(\omega)}\sum_{k=2}^\infty|2^kQ|^{-(1+\varepsilon/n)}\int_{2^kQ}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\omega(x)dx \\
&\leqslant C||\vec{b}||_{BMO}\frac{\omega(2^kQ)}{|2^kQ|}\frac{|Q|}{\omega(Q)}\sum_{k=2}^\infty k2^{-k\varepsilon} \\
&\leqslant C||\vec{b}||_{BMO}.
\end{aligned}$$

so, if

$$\begin{aligned}
&\int_{(2Q)^c} C(x)\omega(x)dx \\
&\leqslant C\sum_{j=1}^m\sum_{\sigma\in C_j^m}\int_{(2Q)^c}|(b(x)-b_Q)_{\sigma^c}| \\
&\quad\times\left(\int\int_{\Gamma(x)}|\int_Q(b(z)-b_Q)_\sigma a(z)dz|^2|\psi_t(y-u)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\omega(x)dx \\
&\leqslant C.
\end{aligned}$$

then

$$\int_{R^n}|S_\psi^{\vec{b}}(a)(x)|\omega(x)dx\leqslant C.$$

This completes the proof of Theorem 2. □

REMARK. S_ψ^b is bounded from $H^1(\omega)$ to weak $L^1(\omega)$ (see[7]).

THEOREM 3. *Let* $1<p<\infty$, $\omega\in A_1$ *and* $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in BMO(R^n)$ *for* $1\leqslant j\leqslant m$. *Then* $S_\psi^{\vec{b}}$ *is bounded from* $B_p(\omega)$ *to* $CMO(\omega)$.

Proof. It is only to prove that there exist constant C_Q, such that

$$\frac{1}{\omega(Q)}\int_Q|S_\psi^{\vec{b}}(f)(x)-C_Q|\omega(x)dx\leqslant C||f||_{B_p(\omega)}.$$

holds for any cube $Q=Q(0,r)$ with $r>1$. Fix a cube $Q=Q(0,r)$ with $r>1$. Set $f_1=f\chi_Q$, $f_2=f\chi_{R^n\setminus Q}$ and $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|dy$,

$1 \leqslant j \leqslant m,$ we have

$$\begin{aligned} F_t^{\vec{b}}(f)(x,y) &= \int_{R^n} \left[\prod_{j=1}^{m}(b_1(x)-b_1(z))\right] \psi_t(y-z)f(z)dz \\ &= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y) \\ &\quad +(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y) \\ &\quad +(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y) \\ &\quad +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y), \end{aligned}$$

thus

$$\begin{aligned} &|S_\psi^{\vec{b}}(f)(x)-S_\psi(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)| \\ &\leqslant ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(y)|| \\ &\leqslant ||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)|| \\ &\quad +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)|| \\ &\quad +||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)|| \\ &\quad +||\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_Q) \\ &\quad \cdots(b_m-(b_m)_Q)f_2)(y)|| \\ &= I_1(x)+I_2(x)+I_3(x)+I_4(x). \end{aligned}$$

For $I_1(x),$ we have

$$\begin{aligned} &\frac{1}{\omega(Q)}\int_Q |I_1(x)|\omega(x)dx \\ &\leqslant \frac{C}{\omega(Q)}\left(\int_Q |(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^{p'}\omega(x)dx\right)^{1/p'} \\ &\quad \times\left(\int_Q |S_\psi(f)(x)|^p\omega(x)dx\right)^{1/p} \\ &\leqslant \frac{C}{\omega(Q)}\left[\left(|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^{p'q'}dx\right)^{1/q'}\left(\int_Q \omega(x)^q dx\right)^{1/q}\right]^{1/p'} \\ &\quad \times\left(\int_Q |f(x)|^p\omega(x)dx\right)^{1/p} \\ &\leqslant \frac{C}{\omega(Q)}|Q|^{1/p'q'}||\vec{b}||_{BMO}|Q|^{1/p'q}\left(\frac{\omega(Q)}{|Q|}\right)^{1/p'}||f\chi_Q||_{L^p(\omega)} \\ &\leqslant C||\vec{b}||_{BMO}\omega(Q)|^{-1/p}|f\chi_Q||_{L^p(\omega)} \\ &\leqslant C||\vec{b}||_{BMO}||f||_{B_p(\omega)}. \end{aligned}$$

For $I_2(x)$, taking $1 < s, s' < \infty$, and $1/s + 1/s' = 1$, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q |I_2(x)|\omega(x)dx \\
&\leqslant \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\omega(Q)}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma||S_\psi((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|\omega(x)dx \\
&\leqslant C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{\omega(Q)}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{s'}\omega(x)dx\right)^{1/s'} \\
&\quad\times\left(\frac{1}{\omega(Q)}\int_Q |S_\psi((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|^{s}\omega(x)dx\right)^{1/s} \\
&= C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} K_1K_2.
\end{aligned}$$

For K_1, by the Hölder's inequality, we have

$$\begin{aligned}
K_1 &\leqslant C\omega(Q)^{-1/s'}\left[\left(\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{s'q'}\right)^{1/q'}\left(\int_Q \omega(x)^q dx\right)^{1/q}\right]^{1/s'} \\
&\leqslant C\omega(Q)^{-1/s'}\left(\int_Q |(\vec{b}(x)-(\vec{b})_Q)_\sigma)|^{s'q'}dx\right)^{1/s'q'}\left[\left(\int_Q \omega(x)^q dx\right)^{1/q}\right]^{1/s'} \\
&\leqslant C\omega(Q)^{-1/s'}|Q|^{1/s'q'+1/s'q}\left(\frac{\omega(Q)}{|Q|}\right)^{1/s'}||\vec{b}_\sigma||_{BMO} \\
&\leqslant C||\vec{b}_\sigma||_{BMO}.
\end{aligned}$$

For K_2, taking $1 < t, t' < \infty$, and $1/t + 1/t' = 1$, we have

$$\begin{aligned}
K_2 &\leqslant C\omega(Q)^{-1/s}\left(\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}f(x)|^s\omega(x)dx\right)^{1/s} \\
&\leqslant C\omega(Q)^{-1/s}\left[\left(\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{sr}dx\right)^{1/r}\left(\int_Q |f(x)|^{r's}\omega(x)^{r'}dx\right)^{1/r'}\right]^{1/s} \\
&\leqslant C\omega(Q)^{-1/s}|Q|^{1/rs}||\vec{b}_{\sigma^c}||_{BMO}\left(\int_Q |f(x)|^{r's}\omega(x)^{r'}dx\right)^{1/r's} \\
&\leqslant C\omega(Q)^{-1/s}|Q|^{1/rs}||\vec{b}_{\sigma^c}||_{BMO}\left(\int_Q |f(x)|^{r'st}\omega(x)dx\right)^{1/r'st}\left(\int_Q \omega(x)^{(r'-1)t'+1}dx\right)^{1/r'st'} \\
&\leqslant C\omega(Q)^{-1/s}|Q|^{1/rs}||\vec{b}_{\sigma^c}||_{BMO}\left(\int_Q |f(x)|^{p}\omega(x)dx\right)^{1/p}\left(\int_Q \omega(x)^q dx\right)^{(p-s)/pqs} \\
&\leqslant C\omega(Q)^{-1/p}||\vec{b}_{\sigma^c}||_{BMO}||f\chi_Q||_{L^p(\omega)} \\
&\leqslant C||\vec{b}_{\sigma^c}||_{BMO}||f||_{B_p(\omega)}.
\end{aligned}$$

so

$$\frac{1}{\omega(Q)}\int_Q |I_2(x)|\omega(x)dx \leqslant C||\vec{b}||_{BMO}||f||_{B_p(\omega)}.$$

For $I_3(x)$, we have

$$\begin{aligned}
&\frac{1}{\omega(Q)}\int_Q |I_3(x)|\omega(x)dx\\
&\leqslant C\left(\frac{1}{\omega(Q)}\int_{R^n}|S_\psi((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s\omega(x)dx\right)^{1/s}\\
&\leqslant C\omega(Q)^{-1/s}\left(\int_Q|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f(x)|^s\omega(x)dx\right)^{1/s}\\
&\leqslant C\omega(Q)^{-1/p}||\vec{b}||_{BMO}||f\chi_Q||_{L^p(\omega)}\\
&\leqslant C||\vec{b}||_{BMO}||f||_{B_p(\omega)}.
\end{aligned}$$

For $I_4(x)$, we have

$$\begin{aligned}
I_4(x) &\leqslant \left[\int\int_{R_+^{n+1}}\left(\int_{Q^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}|\prod_{j=1}^m|b_j(z)-(b_j)_Q||\psi_t(y-z)||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\
&\leqslant C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)||f(z)|dz\\
&\leqslant C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)||f(z)|dz\\
&\leqslant C||\vec{b}||_{BMO}||f||_{B_p(\omega)}\sum_{k=1}^{\infty}k^m\cdot 2^{-km/2}\\
&\leqslant C||\vec{b}||_{BMO}||f||_{B_p(\omega)}.
\end{aligned}$$

so

$$\frac{1}{\omega(Q)}\int_Q |I_4(x)|\omega(x)dx \leqslant C||\vec{b}||_{BMO}||f||_{B_p(\omega)}.$$

This completes the proof of Theorem 3. □

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