

## REMARKS ON SOME WENDROFF TYPE INEQUALITIES

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*Abstract.* In this paper we give new proofs for some recent generalizations of Wendroff inequality (see [1]) and we obtain a representation for the best upper bound in the Wendroff inequality. Moreover we point out that the proofs of theorem 2.1 and 2.2 from [1] contain some errors, hence a new proof is necessary. Our method can be applied to a wide class of Gronwall type inequalities and gives elegant and powerful proofs for most of the known Gronwall type inequalities.

### 1. Introduction

The Wendroff inequality is a generalization of the Gronwall inequality for 2 independent variables, has its origin in the theory of partial differential equations and can be found in many monographs on inequalities ([4], [3], [9], [8]). Recently the authors in [1] gave a sharpened version for a Wendroff type inequality proved by Pachpatte (see [9]) but unfortunately their proof contains some errors. In this paper we prove the inequality given in [1] (theorem 2.2) and we use the abstract comparison Gronwall lemma to obtain new proofs for well known generalizations of the Wendroff inequality. Our method uses an operatorial point of view and can be used to simplify the proofs of many other Wendroff type inequalities.

#### 1.1. Wendroff type inequalities

In what follows we consider  $D = [0, l] \times [0, l] \subset \mathbb{R}^2$ . As a starting point we recall the following generalization of the original Wendroff inequality proved by B. G. Pachpatte:

**THEOREM 1.1.** ([1], [9]) *Let  $u(x, y)$ ,  $w(x, y)$  and  $a(x, y)$  non-negative continuous functions defined for  $(x, y) \in D$ , and let  $w(x, y)$  be non-decreasing in each variable  $x, y \in D$ . If*

$$u(x, y) \leq w(x, y) + \int_0^x \int_0^y a(s, t)u(s, t) dt ds, \quad \forall (x, y) \in D \quad (1.1)$$

then

$$u(x, y) \leq w(x, y) \exp \left( \int_0^x \int_0^y a(s, t) dt ds \right), \quad \forall (x, y) \in D. \quad (1.2)$$

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This inequality was generalized in [1] as follows:

**THEOREM 1.2.** ([1] – Theorem 2.1) *Suppose  $u(x, y)$ ,  $w(x, y)$  and  $a(x, y)$  are non-negative continuous functions defined on a domain  $D$  and  $w$  is nondecreasing in the second variable. If inequality (1.1) is satisfied for all  $(x, y) \in D$ , then*

$$u(x, y) \leq w(x, y) + G^{-1} \left( \int_0^x \int_0^y a(s, t) dt ds \right), \quad \forall (x, y) \in D, \quad (1.3)$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{s+w}, r \geq r_0 > 0, \quad (1.4)$$

$G^{-1}$  is the inverse function of  $G$  and  $\int_0^x \int_0^y a(s, t) dt ds \in \text{Dom}(G^{-1}), \forall (x, y) \in D$ .

**REMARK 1.3.** In [1] the last condition ( $w$  is nondecreasing) is omitted in the statement of the theorem but it is used in the proof (line 7 of the proof).

**THEOREM 1.4.** ([1] – Theorem 2.2) *Suppose  $u(x, y)$ ,  $w(x, y)$  and  $a(x, y)$  are non-negative continuous functions defined on a domain  $D$ , and let  $w(x, y)$  be nondecreasing in each variable  $(x, y) \in D$ . If  $u$  satisfies inequality (1.1), then  $\forall (x, y) \in D$  we have*

$$u(x, y) \leq w(x, y) \left[ 1 + \int_0^x \int_0^y a(s, t) \exp \left( \int_s^x \int_t^y a(\xi, \eta) d\xi d\eta \right) dt ds \right]. \quad (1.5)$$

**REMARK 1.5.** The proof of theorem 2.2 in [1] contains 2 errors. The first error is on line 5-6 of the proof and can be corrected only by adding further assumptions on the functions  $w$  and  $a$ . This motivates the need of a new proof for this theorem. The second error is on page 611, line 11 but this error can be corrected only by replacing the right hand side of the inequality with an other expression and this weakens the inequality.

## 1.2. Picard operators

The Picard operator technique was developed by I. A. Rus (see [11] and the references therein) in order to handle some important problems in the theory of differential equations (existence, uniqueness, differentiability of the solutions, etc.) and can be applied also in the study of Gronwall type inequalities (see [10], [13], [14], [5], [11], [12], [2] and the references therein).

We recall some notations and some properties from [11].

Let  $(X, \rightarrow)$  be an L-space,  $A : X \rightarrow X$  an operator. We denote by  $F_A$  the fixed points of  $A$ . We also denote  $A^0 := 1_X$ ,  $A^1 := A$ ,  $\dots$ ,  $A^{n+1} := A^n \circ A$ ,  $n \in \mathbb{N}$  the iterate operators of the operator  $A$ .

**DEFINITION 1.6.** ([10], [11], [12])  $A$  is a Picard operator (briefly PO), if there exists  $x_A^* \in X$  such that:

- (i)  $F_A = \{x_A^*\}$ ;
- (ii)  $A^n(x) \rightarrow x_A^*$  as  $n \rightarrow \infty$ ,  $\forall x \in X$ .

REMARK 1.7. The Banach fixed point theorem guaranties that if  $A$  is a contraction, then  $A$  is a Picard operator.

The following two abstract Gronwall lemmas can be used in order to give simple and elegant proofs for most of the known Gronwall type inequalities.

LEMMA 1.8. [[10], [11]] (Abstract Gronwall lemma) *Let  $(X, \rightarrow, \leq)$  be an ordered  $L$ -space and  $A : X \rightarrow X$  an operator. We assume that:*

(i)  $A$  is PO;

(ii)  $A$  is increasing.

*If we denote by  $x_A^*$  the unique fixed point of  $A$ , then:*

(a)  $x \leq A(x) \Rightarrow x \leq x_A^*$ ;

(b)  $x \geq A(x) \Rightarrow x \geq x_A^*$ .

LEMMA 1.9. [[10], [11]] (Abstract Gronwall-comparison lemma) *Let  $(X, \rightarrow, \leq)$  be an ordered  $L$ -space and  $A_1, A_2 : X \rightarrow X$  be two operators. We assume that:*

(i)  $A_1$  is increasing;

(ii)  $A_1$  and  $A_2$  are POs;

(iii)  $A_1 \leq A_2$ .

*If we denote by  $x_2^*$  the unique fixed point of  $A_2$ , then*

$$x \leq A_1(x) \Rightarrow x \leq x_2^*.$$

## 2. Main results

The inequality (1.1) is linear in  $u$ , so we can obtain a representation by applying the successive approximation method starting from the  $u_0 = w$ . This representation gives also the solution of the integral equation

$$u(x, y) = w(x, y) + \int_0^x \int_0^y a(s, t)u(s, t)dt ds, \quad \forall (x, y) \in D, \quad (2.1)$$

hence this is the maximal solution of the inequality (1.1). In [5] the authors proved that the right hand side of the classical Wendroff inequality is not the fixed point of the corresponding integral operator (it is not the solution of the associated integral equation). In what follows we prove that this is also valid for the Wendroff type inequalities proved in [1] and we construct the best possible estimation (see theorem 2.1). We use this representation to give a correct proof of theorem 2.2 from [1]. In the last section we use the abstract Gronwall comparison lemma (see [14]) for proving theorem 1.2 and 2.1. from [1].

### 2.1. The representation for the best estimation

THEOREM 2.1. *Suppose  $u(x, y)$ ,  $w(x, y)$  and  $a(x, y)$  are non-negative continuous functions defined on a domain  $D$ . If  $u$  satisfies inequality (1.1), then*

$$u(x, y) \leq w(x, y) + \int_0^x \int_0^y a(s, t)w(s, t)H(x, y, s, t)dt ds, \quad \forall (x, y) \in D \quad (2.2)$$

where

$$H(x, y, s, t) = \sum_{j=0}^{\infty} K_j(x, y, s, t)$$

and

$$K_{j+1}(x, y, s, t) = \int_s^x \int_t^y a(\xi, \eta) K_j(x, y, \xi, \eta) d\eta d\xi, \quad K_0 \equiv 1.$$

*Proof.* The integral operator  $A : C(D) \rightarrow C(D)$  defined by

$$A(u)(x, y) = w(x, y) + \int_0^x \int_0^y a(s, t) u(s, t) dt ds \quad (2.3)$$

is a Picard operator if we consider the Bielecki norm on the set  $C(D)$  :

$$\|u\| = \max_{(x, y) \in D} e^{-\tau(x+y)} |u(x, y)|.$$

Moreover the space  $(C(D), \|\cdot\|)$  is an ordered Banach space with the natural ordering  $u \leq v \Leftrightarrow u(x, y) \leq v(x, y), \forall (x, y) \in D$  and the operator  $A$  is an increasing operator. These observations allow us to apply the abstract Gronwall lemma, so

$$u(x, y) \leq u^*(x, y),$$

where  $u^*(x, y)$  is the solution of the integral equation (2.1). But this solution can be obtained as the limit of the successive approximation sequence starting from  $u_0 = w$  and the terms of this sequence can be calculated as follows:

$$\begin{aligned} u_1(x, y) &= A(u_0)(x, y) \\ &= w(x, y) + \int_0^x \int_0^y a(s, t) w(s, t) dt ds \\ u_2(x, y) &= A(u_1)(x, y) \\ &= w(x, y) + \int_0^x \int_0^y a(s, t) w(s, t) dt ds \\ &\quad + \int_0^x \int_0^y a(s, t) \int_0^s \int_0^t a(\xi, \eta) w(\xi, \eta) d\eta d\xi dt ds \end{aligned}$$

Changing the order of integration in the last integral and renaming the variables we obtain

$$\begin{aligned} u_2(x, y) &= w(x, y) + \int_0^x \int_0^y a(s, t) w(s, t) dt ds \\ &\quad + \int_0^x \int_0^y a(s, t) w(s, t) \int_s^x \int_t^y a(\xi, \eta) d\eta d\xi dt ds \\ &= w(x, y) + \int_0^x \int_0^y a(s, t) w(s, t) [K_0(x, y, s, t) + K_1(x, y, s, t)] dt ds. \end{aligned}$$

Applying the operator  $A$  one more time we obtain

$$\begin{aligned} u_3(x, y) &= A(u_2)(x, y) \\ &= w(x, y) + \int_0^x \int_0^y a(s, t)w(s, t) \sum_{j=0}^2 K_j(x, y, s, t) dt ds \end{aligned}$$

and by an inductive argument we deduce

$$\begin{aligned} u_{k+1}(x, y) &= A(u_k)(x, y) \\ &= w(x, y) + \int_0^x \int_0^y a(s, t)w(s, t) \sum_{j=0}^k K_j(x, y, s, t) dt ds. \end{aligned}$$

Hence the solution can be represented as

$$u^*(x, y) = w(x, y) + \int_0^x \int_0^y a(s, t)w(s, t)H(x, y, s, t) dt ds, \quad \forall (x, y) \in D,$$

where

$$H(x, y, s, t) = \sum_{j=0}^{\infty} K_j(x, y, s, t)$$

and

$$K_{j+1}(x, y, s, t) = \int_s^x \int_t^y a(\xi, \eta)K_j(x, y, \xi, \eta) d\eta d\xi, \quad K_0 \equiv 1. \quad \square$$

**THEOREM 2.2.** (Corrected statement of theorem 2.2 from [1]) *Suppose  $u(x, y)$ ,  $w(x, y)$  and  $a(x, y)$  are non-negative continuous functions defined on a domain  $D$  and  $w$  is nondecreasing in both variables. If  $u$  satisfies inequality (1.1), then*

$$u(x, y) \leq w(x, y) + \int_0^x \int_0^y a(s, t)w(s, t) \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\eta d\xi\right) dt ds, \quad \forall (x, y) \in D. \quad (2.4)$$

*Proof.* Denote by  $\bar{u}(x, y)$  the right hand side of the inequality (2.4). Using the representation from theorem 2.1 it is sufficient to prove that

$$H(x, y, s, t) \leq \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\eta d\xi\right), \quad \forall (x, y) \in D.$$

In order to prove this inequality we proceed by mathematical induction and we prove that for all  $k \in \mathbb{N}$

$$\sum_{j=0}^k K_j(x, y, s, t) \leq \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\eta d\xi\right), \quad \forall (x, y) \in D \text{ and } s \leq x, t \leq y.$$

This inequality is trivial for  $k = 0$ . For a fixed  $k$  by replacing  $s$  with  $\xi$  and  $t$  with  $\eta$ , multiplying with  $a(\xi, \eta)$  and integrating from  $s$  to  $x$  and from  $t$  to  $y$  we obtain

$$\sum_{j=0}^k \int_s^x \int_t^y a(\xi, \eta) K_j(x, y, \xi, \eta) d\eta d\xi \leq \int_s^x \int_t^y a(\xi, \eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha, \beta) d\beta d\alpha\right) d\eta d\xi \quad (2.5)$$

which implies

$$\sum_{j=0}^{k+1} K_j(x, y, s, t) \leq 1 + \int_s^x \int_t^y a(\xi, \eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha, \beta) d\beta d\alpha\right) d\eta d\xi \quad (2.6)$$

In order to complete the inductive argument (and also the proof) it is sufficient to prove that

$$1 + \int_s^x \int_t^y a(\xi, \eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha, \beta) d\beta d\alpha\right) d\eta d\xi \leq \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\eta d\xi\right).$$

Consider the function

$$G(\xi, \eta) = \exp\left(\int_\xi^x \int_\eta^y a(\alpha, \beta) d\beta d\alpha\right), \quad (2.7)$$

where  $x, y$  are fixed parameters. For this function we have

$$\frac{\partial G}{\partial \xi}(\xi, \eta) = - \int_\eta^y a(\xi, \beta) d\beta \cdot G(\xi, \eta) \text{ and}$$

$$\frac{\partial^2 G}{\partial \xi \partial \eta}(\xi, \eta) = a(\xi, \eta) \cdot G(\xi, \eta) + \int_\xi^x a(\alpha, \eta) d\alpha \cdot \int_\eta^y a(\xi, \beta) d\beta \cdot G(\xi, \eta).$$

From this equality and the nonnegativity of  $a$  we obtain

$$a(\xi, \eta) G(\xi, \eta) \leq \frac{\partial^2 G}{\partial x \partial y}(\xi, \eta),$$

hence

$$\int_s^x \int_t^y a(\xi, \eta) G(\xi, \eta) d\eta d\xi \leq \int_s^x \int_t^y \frac{\partial^2 G}{\partial \xi \partial \eta}(\xi, \eta) d\eta d\xi.$$

But calculating the integrals from the right hand side expression we obtain

$$-1 + \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\eta d\xi\right),$$

so the proof is complete.  $\square$

## 2.2. The abstract comparison lemma

The proof of theorem 2.1 in [1] contains an error on line 7 (if  $n(x, y) = \int_0^x \int_0^y a(s, t) u(s, t) dt ds$ , then the quantity  $n_y(x, y) + w_y(x, y)$  is not necessary nonnegative. This error can be corrected if we assume that  $w$  is increasing in  $y$ .

In the following we use the abstract Gronwall-comparison lemma (Lemma 1.9) to prove theorem 1.2 (theorem 2.1 from [1]) and to prove a known version of a nonlinear Gronwall-Bihari inequality.

**THEOREM 2.3.** ([1], [5]) *Suppose  $u(x, y)$ ,  $w(x, y)$  and  $a(x, y)$  are non-negative continuous functions defined on a domain  $D$ ,  $w$  is nondecreasing in  $y$ . If inequality (1.1) is satisfied for all  $(x, y) \in D$ , then*

$$u(x, y) \leq w(x, y) + G^{-1} \left( \int_0^x \int_0^y a(s, t) dt ds \right), \quad \forall (x, y) \in D, \quad (2.8)$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{s+w}, r \geq r_0 > 0, \quad (2.9)$$

$G^{-1}$  is the inverse function of  $G$  and  $\int_0^x \int_0^y a(s, t) dt ds \in \text{Dom}(G^{-1})$ ,  $\forall (x, y) \in D$ .

*Proof.* We can see in [5], that if we take the integral operator  $A_1 : C(D) \rightarrow C(D)$  defined by

$$A_1(u)(x, y) = w(x, y) + \int_0^x \int_0^y a(s, t) u(s, t) dt ds \quad (2.10)$$

it's a PO, but its fixed point is not the function

$$u(x, y) = w(x, y) + G^{-1} \left( \int_0^x \int_0^y a(s, t) dt ds \right), \quad \forall (x, y) \in D, \quad (2.11)$$

Now we have to find a PO  $A_2 : C(D) \rightarrow C(D)$ , with the property  $A_1 \leq A_2$  and with the fixed point defined by (2.11). Due to Lemma 1.9 if we construct  $A_2$ , the proof will be completed.

From (2.11) we obtain:

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial w}{\partial x}(x, y) + (G^{-1})' \left( \int_0^x \int_0^y a(s, t) dt ds \right) \int_0^y a(x, t) dt, \quad \forall (x, y) \in D.$$

But

$$(G^{-1})' \left( \int_0^x \int_0^y a(s, t) dt ds \right) = u(x, y), \quad \forall (x, y) \in D,$$

so we have

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial w}{\partial x}(x, y) + u(x, y) \int_0^y a(x, t) dt, \quad \forall (x, y) \in D.$$

From the basic theorem of the calculus we have:

$$u(x,y) - u(0,y) = \int_0^x \frac{\partial u}{\partial x}(s,y) ds = \int_0^x \left( \frac{\partial w}{\partial x}(s,y) + u(s,y) \int_0^y a(s,t) dt \right) ds.$$

From here we deduce that the Picard operator  $A_2 : C(D) \rightarrow C(D)$ , defined by

$$A_2(u)(x,y) = w(x,y) + \int_0^x \int_0^y u(s,y) a(s,t) dt ds,$$

admits the function defined by (2.11) as a fixed point. If we consider the set

$$X = \{u \in C(D) | u \text{ increasing in } y \text{ and } u(0,y) = w(0,y), u(x,0) = w(x,0)\}$$

and the restrictions of  $A_1, A_2$  to  $X$ , then

$$A_1, A_2 : X \rightarrow X, A_1(u) \leq A_2(u), \quad \forall u \in X$$

because  $u(s,t) \leq u(s,y)$  for  $t \leq y$ . On the other hand  $A_1(u) \in X$ , for all  $u \in C(D)$ ,  $A_1$  and  $A_2$  are Picard operators on  $X$ , and  $X$  is a closed subset of  $C(D)$ , so using Lemma 1.9 the proof is complete.  $\square$

**THEOREM 2.4.** *Let  $u(x,y)$ ,  $w(x,y)$  and  $a(x,y)$  non-negative continuous functions defined on a domain  $D$ ,  $w$  is non-decreasing in both variables and let  $g : [0, \infty) \rightarrow (0, \infty)$  a continuous non-decreasing function. If*

$$u(x,y) \leq w(x,y) + \int_0^x \int_0^y a(s,t) g(u(s,t)) dt ds, \quad \forall (x,y) \in D \tag{2.12}$$

then

$$u(x,y) \leq G^{-1} \left( G(w(x,y)) + \int_0^x \int_0^y a(s,t) dt ds \right), \quad \forall (x,y) \in D, \tag{2.13}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{g(s)}, r \geq r_0 > 0 \tag{2.14}$$

$G^{-1}$  is the inverse function of  $G$  and

$$G(w(x,y)) + \int_0^x \int_0^y a(s,t) dt ds \in \text{Dom}(G^{-1}), \quad \forall (x,y) \in D.$$

*Proof.* Let us consider the integral operator  $A_1 : C(D) \rightarrow C(D)$  defined by the right side of the the inequality (2.12), namely

$$A_1(u)(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t) g(u(s,t)) dt ds, \quad \forall (x,y) \in D,$$

and the function

$$u(x,y) = G^{-1} \left( G(w(x,y)) + \int_0^x \int_0^y a(s,t) dt ds \right), \quad \forall (x,y) \in D. \tag{2.15}$$



We can easily check that  $A_1$  is a PO, but its fixed point is not the function defined by (2.15), so we cannot use the first abstract Gronwall lemma, we have to deal with the abstract Gronwall-comparison lemma, like in the proof of the previous theorem.

From the (2.15) we have:

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= (G^{-1})' \left( G(w(x,y)) + \int_0^x \int_0^y a(s,t) dt ds \right) \\ &\quad \times \left( G'(w(x,y)) \frac{\partial w}{\partial x}(x,y) + \int_0^y a(x,t) dt \right). \end{aligned}$$

But

$$(G^{-1})' \left( G(w(x,y)) + \int_0^x \int_0^y a(s,t) dt ds \right) = g(u(x,y)),$$

so

$$\frac{\partial u}{\partial x}(x,y) = g(u(x,y)) \left( \frac{1}{g(w(x,y))} \frac{\partial w}{\partial x}(x,y) + \int_0^y a(x,t) dt \right).$$

From the basic theorem of calculus we have:

$$\begin{aligned} u(x,y) - u(0,y) &= \int_0^x \frac{\partial u}{\partial x}(s,y) ds \\ &= \int_0^x \frac{g(u(s,y))}{g(w(s,y))} \frac{\partial w}{\partial x}(s,y) ds + \int_0^x \int_0^y a(s,t) g(u(s,y)) dt ds \end{aligned}$$

This relation shows that the function  $u$  defined by (2.15) is the fixed point of the integral operator  $A_2 : C(D) \rightarrow C(D)$ , defined by

$$A_2(u)(x,y) = w(0,y) + \int_0^x \frac{g(u(s,y))}{g(w(s,y))} \frac{\partial w}{\partial x}(s,y) ds + \int_0^x \int_0^y a(s,t) g(u(s,y)) dt ds,$$

$\forall (x,y) \in D$ . If we want to apply the abstract Gronwall-comparison lemma, we need to consider the set

$$X = \{u \in C(D) | u \text{ increasing in } y \text{ and } u(0,y) = w(0,y), u(x,0) = w(x,0)\}$$

and the restrictions of  $A_1, A_2$  to  $X$ . This is necessary in order to obtain  $A_1 u \leq A_2 u$ . But we do not know  $A_2 u \in X$ , and hence we can not apply the abstract Gronwall-comparison lemma as stated in [10] or [14]. This difficulty can be overcome if we observe that:

- from  $u \leq A_1 u$  we deduce  $u \leq u^*$ , where  $u^*$  is the limit of successive approximation sequence for the operator  $A_1$  starting from  $u$ ;
- if  $\bar{u}$  is the fixed point of  $A_2$ , then it is sufficient to have  $A_1(\bar{u}) \leq A_2(\bar{u})$ .

Indeed if  $A_1$  is a PO, then  $u^*$  is the limit of the successive approximation sequence starting from  $\bar{u}$  and from  $A_1(\bar{u}) \leq A_2(\bar{u})$  we can prove by induction that  $A_1^k(\bar{u}) \leq \bar{u}$ , so

$u^* \leq \bar{u}$ . Due to this observation it is sufficient to prove  $A_1 \bar{u} \leq A_2 \bar{u}$ . But  $\bar{u}$  is defined by (2.15), so  $w(x, y) \leq \bar{u}(x, y)$ , hence

$$g(w(s, y)) \leq g(\bar{u}(s, y)), \quad \forall 0 \leq s \leq x,$$

so

$$w(x, y) \leq w(0, y) + \int_0^x \frac{g(\bar{u}(s, y))}{g(w(s, y))} \frac{\partial w}{\partial x}(s, y) ds. \quad (2.16)$$

From (2.15) we can deduce that  $\bar{u}$  is nondecreasing in the second variable, hence

$$\int_0^x \int_0^y a(s, t) g(\bar{u}(s, t)) dt ds \leq \int_0^x \int_0^y a(s, t) g(\bar{u}(s, y)) dt ds, \quad \forall (x, y) \in D. \quad (2.17)$$

From (2.16) and (2.17) we deduce  $A_1(\bar{u}) \leq A_2(\bar{u})$ , so the proof is complete.  $\square$

REMARK 2.5. This inequality generalizes some results from [7].

### 3. Concluding remarks

REMARK 3.1. The abstract Gronwall and the abstract Gronwall-comparison lemma enables us to rewrite the proofs of many Gronwall type inequalities in a unitary, structured, simplified way.

REMARK 3.2. The proof of theorems 2.4 shows that some Gronwall type inequalities can not be proved by using the abstract Gronwall-comparison lemma, because the condition  $A_1 \leq A_2$  is too strong.

REMARK 3.3. The proof of theorems 1.2, 2.2 and 2.4 shows that the abstract Gronwall-comparison lemma needs a revision in order to cover a wider range of concrete Gronwall type inequalities.

### REFERENCES

- [1] A. ABDELDAIM AND M. YAKOUT, *On Wendroff's Inequality and Applications*, Int. Journal of Math. Analysis, **4** (2010), 607–616.
- [2] SZ. ANDRÁS, *Ecuatii integrale Fredholm-Volterra*, Ed. Didactică și Pedagogică, București, 2005.
- [3] D. BAINOV AND P. SIMEONOV, *Integral Inequalities and Applications*, vol. 57 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.
- [4] E. F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer-Verlang, Berlin, 1961.
- [5] C. CRĂCIUN AND N. LUNGU, *Abstract and concrete Gronwall lemmas*, Fixed Point Theory, **10**, 2 (2009), 221–228.
- [6] S. S. DRAGOMIR, *Some Gronwall Type Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: <http://rgmia.vu.edu.au/monographs/>).
- [7] S. S. DRAGOMIR AND YOUNG-HO KIM, *On nonlinear integral inequalities of Gronwall type in two variables*, RGMIA Papers, Victoria University, <http://rgmia.org/papers/v6n2/youngho13-4.pdf>.
- [8] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities: Theory and Applications*, Vol. I-II, Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1969.

- [9] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, New York and London, 1998.
- [10] I. A. RUS, *Fixed points, upper and lower fixed points: abstract Gronwall lemmas*, Carpathian J. Math., **20**, 1 (2004), 125–134.
- [11] I. A. RUS, *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58**, 1 (2003), 191–219.
- [12] I. A. RUS, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.
- [13] I. A. RUS, *Gronwall Lemmas: Ten Open Problems*, Scientiae Math. Japonicae, **22** (2009), Art. 29.
- [14] M. ȘERBAN AND C. CRĂCIUN, *A nonlinear integral equation via Picard operators*, Fixed Point Theory, **12** (2010), to appear.

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