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SOME GEOMETRIC INEQUALITIES INVOLVING ANGLE BISECTORS AND MEDIANS OF A TRIANGLE

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Abstract. In the present paper some geometric inequalities concerning the angle bisectors and the medians of a triangle are established.

1. Introduction

For a given triangle ABC we assume that A,B,C denote its angles, a,b,c denote the lengths of its corresponding sides, w_a,w_b,w_c and m_a,m_b,m_c denote respectively the bisectors of angles A,B,C and the medians. Let R,r and s be the circumradius, the inradius and the semi-perimeter of a triangle respectively, Δ denote the area of triangle ABC.

The angle bisectors and the medians of triangles have many interesting properties. In particular, inequalities for angle bisectors and medians are very attractive subject and play an important role in the study of geometry. A large number of related results can be found in the well-known monographs ([1]–[3]).

In [2, pp. 220], the following inequalities are given.

$$s^2 \leqslant \sum m_a w_a \leqslant 3(2R^2+r^2) \tag{1.1}$$

This is Janous' s extension of a result from [4].

In [2, pp. 219], the following result is also given.

$$\sum \frac{m_a}{w_a} \geqslant \frac{13}{4} - \frac{r}{2R}. \tag{1.2}$$

The main aim of this paper is to establish some geometric inequalities involving angle bisectors and medians of a triangle, as an application, some improvements of (1.1) and (1.2) are given.

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2. Some lemmas

In order to prove our main results, we shall use the following results.

LEMMA 2.1. *In triangle ABC, we have*

$$m_a^2 w_a^2 = s^2(s-a)^2 + \frac{\Delta^2}{(b+c)^2}(b-c)^2. \tag{2.1}$$

Proof. By the formulas for median and angle bisector of triangle ABC: $m_a^2 = \frac{1}{4}\left(2b^2+2c^2-a^2\right)$ and $w_a = \frac{2\sqrt{bcs(s-a)}}{b+c}$, one can easily get

$$m_a^2 = s(s-a) + \frac{1}{4}(b-c)^2, \tag{2.2}$$

and

$$w_a^2 = s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2, \tag{2.3}$$

Based on (2.2) and (2.3), we have

$$\begin{aligned}
m_a^2 w_a^2 &= \left[s(s-a) + \frac{1}{4}(b-c)^2\right]\left[s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2\right] \\
&= s^2(s-a)^2 - \frac{1}{4}\frac{s(s-a)(b-c)^4}{(b+c)^2} + \frac{1}{4}s(s-a)(b-c)^2 - \frac{s^2(s-a)^2}{(b+c)^2}(b-c)^2 \\
&= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}\left[(b+c)^2 - 4s(s-a) - (b-c)^2\right] \\
&= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}\left[(b+c)^2 - (b+c+a)(b+c-a) - (b-c)^2\right] \\
&= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}\left[a^2 - (b-c)^2\right] \\
&= s^2(s-a)^2 + \frac{s(s-a)(b-c)^2}{4(b+c)^2}4(s-b)(s-c) \\
&= s^2(s-a)^2 + \frac{s(s-a)(s-b)(s-c)}{(b+c)^2}(b-c)^2 \\
&= s^2(s-a)^2 + \frac{\Delta^2}{(b+c)^2}(b-c)^2. \qquad \square
\end{aligned}$$

LEMMA 2.2. *In* $\triangle ABC$, *we have*

$$\sum \frac{(b+c)^2}{bc} = \frac{s^2+10Rr+r^2}{2Rr} \tag{2.4}$$

Proof.

$$\sum \frac{(b+c)^2}{bc} = \frac{1}{abc}\sum a(b+c)^2 = \frac{1}{abc}\sum a(2s-a)^2$$
$$= \frac{1}{abc}\left(4s^2\sum a - 4s\sum a^2 + \sum a^3\right)$$

With known identities [2, pp. 52],

$$\sum a^2 = 2(s^2-4Rr-r^2),$$
$$\sum a^3 = 2s(s^2-6Rr-3r^2),$$
$$\sum a = 2s,$$
$$abc = 4Rrs,$$

then we obtain (2.4) immediately. □

3. Main results

THEOREM 3.1. *In $\triangle ABC$, we have*

$$s(s-a) \leqslant m_a w_a \leqslant s(s-a) + \frac{\Delta^2}{2s(s-a)(b+c)^2}(b-c)^2. \tag{3.1}$$

Equality holds if and only if $b=c$.

Proof. By Lemma 2.1 one can easily have

$$m_a^2 w_a^2 \geqslant s^2(s-a)^2,$$

which leads to the left hand side of (3.1).

On the other hand, By Lemma 2.1, we have

$$m_a^2 w_a^2 = s^2(s-a)^2 + \frac{\Delta^2}{(b+c)^2}(b-c)^2,$$
$$= \left[s(s-a) + \frac{\Delta^2(b-c)^2}{2s(s-a)(b+c)^2}\right]^2 - \frac{\Delta^4(b-c)^4}{4s^2(s-a)^2(b+c)^4},$$
$$\leqslant \left[s(s-a) + \frac{\Delta^2(b-c)^2}{2s(s-a)(b+c)^2}\right]^2.$$

this completes the proof of Theorem 3.1. □

COROLLARY 3.1. *In $\triangle ABC$, we have*

$$m_a w_a \leqslant s(s-a) + \frac{1}{8}(b-c)^2. \tag{3.2}$$

Proof. By Theorem 3.1, we only need to prove

$$\frac{\Delta^2}{2s(s-a)(b+c)^2} < \frac{1}{8} \tag{3.3}$$

which is equivalent to

$$\begin{aligned} &4\Delta^2 - s(s-a)(b+c)^2 < 0 \\ &\Leftrightarrow s(s-a)\left[4(s-b)(s-c)-(b+c)^2\right] < 0 \\ &\Leftrightarrow s(s-a)\left[a^2-(b-c)^2-(b+c)^2\right] < 0 \\ &\Leftrightarrow s(s-a)\left[(a+b+c)(a-b-c)-(b-c)^2\right] < 0 \end{aligned}$$

Since $a < b+c$, so (3.3) holds. □

By Corollary 3.1, we can immediately get the following result.

COROLLARY 3.2. *In $\triangle ABC$, we have*

$$\sum m_a w_a \leqslant s^2 + \frac{1}{8}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]. \tag{3.4}$$

Equality holds if and only if triangle ABC is equilateral.

REMARK 3.1. Using the known identities $\sum a^2 = 2(s^2-4Rr-r^2), \sum bc = s^2 + 4Rr + r^2$, then (3.4) is equivalent to

$$\sum m_a w_a \leqslant \frac{5}{4}s^2 - 3Rr - \frac{3}{4}r^2. \tag{3.5}$$

We can find that (3.5) is stronger than the right hand side of (1.1).

REMARK 3.2. In [5, pp. 275], Xiao-Guang Chu proved the following inequality

$$\sum m_a w_a \leqslant s^2 + 2Rr - 4r^2. \tag{3.6}$$

(3.6) is stronger than the right hand side of (1.1) too, but (3.5) and (3.6) cannot be compared between them.

Using Gerretsen's inequality [1, pp. 50]: $s^2 \leqslant 4R^2 + 4Rr + 3r^2$ and (3.5), we have

COROLLARY 3.3.

$$\sum m_a w_a \leqslant 5R^2 + 2Rr + 3r^2.$$

By Corollary 3.1, we can get

COROLLARY 3.4. *In $\triangle ABC$, we have*

$$\sum a m_a w_a \leqslant \frac{1}{4}s(s^2 + 18Rr + 9r^2). \tag{3.7}$$

Equality holds if and only if triangle ABC is equilateral.

By Corollary 3.1 and some known identities, we can get

COROLLARY 3.5. *In $\triangle ABC$, we have*

$$\sum (b+c)m_a w_a \leqslant \frac{1}{2}s(5s^2-22Rr-7r^2). \tag{3.8}$$

Equality holds if and only if $\triangle ABC$ is equilateral.

THEOREM 3.2.

$$\sum \frac{m_a}{w_a} \geqslant \frac{s^2+10Rr+r^2}{8Rr}. \tag{3.9}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. By the left hand side of (3.1) and Lemma 2.2, we have

$$\begin{aligned}\sum \frac{m_a}{w_a} &= \sum \frac{m_a w_a}{w_a^2} \\ &\geqslant \sum \frac{s(s-a)}{w_a^2} = \sum \frac{s(s-a)(b+c)^2}{4bcs(s-a)}, \\ &= \sum \frac{(b+c)^2}{4bc} = \frac{s^2+10Rr+r^2}{8Rr}. \qquad \square\end{aligned}$$

REMARK 3.3. From Gerretsen's inequality [1, pp. 50]: $s^2 \geqslant 16Rr-5r^2$, we can get (1.2) easily from Theorem 3.2, so (3.9) is stronger than (1.2).

COROLLARY 3.6.

$$\sum \frac{m_a}{w_a} \geqslant \frac{5}{3} + \frac{8s^2}{81Rr}. \tag{3.10}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. By Theorem 3.2, we only need to prove

$$\frac{s^2+10Rr+r^2}{8Rr} \geqslant \frac{5}{3} + \frac{8s^2}{81Rr}, \tag{3.11}$$

inequality (3.11) is equivalent to

$$17s^2 \geqslant 270Rr - 81r^2, \tag{3.12}$$

By Gerretsen' s inequality

$$s^2 \geqslant 16Rr-5r^2,$$

and Euler' s inequality ([1, pp. 48])

$$R \geqslant 2r,$$

we can conclude that inequality (3.12) holds. $\square$

In fact, (3.10) was a conjecture proposed by Hua-yan Yin in [5].

THEOREM 3.3. *In $\triangle ABC$, we have*

$$\sum \frac{m_a}{a} \geqslant \frac{3\sqrt{3}}{2} + \frac{3(2-\sqrt{3})}{2} \sum \frac{(b-c)^2}{bc}. \tag{3.13}$$

Equality holds if and only if triangle ABC is equilateral.

Proof. (3.13) is equivalent to

$$\frac{1}{abc} \sum bcm_a \geqslant \frac{3\sqrt{3}}{2} + \frac{3(2-\sqrt{3})}{2} \frac{1}{abc} \sum a(b-c)^2,$$

$$\Leftrightarrow \sum bcm_a \geqslant 6\sqrt{3}Rrs + \frac{3(2-\sqrt{3})}{2} \sum a(b-c)^2. \tag{3.14}$$

By the know identity

$$\sum a(b-c)^2 = 2s(s^2 - 14Rr + r^2),$$

One can get

$$\sum bcm_a \geqslant 3s \left[(2-\sqrt{3})s^2 + (16\sqrt{3}-28)Rr + (2-\sqrt{3})r^2\right]. \tag{3.15}$$

Notice that the following identity and inequality have been given in [5, pp. 242, pp. 244]:

$$\sum b^2c^2m_a^2 = s^6 - 12Rrs^4 + r^2s^4 + r^2s^2(12R^2 + 8Rr - r^2) - r^3(4R+r)^3, \tag{3.16}$$

$$abc \sum am_bm_c \geqslant 4Rrs^2(s^2 - 7Rr + 5r^2), \tag{3.17}$$

By (3.16) and (3.17), we have

$$(\sum bcm_a)^2 \geqslant s^6 - s^4(4Rr - r^2) + s^2(-44R^2r^2 + 48Rr^3 - r^4) - r^3(4R+r)^3. \tag{3.18}$$

In order to prove (3.13), by (3.18), we only need to prove

$$\begin{aligned}
&s^6 - s^4(4Rr - r^2) + s^2(-44R^2r^2 + 48Rr^3 - r^4) - r^3(4R+r)^3 \\
&\qquad \geqslant 9s^2 \left[(2-\sqrt{3})s^2 + (16\sqrt{3}-28)Rr + (2-\sqrt{3})r^2\right]^2 \\
&\Leftrightarrow (36\sqrt{3}-62)s^6 + s^4\left[(1868 - 1080\sqrt{3})Rr + (72\sqrt{3}-125)r^2\right] \\
&\qquad\quad + s^2\left[(8064\sqrt{3} - 14012)R^2r^2 + (1920 - 1080\sqrt{3})Rr^3 + (36\sqrt{3}-64)r^4\right] \\
&\qquad\quad - r^3(4R+r)^3 \geqslant 0 \\
&\Leftrightarrow \Big\{(36\sqrt{3}-62)s^4 + \left[(876 - 504\sqrt{3})Rr + (123 - 72\sqrt{3})r^2 + (124 - 72\sqrt{3})\frac{r^3}{R}\right] s^2
\end{aligned}$$

$$+4R^2r^2+(384-216\sqrt{3})Rr^3+(180\sqrt{3}-324)r^4+(432\sqrt{3}-742)\frac{r^5}{R}$$
$$+(144\sqrt{3}-248)\frac{r^6}{R}\Big\}\cdot\left(s^2-16Rr+4r^2+\frac{2r^3}{R}\right)$$
$$+\frac{r^4}{R^3}(R-2r)\cdot\Big\{(6080-3456\sqrt{3})R^4+(5420-3168\sqrt{3})R^3r$$
$$+(288\sqrt{3}-505)R^2r^2+(792\sqrt{3}-1362)Rr^3+(144\sqrt{3}-248)r^4\Big\}\geqslant 0.$$

Since $s^2-16Rr+4r^2+\frac{2r^3}{R}\geqslant 0$ (see [6]), $R\geqslant 2r$ and the coefficients in parentheses are non negative, this completes the proof of Theorem 3.3. □

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REFERENCES

[1] O. BOTTEMA, R. Z. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ AND P. M. VASIĆ, *Geometirc inequalities.*, Groningen, Wolters-Noordhoff, 1969.
[2] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND V. VOLENEC, *Recent Advances in geometric inequalities*, Dordrecht, Netherlands, Kluwer Academic Publishers, 1989.
[3] D. S. MITRINOVIĆ, J. E. PEČARIĆ, V. VOLENEC AND J. CHEN, *Addenda to the monograph: Recent Advances in geometric inequalities(I)*, Journal of Ningbo University **4**, 2 (1991), 1–145.
[4] L. PANAITOPOL AND E. VASILE, *Problem 0: 139*, Gaz. Mat. (Bucharest) **85** (1980), **223** and **86** (1981), 223.
[5] XUE-ZHI YANG(CHIEF EDITOR), *Research in Inequalities*(in Chinese), Xizang People's Press, 2000.
[6] WEI-DONG JIANG, YUN HUA, *Refinement of an inequality*, shu xue tong bao(in Chinese) **10** (1997).

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