

GROWTH OF POLYNOMIALS WITH PRESCRIBED ZEROS

K. K. DEWAN AND ARTY AHUJA

(Communicated by A. Horwitz)

Abstract. In this paper, we study the growth of polynomials of degree n having all its zeros on $|z| = k$, $k \leq 1$. Using the notation $M(p, t) = \max_{|z|=t} |p(z)|$, we measure the growth of p by estimating $\left\{ \frac{M(p, t)}{M(p, 1)} \right\}^s$ from above for any $t \geq 1$, s being an arbitrary positive integer.

1. Introduction and Statement of Results

For an arbitrary entire function $f(z)$, let $M(f, r) = \max_{|z|=r} |f(z)|$ and $m(f, k) = \min_{|z|=k} |f(z)|$. Then for a polynomial $p(z)$ of degree n , it is a simple consequence of maximum modulus principle (for reference see [4, Vol. I, p. 137, Problem III, 269]) that

$$M(p, R) \leq R^n M(p, 1), \quad \text{for } R \geq 1. \tag{1.1}$$

The result is best possible and equality holds in (1.1) for $p(z) = \lambda z^n$, where $|\lambda| = 1$.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequalities (1.1) can be sharpened. In fact, it was shown by Ankeny and Rivlin [1] that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$M(p, R) \leq \left(\frac{R^n + 1}{2} \right) M(p, 1), \quad R \geq 1. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

For the class of polynomials not vanishing in the disk $|z| < k$, $k \geq 1$ Shah [6] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for every real number $R > k$,

$$M(p, R) \leq \left(\frac{R^n + k}{1 + k} \right) M(p, 1) - \left(\frac{R^n - 1}{1 + k} \right) m(p, k). \tag{1.3}$$

The result is best possible in case $k=1$ and equality holds for the polynomial $p(z) = z^n + 1$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \leq 1$, we have been able to prove the following results.

Mathematics subject classification (2010): 30A10, 30C10, 30C15.

Keywords and phrases: Polynomials, maximum modulus, zeros, extremal problems.

THEOREM 1. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \left(\frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n} \right) \{M(p,1)\}^s, \quad R \geq 1. \tag{1.4}$$

If we take $s = 1$ in Theorem 1, we get the following result.

COROLLARY 1. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then

$$M(p,R) \leq \left(\frac{k^{n-1}(1+k) + (R^n - 1)}{k^{n-1} + k^n} \right) M(p,1), \quad R \geq 1. \tag{1.5}$$

The following corollary immediately follows from inequality (1.5) by taking $k = 1$.

COROLLARY 2. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = 1$, then

$$M(p,R) \leq \left(\frac{R^n + 1}{2} \right) M(p,1), \quad R \geq 1. \tag{1.6}$$

If we involve the coefficients of $p(z)$ also, then we are able to obtain a bound which is better than the bound of Theorem 1. More precisely, we prove

THEOREM 2. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k, k \leq 1$, then for every positive integer s

$$\{M(p,R)\}^s \leq \frac{1}{k^n} \left(\frac{n|c_n|\{k^n(1+k^2) + k^2(R^{ns} - 1)\} + |c_{n-1}|\{2k^n + R^{ns} - 1\}}{2|c_{n-1}| + n|c_n|(1+k^2)} \right) \times \{M(p,1)\}^s, \quad R \geq 1. \tag{1.7}$$

To prove that the bound obtained in the above theorem is, in general, better than the bound obtained in Theorem 1, we show that

$$\begin{aligned} & \frac{1}{k^n} \left(\frac{n|c_n|\{k^n(1+k^2) + k^2(R^{ns} - 1)\} + |c_{n-1}|\{2k^n + R^{ns} - 1\}}{2|c_{n-1}| + n|c_n|(1+k^2)} \right) \\ & \leq \frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n} \end{aligned}$$

which is equivalent to

$$\begin{aligned} & n|c_n|(k^{n-1} + k^n)(k^n + k^{n+2} + k^2R^{ns} - k^2) + |c_{n-1}|\{k^{n-1} + k^n\}(2k^n + R^{ns} - 1) \\ & \leq n|c_n|\{k^{n+2} + k^n\}(k^{n-1} + k^n + R^{ns} - 1) + 2k^n|c_{n-1}|\{k^{n-1} + k^n + R^{ns} - 1\} \end{aligned}$$

which implies

$$\begin{aligned} & n|c_n|(k^{2n-1} + k^{2n+1} + k^{n+1}R^{ns} - k^{n+1} + k^{2n} + k^{2n+2} + k^{n+2}R^{ns} - k^{n+2}) \\ & \quad + |c_{n-1}|(2k^{2n-1} + k^{n-1}R^{ns} - k^{n-1} + 2k^{2n} + k^nR^{ns} - k^n) \\ & \leq n|c_n|(k^{2n-1} + k^{2n+1} + k^nR^{ns} - k^n + k^{2n} + k^{2n+2} + k^{n+2}R^{ns} - k^{n+2}) \\ & \quad + |c_{n-1}|(2k^{2n-1} + 2k^{2n} + 2k^nR^{ns} - 2k^n) \end{aligned}$$

or

$$\begin{aligned} n|c_n|\{k^{n+1}(R^{ns}-1)\} + |c_{n-1}|\{k^{n-1}(R^{ns}-1)\} & \leq n|c_n|\{k^n(R^{ns}-1)\} + |c_{n-1}|\{k^n(R^{ns}-1)\}, \\ |c_{n-1}|k^{n-1}(1-k) & \leq n|c_n|k^n(1-k), \\ \frac{|c_{n-1}|}{n|c_n|} & \leq k, \end{aligned}$$

which is always true (see Lemma 4).

We illustrate by means of following example that the bound obtained in Theorem 2 is better than the bound obtained in Theorem 1.

EXAMPLE 1. Let $P(z) = z^4 - \frac{1}{50}z^2 + \left(\frac{1}{100}\right)^2$ and $k = 1/10$, $R = 1.5$ and $s = 2$.

Then by Theorem 1, we have

$$\{M(P, R)\}^s \leq 22390.91477\{M(P, 1)\}^s,$$

while by Theorem 2, we get

$$\{M(P, R)\}^s \leq 2439.505569\{M(P, 1)\}^s.$$

For $s = 1$ in Theorem 2, we get the following result.

COROLLARY 3. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then

$$M(p, R) \leq \frac{1}{k^n} \left(\frac{n|c_n|\{k^n(1+k^2) + k^2(R^n - 1)\} + |c_{n-1}|\{2k^n + R^n - 1\}}{2|c_{n-1}| + n|c_n|(1+k^2)} \right) M(p, 1), \quad R \geq 1. \tag{1.8}$$

REMARK 1. If we take $k = 1$ in inequality (1.8), it reduces to Corollary 2.

2. Lemmas

We need the following lemmas for the proof of these theorems. The first lemma is due to Govil [3].

LEMMA 1. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-1} + k^n} \max_{|z|=1} |p(z)|. \quad (2.1)$$

LEMMA 2. If $p(z) = \sum_{j=0}^n c_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n} \left(\frac{n|c_n|k^2 + |c_{n-1}|}{2|c_{n-1}| + n|c_n|(1+k^2)} \right) \max_{|z|=1} |p(z)|. \quad (2.2)$$

The above lemma is due to Dewan and Mir [2].

LEMMA 3. Let $p(z) = c_0 + \sum_{v=\mu}^n c_v z^v$, $1 \leq \mu \leq n$, be a polynomial of degree n having no zero in the disk $|z| < k$, $k \geq 1$. Then

$$\frac{\mu}{n} \left| \frac{c_\mu}{c_0} \right| k^\mu \leq 1. \quad (2.3)$$

The above lemma was given by Qazi [5, Remark 1].

LEMMA 4. Let $p(z) = \sum_{v=0}^n c_v z^v$ be a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$. Then

$$\frac{1}{n} \left| \frac{c_{n-1}}{c_n} \right| \leq k. \quad (2.4)$$

Proof of Lemma 4. If $p(z)$ has all its zeros on $|z| = k$, $k \leq 1$, then $q(z) = \left(z^n p\left(\frac{1}{\bar{z}}\right) \right)$ has all its zeros on $|z| = \frac{1}{k}$, $\frac{1}{k} \geq 1$. Now applying Lemma 3 for $\mu = 1$ to the polynomial $q(z)$, Lemma 4 follows. \square

3. Proof of the Theorems

Proof of Theorem 1. Let $M(p, 1) = \max_{|z|=1} |p(z)|$. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, therefore, by Lemma 1, we have

$$|p'(z)| \leq \frac{n}{k^{n-1} + k^n} M(p, 1) \quad \text{for } |z| = 1. \quad (3.1)$$

Now applying inequality (1.1) to the polynomial $p'(z)$ which is of degree $n - 1$ and noting (3.1), it follows that for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq \frac{nr^{n-1}}{k^{n-1} + k^n} M(p, 1). \quad (3.2)$$

Also for each θ , $0 \leq \theta < 2\pi$ and $R \geq 1$, we obtain

$$\begin{aligned} \{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s &= \int_1^R \frac{d}{dt} \{p(te^{i\theta})\}^s dt, \\ &= \int_1^R s \{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt. \end{aligned}$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt,$$

which on combining with inequalities (3.2) and (1.1), we get

$$\begin{aligned} |\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leq \frac{ns}{k^{n-1} + k^n} \{M(p, 1)\}^s \int_1^R t^{ns-1} dt, \\ &= \left(\frac{R^{ns} - 1}{k^{n-1} + k^n} \right) \{M(p, 1)\}^s. \end{aligned}$$

Which implies

$$\begin{aligned} |p(Re^{i\theta})|^s &\leq |p(e^{i\theta})|^s + \left(\frac{R^{ns} - 1}{k^{n-1} + k^n} \right) \{M(p, 1)\}^s, \\ &\leq \{M(p, 1)\}^s + \left(\frac{R^{ns} - 1}{k^{n-1} + k^n} \right) \{M(p, 1)\}^s. \end{aligned} \quad (3.3)$$

Hence, from (3.3), we conclude that

$$\{M(p, R)\}^s \leq \left(\frac{k^{n-1} + k^n + R^{ns} - 1}{k^{n-1} + k^n} \right) \{M(p, 1)\}^s.$$

This completes the proof of Theorem 1. \square

Proof of Theorem 2. The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using Lemma 2 instead of Lemma 1. But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, therefore, by Lemma 2, we have

$$|p'(z)| \leq \frac{n}{k^n} \left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|} \right) M(p, 1) \quad \text{for } |z| = 1. \quad (3.4)$$

Now $p'(z)$ is a polynomial of degree $n - 1$, therefore, it follows by (1.1) that for all $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq \frac{nr^{n-1}}{k^n} \left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|} \right) M(p, 1). \quad (3.5)$$

Also for each $\theta, 0 \leq \theta < 2\pi$ and $R \geq 1$, we have

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt,$$

which on combining with inequalities (1.1) and (3.5), we get

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq \left(\frac{R^{ns} - 1}{k^n} \right) \left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|} \right) \{M(p, 1)\}^s,$$

which implies

$$|p(Re^{i\theta})|^s \leq \{M(p, 1)\}^s + \left(\frac{R^{ns} - 1}{k^n} \right) \left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|} \right) \{M(p, 1)\}^s,$$

from which the proof of Theorem 2 follows. \square

REFERENCES

- [1] N. C. ANKENY AND T. J. RIVLIN, *On a Theorem of S. Bernstein*, Pacific J. Math., **5** (1955), 849–852.
- [2] K. K. DEWAN AND A. MIR, *Note on a Theorem of S. Bernstein*, Southeast Asian Bulletin of Math., **31** (2007), 691–695.
- [3] N. K. GOVIL, *On the Theorem of S. Bernstein*, J. Math. and Phy. Sci., **14** (1980), 183–187.
- [4] G. PÓLYA AND G. SZEGÖ, *Aufgaben and Lehrsätze aus der Analysis*, Springer-Verlag, Berlin, 1925.
- [5] M. A. QAZI, *On the maximum modulus of polynomials*, Proc. Amer. Math. Soc., **115** (1992), 337–343.
- [6] W. M. SHAH, *Extremal Properties and Bounds for the Zeros of Polynomials*, Ph. D. Thesis, University of Kashmir, 1998.

(Received April 27, 2010)

K. K. Dewan

Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia (Central University)
New Delhi – 110025 India
e-mail: kkdewan123@yahoo.co.in

Arty Ahuja

Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia (Central University)
New Delhi – 110025 India
e-mail: aarty_ahuja@yahoo.com