

SOME GEOMETRIC INEQUALITIES INVOLVING ANGLE BISECTORS AND MEDIANS OF A TRIANGLE

WEI-DONG JIANG AND MIHÁLY BENCZE

(Communicated by S. Segura Gomis)

Abstract. In the present paper some geometric inequalities concerning the angle bisectors and the medians of a triangle are established.

1. Introduction

For a given triangle ABC we assume that A, B, C denote its angles, a, b, c denote the lengths of its corresponding sides, w_a, w_b, w_c and m_a, m_b, m_c denote respectively the bisectors of angles A, B, C and the medians. Let R, r and s be the circumradius, the inradius and the semi-perimeter of a triangle respectively, Δ denote the area of triangle ABC .

The angle bisectors and the medians of triangles have many interesting properties. In particular, inequalities for angle bisectors and medians are very attractive subject and play an important role in the study of geometry. A large number of related results can be found in the well-known monographs ([1]–[3]).

In [2, pp. 220], the following inequalities are given.

$$s^2 \leq \sum m_a w_a \leq 3(2R^2 + r^2) \quad (1.1)$$

This is Janous' s extension of a result from [4].

In [2, pp. 219], the following result is also given.

$$\sum \frac{m_a}{w_a} \geq \frac{13}{4} - \frac{r}{2R}. \quad (1.2)$$

The main aim of this paper is to establish some geometric inequalities involving angle bisectors and medians of a triangle, as an application, some improvements of (1.1) and (1.2) are given.

Mathematics subject classification (2010): Primary 26D15.

Keywords and phrases: inequality, triangle, angle bisector, median.

2. Some lemmas

In order to prove our main results, we shall use the following results.

LEMMA 2.1. *In triangle ABC, we have*

$$m_a^2 w_a^2 = s^2(s-a)^2 + \frac{\Delta^2}{(b+c)^2}(b-c)^2. \quad (2.1)$$

Proof. By the formulas for median and angle bisector of triangle ABC : $m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$ and $w_a = \frac{2\sqrt{bcs(s-a)}}{b+c}$, one can easily get

$$m_a^2 = s(s-a) + \frac{1}{4}(b-c)^2, \quad (2.2)$$

and

$$w_a^2 = s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2, \quad (2.3)$$

Based on (2.2) and (2.3), we have

$$\begin{aligned} m_a^2 w_a^2 &= \left[s(s-a) + \frac{1}{4}(b-c)^2 \right] \left[s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2 \right] \\ &= s^2(s-a)^2 - \frac{1}{4} \frac{s(s-a)(b-c)^4}{(b+c)^2} + \frac{1}{4} s(s-a)(b-c)^2 - \frac{s^2(s-a)^2}{(b+c)^2}(b-c)^2 \\ &= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2} [(b+c)^2 - 4s(s-a) - (b-c)^2] \\ &= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2} [(b+c)^2 - (b+c+a)(b+c-a) - (b-c)^2] \\ &= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2} [a^2 - (b-c)^2] \\ &= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2} 4(s-b)(s-c) \\ &= s^2(s-a)^2 + \frac{s(s-a)(s-b)(s-c)}{(b+c)^2}(b-c)^2 \\ &= s^2(s-a)^2 + \frac{\Delta^2}{(b+c)^2}(b-c)^2. \quad \square \end{aligned}$$

LEMMA 2.2. *In $\triangle ABC$, we have*

$$\sum \frac{(b+c)^2}{bc} = \frac{s^2 + 10Rr + r^2}{2Rr} \quad (2.4)$$

Proof.

$$\begin{aligned} \sum \frac{(b+c)^2}{bc} &= \frac{1}{abc} \sum a(b+c)^2 = \frac{1}{abc} \sum a(2s-a)^2 \\ &= \frac{1}{abc} (4s^2 \sum a - 4s \sum a^2 + \sum a^3) \end{aligned}$$

With known identities [2, pp. 52],

$$\begin{aligned} \sum a^2 &= 2(s^2 - 4Rr - r^2), \\ \sum a^3 &= 2s(s^2 - 6Rr - 3r^2), \\ \sum a &= 2s, \\ abc &= 4Rrs, \end{aligned}$$

then we obtain (2.4) immediately. \square

3. Main results

THEOREM 3.1. *In $\triangle ABC$, we have*

$$s(s-a) \leq m_a w_a \leq s(s-a) + \frac{\Delta^2}{2s(s-a)(b+c)^2} (b-c)^2. \tag{3.1}$$

Equality holds if and only if $b = c$.

Proof. By Lemma 2.1 one can easily have

$$m_a^2 w_a^2 \geq s^2 (s-a)^2,$$

which leads to the left hand side of (3.1).

On the other hand, By Lemma 2.1, we have

$$\begin{aligned} m_a^2 w_a^2 &= s^2 (s-a)^2 + \frac{\Delta^2}{(b+c)^2} (b-c)^2, \\ &= \left[s(s-a) + \frac{\Delta^2 (b-c)^2}{2s(s-a)(b+c)^2} \right]^2 - \frac{\Delta^4 (b-c)^4}{4s^2 (s-a)^2 (b+c)^4}, \\ &\leq \left[s(s-a) + \frac{\Delta^2 (b-c)^2}{2s(s-a)(b+c)^2} \right]^2. \end{aligned}$$

this completes the proof of Theorem 3.1. \square

COROLLARY 3.1. *In $\triangle ABC$, we have*

$$m_a w_a \leq s(s-a) + \frac{1}{8} (b-c)^2. \tag{3.2}$$

Proof. By Theorem 3.1, we only need to prove

$$\frac{\Delta^2}{2s(s-a)(b+c)^2} < \frac{1}{8} \quad (3.3)$$

which is equivalent to

$$\begin{aligned} 4\Delta^2 - s(s-a)(b+c)^2 &< 0 \\ \Leftrightarrow s(s-a) [4(s-b)(s-c) - (b+c)^2] &< 0 \\ \Leftrightarrow s(s-a) [a^2 - (b-c)^2 - (b+c)^2] &< 0 \\ \Leftrightarrow s(s-a) [(a+b+c)(a-b-c) - (b-c)^2] &< 0 \end{aligned}$$

Since $a < b + c$, so (3.3) holds. \square

By Corollary 3.1, we can immediately get the following result.

COROLLARY 3.2. *In $\triangle ABC$, we have*

$$\sum m_a w_a \leq s^2 + \frac{1}{8} [(a-b)^2 + (b-c)^2 + (c-a)^2]. \quad (3.4)$$

Equality holds if and only if triangle ABC is equilateral.

REMARK 3.1. Using the known identities $\sum a^2 = 2(s^2 - 4Rr - r^2)$, $\sum bc = s^2 + 4Rr + r^2$, then (3.4) is equivalent to

$$\sum m_a w_a \leq \frac{5}{4}s^2 - 3Rr - \frac{3}{4}r^2. \quad (3.5)$$

We can find that (3.5) is stronger than the right hand side of (1.1).

REMARK 3.2. In [5, pp. 275], Xiao-Guang Chu proved the following inequality

$$\sum m_a w_a \leq s^2 + 2Rr - 4r^2. \quad (3.6)$$

(3.6) is stronger than the right hand side of (1.1) too, but (3.5) and (3.6) cannot be compared between them.

Using Gerretsen's inequality [1, pp. 50]: $s^2 \leq 4R^2 + 4Rr + 3r^2$ and (3.5), we have

COROLLARY 3.3.

$$\sum m_a w_a \leq 5R^2 + 2Rr + 3r^2.$$

By Corollary 3.1, we can get

COROLLARY 3.4. *In $\triangle ABC$, we have*

$$\sum a m_a w_a \leq \frac{1}{4}s(s^2 + 18Rr + 9r^2). \quad (3.7)$$

Equality holds if and only if triangle ABC is equilateral.

By Corollary 3.1 and some known identities, we can get

COROLLARY 3.5. *In $\triangle ABC$, we have*

$$\sum (b+c)m_a w_a \leq \frac{1}{2}s(5s^2 - 22Rr - 7r^2). \tag{3.8}$$

Equality holds if and only if $\triangle ABC$ is equilateral.

THEOREM 3.2.

$$\sum \frac{m_a}{w_a} \geq \frac{s^2 + 10Rr + r^2}{8Rr}. \tag{3.9}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. By the left hand side of (3.1) and Lemma 2.2, we have

$$\begin{aligned} \sum \frac{m_a}{w_a} &= \sum \frac{m_a w_a}{w_a^2} \\ &\geq \sum \frac{s(s-a)}{w_a^2} = \sum \frac{s(s-a)(b+c)^2}{4bcs(s-a)}, \\ &= \sum \frac{(b+c)^2}{4bc} = \frac{s^2 + 10Rr + r^2}{8Rr}. \quad \square \end{aligned}$$

REMARK 3.3. From Gerretsen’s inequality [1, pp. 50]: $s^2 \geq 16Rr - 5r^2$, we can get (1.2) easily from Theorem 3.2, so (3.9) is stronger than (1.2).

COROLLARY 3.6.

$$\sum \frac{m_a}{w_a} \geq \frac{5}{3} + \frac{8s^2}{81Rr}. \tag{3.10}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. By Theorem 3.2, we only need to prove

$$\frac{s^2 + 10Rr + r^2}{8Rr} \geq \frac{5}{3} + \frac{8s^2}{81Rr}, \tag{3.11}$$

inequality (3.11) is equivalent to

$$17s^2 \geq 270Rr - 81r^2, \tag{3.12}$$

By Gerretsen’s inequality

$$s^2 \geq 16Rr - 5r^2,$$

and Euler’s inequality ([1, pp. 48])

$$R \geq 2r,$$

we can conclude that inequality (3.12) holds. \square

In fact, (3.10) was a conjecture proposed by Hua-yan Yin in [5].

THEOREM 3.3. *In $\triangle ABC$, we have*

$$\sum \frac{m_a}{a} \geq \frac{3\sqrt{3}}{2} + \frac{3(2-\sqrt{3})}{2} \sum \frac{(b-c)^2}{bc}. \tag{3.13}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. (3.13) is equivalent to

$$\begin{aligned} \frac{1}{abc} \sum bcm_a &\geq \frac{3\sqrt{3}}{2} + \frac{3(2-\sqrt{3})}{2} \frac{1}{abc} \sum a(b-c)^2, \\ \Leftrightarrow \sum bcm_a &\geq 6\sqrt{3}Rrs + \frac{3(2-\sqrt{3})}{2} \sum a(b-c)^2. \end{aligned} \tag{3.14}$$

By the know identity

$$\sum a(b-c)^2 = 2s(s^2 - 14Rr + r^2),$$

One can get

$$\sum bcm_a \geq 3s \left[(2-\sqrt{3})s^2 + (16\sqrt{3}-28)Rr + (2-\sqrt{3})r^2 \right]. \tag{3.15}$$

Notice that the following identity and inequality have been given in [5, pp. 242, pp. 244]:

$$\sum b^2c^2m_a^2 = s^6 - 12Rrs^4 + r^2s^4 + r^2s^2(12R^2 + 8Rr - r^2) - r^3(4R+r)^3, \tag{3.16}$$

$$abc \sum am_b m_c \geq 4Rrs^2(s^2 - 7Rr + 5r^2), \tag{3.17}$$

By (3.16) and (3.17), we have

$$\left(\sum bcm_a \right)^2 \geq s^6 - s^4(4Rr - r^2) + s^2(-44R^2r^2 + 48Rr^3 - r^4) - r^3(4R+r)^3. \tag{3.18}$$

In order to prove (3.13), by (3.18), we only need to prove

$$\begin{aligned} &s^6 - s^4(4Rr - r^2) + s^2(-44R^2r^2 + 48Rr^3 - r^4) - r^3(4R+r)^3 \\ &\geq 9s^2 \left[(2-\sqrt{3})s^2 + (16\sqrt{3}-28)Rr + (2-\sqrt{3})r^2 \right]^2 \\ \Leftrightarrow &(36\sqrt{3}-62)s^6 + s^4 \left[(1868-1080\sqrt{3})Rr + (72\sqrt{3}-125)r^2 \right] \\ &+ s^2 \left[(8064\sqrt{3}-14012)R^2r^2 + (1920-1080\sqrt{3})Rr^3 + (36\sqrt{3}-64)r^4 \right] \\ &- r^3(4R+r)^3 \geq 0 \\ \Leftrightarrow &\left\{ (36\sqrt{3}-62)s^4 + \left[(876-504\sqrt{3})Rr + (123-72\sqrt{3})r^2 + (124-72\sqrt{3})\frac{r^3}{R} \right] \right\} s^2 \end{aligned}$$

$$\begin{aligned}
& + 4R^2r^2 + (384 - 216\sqrt{3})Rr^3 + (180\sqrt{3} - 324)r^4 + (432\sqrt{3} - 742)\frac{r^5}{R} \\
& + (144\sqrt{3} - 248)\frac{r^6}{R} \} \cdot \left(s^2 - 16Rr + 4r^2 + \frac{2r^3}{R} \right) \\
& + \frac{r^4}{R^3}(R - 2r) \cdot \left\{ (6080 - 3456\sqrt{3})R^4 + (5420 - 3168\sqrt{3})R^3r \right. \\
& \left. + (288\sqrt{3} - 505)R^2r^2 + (792\sqrt{3} - 1362)Rr^3 + (144\sqrt{3} - 248)r^4 \right\} \geq 0.
\end{aligned}$$

Since $s^2 - 16Rr + 4r^2 + \frac{2r^3}{R} \geq 0$ (see [6]), $R \geq 2r$ and the coefficients in parentheses are non negative, this completes the proof of Theorem 3.3. \square

Acknowledgement. The authors are very grateful to the referee for his (her) suggestions for the improvement of this paper.

REFERENCES

- [1] O. BOTTEMA, R. Z. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ AND P. M. VASTIĆ, *Geometric inequalities.*, Groningen, Wolters-Noordhoff, 1969.
- [2] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND V. VOLENEC, *Recent Advances in geometric inequalities*, Dordrecht, Netherlands, Kluwer Academic Publishers, 1989.
- [3] D. S. MITRINOVIĆ, J. E. PEČARIĆ, V. VOLENEC AND J. CHEN, *Addenda to the monograph: Recent Advances in geometric inequalities(I)*, Journal of Ningbo University **4**, 2 (1991), 1–145.
- [4] L. PANAITOPOL AND E. VASILE, *Problem 0: 139*, Gaz. Mat. (Bucharest) **85** (1980), **223** and **86** (1981), 223.
- [5] XUE-ZHI YANG (CHIEF EDITOR), *Research in Inequalities* (in Chinese), Xizang People's Press, 2000.
- [6] WEI-DONG JIANG, YUN HUA, *Refinement of an inequality*, shu xue tong bao (in Chinese) **10** (1997).

(Received June 10, 2010)

Wei-dong Jiang
 Department of Information Engineering
 Weihai Vocational College
 Weihai 264210, ShanDong province
 P. R. CHINA
 e-mail: jackjwd@163.com

Mihály Bencze
 Str Harmanului 6
 505600 Sacele-Négyfalu
 Jud. Brasov, Romania
 e-mail: benczemihaly@yahoo.com