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GROWTH OF POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. In this paper, we study the growth of polynomials of degree n having all its zeros on $|z| = k$, $k \leqslant 1$. Using the notation $M(p,t) = \max_{|z|=t} |p(z)|$, we measure the growth of p by estimating $\left\{ \frac{M(p,t)}{M(p,1)} \right\}^s$ from above for any $t \geqslant 1$, s being an arbitrary positive integer.

1. Introduction and Statement of Results

For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$ and $m(f,k) = \min_{|z|=k} |f(z)|$. Then for a polynomial $p(z)$ of degree n, it is a simple consequence of maximum modulus principle (for reference see [4 , Vol. I, p. 137, Problem III, 269]) that

$$M(p,R) \leqslant R^n M(p,1), \quad \text{for } R \geqslant 1. \tag{1.1}$$

The result is best possible and equality holds in (1.1) for $p(z) = \lambda z^n$, where $|\lambda| = 1$.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequalities (1.1) can be sharpened. In fact, it was shown by Ankeny and Rivlin [1] that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$M(p,R) \leqslant \left(\frac{R^n + 1}{2} \right) M(p,1), \qquad R \geqslant 1. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

For the class of polynomials not vanishing in the disk $|z| < k$, $k \geqslant 1$ Shah [6] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geqslant 1$, then for every real number $R > k$,

$$M(p,R) \leqslant \left(\frac{R^n + k}{1 + k} \right) M(p,1) - \left(\frac{R^n - 1}{1 + k} \right) m(p,k). \tag{1.3}$$

The result is best possible in case k=1 and equality holds for the polynomial $p(z) = z^n + 1$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \leqslant 1$, we have been able to prove the following results.

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THEOREM 1. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leqslant 1$, *then for every positive integer* s

$$\{M(p,R)\}^s \leqslant \left(\frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}\right)\{M(p,1)\}^s, \qquad R \geqslant 1. \tag{1.4}$$

If we take $s = 1$ *in Theorem* 1, *we get the following result.*

COROLLARY 1. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leqslant 1$, *then*

$$M(p,R) \leqslant \left(\frac{k^{n-1}(1+k)+(R^{n}-1)}{k^{n-1}+k^n}\right)M(p,1), \qquad R \geqslant 1. \tag{1.5}$$

The following corollary immediately follows from inequality (1.5) *by taking* $k = 1$.

COROLLARY 2. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z| = 1$, *then*

$$M(p,R) \leqslant \left(\frac{R^n+1}{2}\right)M(p,1), \qquad R \geqslant 1. \tag{1.6}$$

If we involve the coefficients of $p(z)$ *also, then we are able to obtain a bound which is better than the bound of Theorem* 1. *More precisely, we prove*

THEOREM 2. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leqslant 1$, *then for every positive integer* s

$$\{M(p,R)\}^s \leqslant \frac{1}{k^n}\left(\frac{n|c_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|c_{n-1}|\{2k^n+R^{ns}-1\}}{2|c_{n-1}|+n|c_n|(1+k^2)}\right) \times\{M(p,1)\}^s, \quad R \geqslant 1. \tag{1.7}$$

To prove that the bound obtained in the above theorem is, in general, better than the bound obtained in Theorem 1, we show that

$$\frac{1}{k^n}\left(\frac{n|c_n|\{k^n(1+k^2)+k^2(R^{ns}-1)\}+|c_{n-1}|\{2k^n+R^{ns}-1\}}{2|c_{n-1}|+n|c_n|(1+k^2)}\right) \leqslant \frac{k^{n-1}(1+k)+(R^{ns}-1)}{k^{n-1}+k^n}$$

which is equivalent to

$$n|c_n|(k^{n-1}+k^n)(k^n+k^{n+2}+k^2R^{ns}-k^2)+|c_{n-1}|(k^{n-1}+k^n)(2k^n+R^{ns}-1) \leqslant n|c_n|(k^{n+2}+k^n)(k^{n-1}+k^n+R^{ns}-1)+2k^n|c_{n-1}|(k^{n-1}+k^n+R^{ns}-1)$$

which implies

$$\begin{aligned} &n|c_n|(k^{2n-1}+k^{2n+1}+k^{n+1}R^{ns}-k^{n+1}+k^{2n}+k^{2n+2}+k^{n+2}R^{ns}-k^{n+2})\\ &\quad +|c_{n-1}|(2k^{2n-1}+k^{n-1}R^{ns}-k^{n-1}+2k^{2n}+k^{n}R^{ns}-k^{n})\\ &\leqslant n|c_n|(k^{2n-1}+k^{2n+1}+k^{n}R^{ns}-k^{n}+k^{2n}+k^{2n+2}+k^{n+2}R^{ns}-k^{n+2})\\ &\quad +|c_{n-1}|(2k^{2n-1}+2k^{2n}+2k^{n}R^{ns}-2k^{n}) \end{aligned}$$

or

$$\begin{aligned} n|c_n|\{k^{n+1}(R^{ns}-1)\}+|c_{n-1}|\{k^{n-1}(R^{ns}-1)\} &\leqslant n|c_n|\{k^{n}(R^{ns}-1)\}+|c_{n-1}|\{k^{n}(R^{ns}-1)\},\\ |c_{n-1}|k^{n-1}(1-k) &\leqslant n|c_n|k^{n}(1-k),\\ \frac{|c_{n-1}|}{n|c_n|} &\leqslant k, \end{aligned}$$

which is always true (see Lemma 4).

We illustrate by means of following example that the bound obtained in Theorem 2 is better than the bound obtained in Theorem 1.

EXAMPLE 1. Let $P(z)=z^4-\frac{1}{50}z^2+\left(\frac{1}{100}\right)^2$ and $k=1/10$, $R=1.5$ and $s=2$. Then by Theorem 1, we have

$$\{M(P,R)\}^s \leqslant 22390.91477\{M(P,1)\}^s,$$

while by Theorem 2, we get

$$\{M(P,R)\}^s \leqslant 2439.505569\{M(P,1)\}^s.$$

For $s=1$ in Theorem 2, we get the following result.

COROLLARY 3. *If* $p(z)=\sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z|=k,\ k\leqslant 1$*, then*

$$M(p,R)\leqslant \frac{1}{k^n}\left(\frac{n|c_n|\{k^n(1+k^2)+k^2(R^n-1)\}+|c_{n-1}|\{2k^n+R^n-1\}}{2|c_{n-1}|+n|c_n|(1+k^2)}\right)M(p,1), \quad R\geqslant 1. \tag{1.8}$$

REMARK 1. If we take $k=1$ in inequality (1.8), it reduces to Corollary 2.

2. Lemmas

We need the following lemmas for the proof of these theorems. The first lemma is due to Govil [3].

LEMMA 1. *If* $p(z)=\sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z|=k,\ k\leqslant 1$, *then*

$$\max_{|z|=1}|p'(z)|\leqslant \frac{n}{k^{n-1}+k^{n}}\max_{|z|=1}|p(z)|. \tag{2.1}$$

LEMMA 2. *If* $p(z)=\sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros on* $|z|=k,\ k\leqslant 1$, *then*

$$\max_{|z|=1}|p'(z)|\leqslant \frac{n}{k^{n}}\left(\frac{n|c_n|k^2+|c_{n-1}|}{2|c_{n-1}|+n|c_n|(1+k^2)}\right)\max_{|z|=1}|p(z)|. \tag{2.2}$$

The above lemma is due to Dewan and Mir [2].

LEMMA 3. *Let* $p(z)=c_0+\sum_{\upsilon=\mu}^{n} c_\upsilon z^\upsilon,\ 1\leqslant\mu\leqslant n$, *be a polynomial of degree* n *having no zero in the disk* $|z|<k,\ k\geqslant 1$. *Then*

$$\frac{\mu}{n}\left|\frac{c_\mu}{c_0}\right|k^{\mu}\leqslant 1. \tag{2.3}$$

The above lemma was given by Qazi [5, Remark 1].

LEMMA 4. *Let* $p(z)=\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ *be a polynomial of degree n having all its zeros on* $|z|=k,\ k\leqslant 1$. *Then*

$$\frac{1}{n}\left|\frac{c_{n-1}}{c_n}\right|\leqslant k. \tag{2.4}$$

Proof of Lemma 4. If $p(z)$ has all its zeros on $|z|=k,\ k\leqslant 1$, then $q(z)=\left(z^n\overline{p\left(\frac{1}{\overline{z}}\right)}\right)$ has all its zeros on $|z|=\frac{1}{k},\ \frac{1}{k}\geqslant 1$. Now applying Lemma 3 for $\mu=1$ to the polynomial $q(z)$, Lemma 4 follows. □

3. Proof of the Theorems

Proof of Theorem 1. Let $M(p,1) = \max_{|z|=1} |p(z)|$. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leqslant 1$, therefore, by Lemma 1, we have

$$|p'(z)| \leqslant \frac{n}{k^{n-1}+k^n} M(p,1) \quad \text{for } |z| = 1. \tag{3.1}$$

Now applying inequality (1.1) to the polynomial $p'(z)$ which is of degree $n-1$ and noting (3.1), it follows that for all $r \geqslant 1$ and $0 \leqslant \theta < 2\pi$

$$|p'(re^{i\theta})| \leqslant \frac{nr^{n-1}}{k^{n-1}+k^n} M(p,1). \tag{3.2}$$

Also for each θ, $0 \leqslant \theta < 2\pi$ and $R \geqslant 1$, we obtain

$$\begin{aligned}\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s &= \int_1^R \frac{d}{dt}\{p(te^{i\theta})\}^s dt, \\ &= \int_1^R s\{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt.\end{aligned}$$

This implies

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leqslant s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt,$$

which on combining with inequalities (3.2) and (1.1), we get

$$\begin{aligned}|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leqslant \frac{ns}{k^{n-1}+k^n} \{M(p,1)\}^s \int_1^R t^{ns-1} dt, \\ &= \left(\frac{R^{ns}-1}{k^{n-1}+k^n}\right) \{M(p,1)\}^s.\end{aligned}$$

Which implies

$$\begin{aligned}|p(Re^{i\theta})|^s &\leqslant |p(e^{i\theta})|^s + \left(\frac{R^{ns}-1}{k^{n-1}+k^n}\right)\{M(p,1)\}^s, \\ &\leqslant \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^{n-1}+k^n}\right)\{M(p,1)\}^s. \end{aligned} \tag{3.3}$$

Hence, from (3.3), we conclude that

$$\{M(p,R)\}^s \leqslant \left(\frac{k^{n-1}+k^n+R^{ns}-1}{k^{n-1}+k^n}\right)\{M(p,1)\}^s.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 2 follows on the same lines as that of Theorem 1 by using Lemma 2 instead of Lemma 1. But for the sake of completeness we give a brief outline of the proof. Since $p(z)$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leqslant 1$, therefore, by Lemma 2, we have

$$|p'(z)| \leqslant \frac{n}{k^n}\left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|}\right)M(p,1) \quad \text{for} \ |z| = 1. \tag{3.4}$$

Now $p'(z)$ is a polynomial of degree $n-1$, therefore, it follows by (1.1) that for all $r \geqslant 1$ and $0 \leqslant \theta < 2\pi$

$$|p'(re^{i\theta})| \leqslant \frac{nr^{n-1}}{k^n}\left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|}\right)M(p,1). \tag{3.5}$$

Also for each $\theta, 0 \leqslant \theta < 2\pi$ and $R \geqslant 1$, we have

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leqslant s\int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|dt,$$

which on combining with inequalities (1.1) and (3.5), we get

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leqslant \left(\frac{R^{ns}-1}{k^n}\right)\left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|}\right)\{M(p,1)\}^s,$$

which implies

$$|p(Re^{i\theta})|^s \leqslant \{M(p,1)\}^s + \left(\frac{R^{ns}-1}{k^n}\right)\left(\frac{n|c_n|k^2 + |c_{n-1}|}{n|c_n|(1+k^2) + 2|c_{n-1}|}\right)\{M(p,1)\}^s,$$

from which the proof of Theorem 2 follows. □

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