

A NOTE ON THE ISOPERIMETRIC INEQUALITY AND ITS STABILITY

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(Communicated by S. Segura Gomis)

Abstract. In this paper, we deal with isoperimetric-type inequalities for closed convex curves in the Euclidean plane \mathbb{R}^2 . We derive a family of parametric inequalities involving the following geometric functionals associated to a given convex curve with a simple Fourier series proof: length, area of the region included by the curve, area of the domain enclosed by the locus of curvature centers and integral of the radius of curvature. By using our isoperimetric-type inequalities, we also obtain some new geometric Bonnesen-type inequalities. Furthermore we investigate stability properties of such inequalities (near equality implies curve nearly circular).

1. Introduction

The classical isoperimetric inequality in the Euclidean plane \mathbb{R}^2 states that:

THEOREM 1.1. [Isoperimetric Inequality] *If γ is a simple closed curve of length L , enclosing a region of area A , then*

$$L^2 - 4\pi A \geq 0, \quad (1)$$

and the equality holds if and only if γ is a circle.

This fact was known to the ancient Greeks, and the first mathematical proof was only given in the 19th century by Steiner. Since then, there have been many new proofs, sharpened forms, generalizations, and applications of this famous inequality.

Suppose that γ is a \mathcal{C}_+^2 closed and strictly convex curve in the Euclidean plane \mathbb{R}^2 with length L , area of the region included by the curve A , and area of the domain enclosed by the locus of curvature centers \tilde{A} . Then there are also some interesting reverse isoperimetric inequalities, such as the inequality

$$L^2 \leq 4\pi (A + |\tilde{A}|), \quad (2)$$

proved by S. L. Pan and H. Zhang in [1], and the inequality (3) proved by S. L. Pan and J. N. Yang in [2]:

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geq \frac{L^2 - 2\pi A}{\pi}, \quad (3)$$

Mathematics subject classification (2010): Primary 52A38, Secondary 52A40.

Keywords and phrases: Isoperimetric inequality, Fourier series, stability.

where ρ is the radius of curvature and θ is the angle between x -axis and the outward normal vector at the corresponding point p . Moreover the equalities in (2) and (3) hold if and only if γ is a circle.

It is obvious that if γ is a circle, then the locus of its curvature centers is only a point, and thus its area $\tilde{A} = 0$. Conversely, if $\tilde{A} = 0$, then from the classical isoperimetric inequality (1) and the reverse isoperimetric inequality (2), it follows that the area A and the length L of γ satisfy $L^2 = 4\pi A$, which implies that γ is a circle, and therefore the locus of curvature centers of γ is a point.

In this paper we deal with a family of parametric isoperimetric-type inequalities for closed convex plane curves, which is actually an improved version of the reverse isoperimetric inequalities (2) and (3), and one of the main results is as follows:

THEOREM 1.2. [Main Theorem] *Let γ be a \mathcal{C}_+^2 closed and strictly convex curve in the Euclidean plane \mathbb{R}^2 with length L and enclosing an area A , then for arbitrary constants $\alpha, \beta, \lambda, \delta$ satisfying*

$$\begin{cases} 2\alpha + \delta \geq 0 \\ 2\alpha + 4\pi\beta + \lambda \geq 0 \\ 6\alpha - \lambda + 4\delta \geq 0, \end{cases} \tag{4}$$

we have

$$\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2 + \lambda A + \delta |\tilde{A}| \geq 0, \tag{5}$$

where ρ is the curvature radius of γ and \tilde{A} is the area of the domain enclosed by the locus of curvature centers. The equality holds if γ is a circle and the parameters $\alpha, \beta, \lambda, \delta$ satisfy

$$2\alpha + 4\pi\beta + \lambda = 0. \tag{6}$$

Moreover if the equality in (5) holds and the parameters $\alpha, \beta, \lambda, \delta$ satisfy

$$\begin{cases} 2\alpha + \delta > 0 \\ 2\alpha + 4\pi\beta + \lambda = 0 \\ 6\alpha - \lambda + 4\delta = 0, \end{cases} \tag{7}$$

then the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$.

REMARK 1. When $\alpha = 0, \beta = -1, \lambda = \delta = 4\pi$, (4) satisfies and the isoperimetric inequality (5) turns into (2). When $\alpha = 1, \beta = -\frac{1}{\pi}, \lambda = 2, \delta = 0$, (4) also satisfies and we obtain (3). Hence (5) could also be regarded as a reverse isoperimetric inequality. Furthermore, if we select other values of the parameters $\alpha, \beta, \lambda, \delta$ satisfying (4), we can obtain some new geometric Bonnesen-type inequalities [3]:

COROLLARY 1.3. *Let γ be a \mathcal{C}_+^2 closed and strictly convex curve in the Euclidean plane \mathbb{R}^2 with length L and enclosing an area A , we have*

$$L^2 \leq 4\pi A + \pi |\tilde{A}|, \tag{8}$$

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geq \frac{L^2}{\pi} - 2A + |\tilde{A}|, \tag{9}$$

and

$$\max_{\theta \in [0, 2\pi]} \rho(\theta)^2 \geq \frac{1}{2\pi} \left(\frac{L^2}{\pi} - 2A + |\tilde{A}| \right), \tag{10}$$

where ρ is the curvature radius of γ and \tilde{A} is the area of the domain enclosed by the locus of curvature centers. Furthermore, (8) and (9) are improved versions of (2) and (3), and the equalities in (8), (9) and (10) hold if γ is a circle. Moreover if the equalities in (8), (9) and (10) hold, then the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$.

The stability problem associated with isoperimetric inequality is also interesting and significant.

Recently in [4], S. L. Pan and H. P. Xu obtained the following stability estimates for the reverse isoperimetric inequality (2) by comparing a convex body K with its Steiner disk.

$$\begin{aligned} h_1(K, S(K))^2 &= \left(\max_u |p_K(u) - p_{S(K)}(u)| \right)^2 \\ &\leq \frac{4\pi^2 - 33}{96\pi^2} (4\pi(A(K) + |\tilde{A}(K)|) - L^2(K)), \end{aligned}$$

$$\begin{aligned} h_2(K, S(K))^2 &= \int_0^{2\pi} |p_K(\theta) - p_{S(K)}(\theta)|^2 d\theta \\ &\leq \frac{1}{18\pi} (4\pi(A(K) + |\tilde{A}(K)|) - L^2(K)), \end{aligned}$$

where $p_K(\theta)$ denotes the Minkowski support function of a given convex body K , and $S(K)$ denotes the Steiner disc associated with K (see section 4 for the definition) which satisfies

$$4\pi(A(S(K)) + |\tilde{A}(S(K))|) - L^2(S(K)) = 0. \tag{11}$$

For arbitrary $\varepsilon > 0$ such that $\varphi(K) = 4\pi(A(K) + |\tilde{A}(K)|) - L^2(K) < \varepsilon$, by the stability estimates for inequality above and (11) it follows that

$$\max \left\{ h_1(K, S(K))^2, h_2(K, S(K))^2 \right\} \leq C|\varphi(K) - \varphi(S(K))| < C\varepsilon,$$

which implies that the reverse isoperimetric inequality (2) does have a good stability behaviour with respect to both Hausdorff distance and L^2 -metric.

The paper is organized as follows. In section 2, we recall some basic facts about plane convex geometry. In section 3, we provide a simpler proof of Theorem 1.2 by using Fourier series, which is different from the approach in [1] and [2]. In section 4, we investigate stability properties of inequality (5) (near equality implies curve nearly circular). We believe that our trick could be used to derive more interesting isoperimetric inequalities.

2. Geometric quantities and their fourier series

In this section, we recall some basic facts about convex plane curve which will be used later. In this paper we always assume that γ is a closed and convex plane curve which is sufficiently regular, actually it should be a \mathcal{C}_+^2 closed and strictly convex curve in the plane \mathbb{R}^2 , such that the radius of curvature can be defined and the Fourier series needed in the proof convergent uniformly. The details can be found in the classical literature [5].

Let $p(\theta)$ denote the Minkowski support function of curve $\gamma(\theta)$, where θ is the angle between x -axis and the outward normal vector at the corresponding point p . It gives us the parametrization of $\gamma(\theta)$ in terms of θ as follows:

$$\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = (p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta).$$

Therefore the curvature $k(\theta)$ and the radius of curvature $\rho(\theta)$ of $\gamma(\theta)$ can be calculated by

$$k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0$$

and

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta) > 0.$$

The length L of $\gamma(\theta)$ and the area A it bounds can be also calculated respectively by

$$L = \int_{\gamma} ds = \int_0^{2\pi} p(\theta) d\theta$$

and

$$A = \frac{1}{2} \int_{\gamma} p(\theta) ds = \frac{1}{2} \int_0^{2\pi} (p(\theta)^2 - p'(\theta)^2) d\theta.$$

At the same time, we could obtain the locus of centers of curvature of $\gamma(\theta)$ as follow

$$\beta(\theta) = \gamma(\theta) + \rho(\theta)N(\theta) = (-p'(\theta) \sin \theta - p''(\theta) \cos \theta, p'(\theta) \cos \theta - p''(\theta) \sin \theta),$$

and the oriented area of the domain enclosed by $\beta(\theta)$ is given by

$$\tilde{A} = \frac{1}{2} \int_0^{2\pi} (p'(\theta)^2 - p''(\theta)^2) d\theta.$$

Since the Minkowski support function of a given convex body K is always continuous, bounded and 2π -periodic, it has a Fourier series of the form

$$p(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \quad (12)$$

Differentiation of (12) with respect to θ gives us

$$p'(\theta) = \sum_{n=1}^{\infty} n(-a_n \sin n\theta + b_n \cos n\theta) \quad (13)$$

and

$$p''(\theta) = - \sum_{n=1}^{\infty} n^2 (a_n \cos n\theta + b_n \sin n\theta). \tag{14}$$

Thus by (12), (13), (14) and the Parseval equality we could express these geometric quantities in terms of the Fourier coefficients of $p(\theta)$

$$\begin{aligned} \rho(\theta) &= p(\theta) + p''(\theta) \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) - \sum_{n=1}^{\infty} n^2 (a_n \cos n\theta + b_n \sin n\theta), \end{aligned} \tag{15}$$

$$L(K) = 2\pi a_0, \tag{16}$$

$$A = \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1) (a_n^2 + b_n^2), \tag{17}$$

$$|\tilde{A}| = \frac{\pi}{2} \sum_{n=2}^{\infty} n^2 (n^2 - 1) (a_n^2 + b_n^2). \tag{18}$$

3. Proof of the main theorems

Proof of Theorem 1.2]. Firstly from (15), one can easily get

$$\begin{aligned} &\int_0^{2\pi} \rho(\theta)^2 d\theta \\ &= 2 \left(\pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1) (a_n^2 + b_n^2) + \frac{\pi}{2} \sum_{n=2}^{\infty} n^2 (n^2 - 1) (a_n^2 + b_n^2) \right) \\ &= 2\pi \left(a_0^2 + \frac{1}{2} \sum_{n=2}^{\infty} (n^2 - 1)^2 (a_n^2 + b_n^2) \right), \end{aligned}$$

thus by using (16), (17) and (18) we have

$$\begin{aligned} &\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2 + \lambda A + \delta |\tilde{A}| \\ &= 2\pi\alpha \left(a_0^2 + \frac{1}{2} \sum_{n=2}^{\infty} (n^2 - 1)^2 (a_n^2 + b_n^2) \right) + \beta (2\pi a_0)^2 \\ &\quad + \lambda \left(\pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty} (n^2 - 1) (a_n^2 + b_n^2) \right) + \frac{\pi}{2} \delta \sum_{n=2}^{\infty} n^2 (n^2 - 1) (a_n^2 + b_n^2) \\ &= \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \frac{\pi}{2} \sum_{n=2}^{\infty} (2\alpha (n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) (a_n^2 + b_n^2) \end{aligned} \tag{19}$$

It follows from (4) that (19) is nonnegative, which completes the proof of inequality (5).

Furthermore, if γ is a circle, by the equality conditions in (2) and (3) we have

$$L^2 = 4\pi (A + |\tilde{A}|) = 4\pi A$$

and

$$\int_0^{2\pi} \rho(\theta)^2 d\theta = \frac{L^2 - 2\pi A}{\pi} = 2A.$$

Hence

$$\begin{aligned} & \alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2 + \lambda A + \delta |\tilde{A}| \\ &= 2\alpha A + 4\pi\beta A + \lambda A \\ &= (2\alpha + 4\pi\beta + \lambda)A \end{aligned}$$

then for the parameters $\alpha, \beta, \lambda, \delta$ satisfying (6) we have

$$\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2 + \lambda A + \delta |\tilde{A}| = 0$$

On the other hand, if equality holds in (5):

$$\begin{aligned} 0 &= \alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2 + \lambda A + \delta |\tilde{A}| \\ &= \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \frac{3\pi}{2} (6\alpha - \lambda + 4\delta) (a_2^2 + b_2^2) \\ &\quad + \frac{\pi}{2} \sum_{n=3}^{\infty} (2\alpha (n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) (a_n^2 + b_n^2) \end{aligned}$$

then by the condition (7):

$$\begin{cases} 2\alpha + \delta > 0 \\ 2\alpha + 4\pi\beta + \lambda = 0 \\ 6\alpha - \lambda + 4\delta = 0 \end{cases}$$

we have

$$0 = \frac{\pi}{2} \sum_{n=3}^{\infty} (2\alpha (n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) (a_n^2 + b_n^2)$$

and

$$(2\alpha (n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) > 0$$

for $n \geq 3$. Thus $a_n = b_n = 0$ for $n \geq 3$ and the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$. This completes the proof of Theorem 1.2. \square

Proof of Corollary 1.3. Let $\alpha = 0, \beta = -1, \lambda = 4\pi, \delta = \pi$ we obtain (8), and let $\alpha = 1, \beta = -\frac{1}{\pi}, \lambda = 2, \delta = -1$, we can derive (9). Moreover the equality conditions in (8) and (9) follows directly from the equality conditions in (5).

On the other hand, inequality (10) is an easy consequence of (9), and if γ is a circle, the equality holds directly. Conversely, since

$$\max_{\theta \in [0, 2\pi]} \rho(\theta)^2 \geq \frac{1}{2\pi} \int_0^{2\pi} \rho(\theta)^2 d\theta \geq \frac{1}{2\pi} \left(\frac{L^2}{\pi} - 2A + |\tilde{A}| \right),$$

if equality holds in (10), we have

$$\int_0^{2\pi} \rho(\theta)^2 d\theta = \frac{L^2}{\pi} - 2A + |\tilde{A}|.$$

By the equality condition of (9), it follows that the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$. \square

4. The stability property of the isoperimetric inequality

Let K and M be two convex bodies with respective Minkowski support functions p_K and p_M . The most frequently used function to measure the deviation between K and M is the Hausdorff distance:

$$h_1(K, M) = \max_u |p_K(u) - p_M(u)|.$$

Another distance is defined by means of the L^2 -norm of the support functions, that is

$$h_2(K, M) = \left(\int_0^{2\pi} |p_K(\theta) - p_M(\theta)|^2 d\theta \right)^{\frac{1}{2}},$$

where θ is the angle between x -axis and the outward normal vector at the corresponding point p . It is obvious that $h_1(K, M) = 0$ or $h_2(K, M) = 0$ if and only if $K = M$.

We also recall the definition of Steiner disc $S(K)$ of a planar convex body K .

DEFINITION 4.1. The Steiner disc of a convex body K , denoted by $S(K)$ is the circular disc with radius $\frac{L(K)}{2\pi}$ and center at the Steiner point $\vec{s}(K)$ which can be defined in terms of the Minkowski support function $p_K(\theta)$:

$$\vec{s}(K) = \frac{1}{\pi} \int_0^{2\pi} \vec{u}(\theta) p_K(\theta) d\theta,$$

where $\vec{u}(\theta)$ is a unit tangent vector at the corresponding point p , and $L(K)$ denotes the perimeter of the domain K .

We now derive a stability version of (5) with respect to both Hausdorff distance h_1 and h_2 metric.

THEOREM 4.2. Let K be a domain enclosed by a \mathcal{C}_+^2 closed and strictly convex plane curve γ with area $A(K)$ and perimeter $L(K)$, and let $\tilde{A}(K)$ denote the oriented area of the domain enclosed by the locus of curvature centers of γ , $S(K)$ denotes the Steiner disc associated with K . Then for arbitrary constants $\alpha, \beta, \lambda, \delta$ which satisfy

$$\begin{cases} 2\alpha + \delta \geq 0 \\ 2\alpha + 4\pi\beta + \lambda \geq 0 \\ 6\alpha - \lambda + 4\delta > 0, \end{cases} \tag{20}$$

we have

$$h_1(K, S(K))^2 \leq C(\alpha, \lambda, \delta) \left(\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \right), \tag{21}$$

where $C(\alpha, \lambda, \delta) = \max \left\{ 1, \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1}{(2\alpha(n^2-1) - \lambda + \delta n^2)(n^2-1)} \right\}$. The equality holds if γ is a circle and the parameters $\alpha, \beta, \lambda, \delta$ satisfy

$$2\alpha + 4\pi\beta + \lambda = 0.$$

Proof. We may assume $\vec{s}(K) = 0$, because of (12) and (16), the support functions p_K and $p_{S(K)}$ have the following Fourier series:

$$p_K(\theta) = \frac{L(K)}{2\pi} + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{22}$$

and

$$p_{S(K)}(\theta) = \frac{L(K)}{2\pi}. \tag{23}$$

One can observe that (22) and (23) yield an explicit expression (in terms of the Fourier coefficients) for the quantity:

$$\begin{aligned} & \alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \\ &= \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \frac{\pi}{2} \sum_{n=2}^{\infty} (2\alpha(n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) (a_n^2 + b_n^2). \end{aligned} \tag{24}$$

Since it is easily seen that

$$|a_n \cos n\theta + b_n \sin n\theta| \leq \sqrt{a_n^2 + b_n^2},$$

it follows that

$$\begin{aligned} |p_K(\theta) - p_{S(K)}(\theta)| &= \left| \frac{L(K)}{2\pi} + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) - \frac{L(K)}{2\pi} \right| \\ &\leq \sum_{n=2}^{\infty} |a_n \cos n\theta + b_n \sin n\theta| \\ &\leq \sum_{n=2}^{\infty} \sqrt{a_n^2 + b_n^2}. \end{aligned}$$

Using Holder’s inequality, together with (24) we have

$$\begin{aligned}
 h_1(K, S(K))^2 &\leq \left(\sum_{n=2}^{\infty} \sqrt{a_n^2 + b_n^2} \right)^2 \\
 &\leq \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \left(\frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1}{(2\alpha(n^2 - 1) - \lambda + \delta n^2)(n^2 - 1)} \right) \\
 &\quad \left(\frac{\pi}{2} \sum_{n=2}^{\infty} (2\alpha(n^2 - 1) - \lambda + \delta n^2)(n^2 - 1)(a_n^2 + b_n^2) \right) \\
 &\leq \max \left\{ 1, \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1}{(2\alpha(n^2 - 1) - \lambda + \delta n^2)(n^2 - 1)} \right\} \\
 &\quad \left(\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \right),
 \end{aligned}$$

for arbitrary constants $\alpha, \beta, \lambda, \delta$ satisfying (20).

Furthermore, if γ is a circle, as the proof of Theorem 1.2 we have

$$\begin{aligned}
 &\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \\
 &= (2\alpha + 4\pi\beta + \lambda)A
 \end{aligned}$$

If the parameters $\alpha, \beta, \lambda, \delta$ satisfy $2\alpha + 4\pi\beta + \lambda = 0$, then we have

$$\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| = 0$$

It is obvious that $h_1(K, S(K)) = 0$, thus equality holds in (21). \square

THEOREM 4.3. *Under the same assumptions of Theorem 4.2, then for arbitrary constants $\alpha, \beta, \lambda, \delta$ which satisfy*

$$\begin{cases} 2\alpha + \delta \geq 0 \\ 2\alpha + 4\pi\beta + \lambda \geq 0 \\ 18\alpha - 3\lambda + 12\delta - 2 \geq 0, \end{cases} \tag{25}$$

we have

$$h_2(K, S(K))^2 \leq \alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2 + \lambda A + \delta |\tilde{A}|. \tag{26}$$

The equality holds if γ is a circle and the parameters $\alpha, \beta, \lambda, \delta$ satisfy

$$2\alpha + 4\pi\beta + \lambda = 0.$$

Moreover if the equality in (26) holds and the parameters $\alpha, \beta, \lambda, \delta$ satisfy

$$\begin{cases} 2\alpha + \delta > 0 \\ 2\alpha + 4\pi\beta + \lambda = 0 \\ 18\alpha - 3\lambda + 12\delta - 2 = 0, \end{cases} \tag{27}$$

then the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$.

Proof. As the proof of Theorem 4.2, we use Parseval's equality, (22) and (23) to deduce that

$$h_2(K, S(K))^2 = \int_0^{2\pi} |p_K(\theta) - p_{S(K)}(\theta)|^2 d\theta = \pi \sum_{n=2}^{\infty} (a_n^2 + b_n^2),$$

together with (24) one gets that

$$\begin{aligned} & \left(\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \right) - h_2(K, S(K))^2 \\ &= \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \frac{\pi}{2} \sum_{n=2}^{\infty} (2\alpha(n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) (a_n^2 + b_n^2) \\ & \quad - \pi \sum_{n=2}^{\infty} (a_n^2 + b_n^2) \\ &= \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \frac{\pi}{2} \sum_{n=2}^{\infty} ((2\alpha(n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) - 2) (a_n^2 + b_n^2). \end{aligned}$$

Hence for arbitrary constants $\alpha, \beta, \lambda, \delta$ satisfying (25), we have

$$\left(\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \right) - h_2(K, S(K))^2 \geq 0,$$

which implies the following stability result:

$$h_2(K, S(K))^2 \leq \alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)|.$$

Furthermore, if γ is a circle, as the proof of Theorem 4.2, we have equality in (26). Conversely, if equality holds in (26):

$$\begin{aligned} 0 &= \left(\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \right) - h_2(K, S(K))^2 \\ &= \pi a_0^2 (2\alpha + 4\pi\beta + \lambda) + \frac{\pi}{2} \sum_{n=2}^{\infty} ((2\alpha(n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) - 2) (a_n^2 + b_n^2) \end{aligned}$$

then by the condition (27), we have

$$0 = \frac{\pi}{2} \sum_{n=3}^{\infty} ((2\alpha(n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) - 2) (a_n^2 + b_n^2)$$

and

$$(2\alpha(n^2 - 1) - \lambda + \delta n^2) (n^2 - 1) - 2 > 0$$

for $n \geq 3$. Thus $a_n = b_n = 0$ for $n \geq 3$ and the Minkowski support function of γ is of the form $p(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta$. This completes the proof of Theorem 4.3. \square

REMARK 2. The combination of Theorem 4.2 and 4.3 leads to

$$\begin{aligned} & \max \left\{ h_1(K, S(K))^2, h_2(K, S(K))^2 \right\} \\ & \leq C(\alpha, \lambda, \delta) \left(\alpha \int_0^{2\pi} \rho(\theta)^2 d\theta + \beta L^2(K) + \lambda A(K) + \delta |\tilde{A}(K)| \right), \end{aligned} \quad (30)$$

where $C(\alpha, \lambda, \delta) = \max \left\{ 1, \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1}{(2\alpha(n^2-1) - \lambda + \delta n^2)(n^2-1)} \right\}$, which states that the isoperimetric inequality (5) does have a good stability behaviour with respect to both Hausdorff distance and L^2 -metric.

Acknowledgement. I would especially like to express my appreciation to my advisor professor Yu Zheng for longtime encouragement and meaningful discussions. I would also especially like to thank the referee for meaningful suggestions that led to improvement of the article.

REFERENCES

- [1] S. L. PAN, H. ZHANG, *A reverse isoperimetric inequality for convex plane curves*, Beitrage Algebra Geom. **48** (2007), 303–308.
- [2] S. L. PAN, J. N. YANG, *On a non-local perimeter-preserving curve evolution problem for convex plane curves*, Manuscripta Math. **127** (2008), 469–484.
- [3] R. SCHNEIDER, *Convex bodies: The Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1993.
- [4] S. L. PAN, H. P. XU, *Stability of a reverse isoperimetric inequality*, J. Math. Anal. Appl. **350** (2009), 348–353.
- [5] H. GROEMER, *Geometric applications of Fourier series and spherical harmonics*, Encyclopedia of Mathematics and its Applications, 61. Cambridge University Press, Cambridge, 1996.

(Received June 22, 2010)

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