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QUADRATIC INTERPOLATION AND SOME OPERATOR INEQUALITIES

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Abstract. We investigate some properties of Hilbert spaces and bounded linear operators under quadratic interpolation in both qualitative and quantitative ways. Interpolation type, reiteration, interpolation methods associated with quasi-power function parameters, nonlinear commutator estimates, and interpolation of certain operators and spectral properties are under consideration.

1. Introduction and preliminaries

The interpolation theory for Hilbert spaces has a very simple character because of the fine geometric structure of those spaces. In 1967, Donoghue gave a complete description of exact quadratic interpolation methods which classified all Hilbert spaces that can be interpolated exactly between two given Hilbert spaces [9]. Recently, Ameur showed that all exact interpolation spaces for a couple of Hilbert spaces can be obtained in terms of Peetre's K-functional and the real interpolation methods with function parameters, which leads to a new interpretation of Donoghue's interpolation theorem [3, 4, 5].

The aim of the present paper is to investigate some properties of Hilbert spaces and bounded linear operators under quadratic interpolation in both qualitative and quantitative ways. Following this introductory section, we formulate an inequality concerning the interpolation type and the reiteration result in Section 2. In section 3, we consider the quadratic interpolation methods associated with quasi-power function parameters. Section 4 is devoted to some operator inequalities concerning nonlinear commutator estimates, invertible operators, measures of noncompactness, and some spectral properties.

Let $\overline{\mathscr{H}} = \left(\mathscr{H}_0, \mathscr{H}_1\right)$ be a compatible couple of separable Hilbert spaces in the sense that they are both embedded continuously in some Hausdorff topological vector space. Throughout the paper, we assume that $\Delta\overline{\mathscr{H}} = \mathscr{H}_0 \cap \mathscr{H}_1$ is dense in both $\mathscr{H}_0$ and $\mathscr{H}_1$. Such a couple $\overline{\mathscr{H}}$ is said to be a regular Hilbert couple. We denote by $(\cdot,\cdot)_j$ and $\|\cdot\|_j$ the inner product and norm on $\mathscr{H}_j$ $(j = 0, 1)$, respectively, unless otherwise

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mentioned. Observe that $\|\cdot\|_1^2$ is a quadratic form on $\mathscr{H}_0$, and thus there exists a positive, injective, densely defined linear operator $A = \int_{[0,\infty]} \lambda \, dE_\lambda$ in $\mathscr{H}_0$ such that

$$\|x\|_1^2 = (Ax,x)_0 \qquad \text{for} \quad x \in \Delta\overline{\mathscr{H}}.$$

The operator A is bounded on $\mathscr{H}_0$ iff $\mathscr{H}_0 \subseteq \mathscr{H}_1$.

For $t > 0$, we consider the following functionals:

$$J_2\left(t,x;\overline{\mathscr{H}}\right) = J_2(t,x) = \left(\|x\|_0^2 + t\|x\|_1^2\right)^{1/2}$$

for $x \in \Delta\overline{\mathscr{H}}$, and

$$\begin{aligned} &K_2\left(t,x;\overline{\mathscr{H}}\right) = K_2(t,x) \\ &\quad = \inf\left\{ \left(\|x_0\|_0^2 + t\|x_1\|_1^2\right)^{1/2} \middle| x = x_0 + x_1,\, x_j \in \mathscr{H}_j\, (j = 0,1) \right\} \end{aligned}$$

for $x \in \Sigma\overline{\mathscr{H}} = \mathscr{H}_0 + \mathscr{H}_1$. Now $\Delta\overline{\mathscr{H}}$ and $\Sigma\overline{\mathscr{H}}$ are also Hilbert spaces with the norm

$$\|x\|_\Delta^2 = J_2(1,x)^2 = \big((I+A)x,x\big)_0$$

for $x \in \Delta\overline{\mathscr{H}}$, and

$$\|x\|_\Sigma^2 = K_2(1,x)^2$$

for $x \in \Sigma\overline{\mathscr{H}}$. Ameur calculated the K_2-functional as follows [3, (3.1)]:

$$K_2(t,x)^2 = \left(tA(I+tA)^{-1}x,x\right)_0 = \int_{[0,\infty]} \frac{t\lambda}{1+t\lambda}\, d\left(E_\lambda x,x\right)_0 \tag{1.1}$$

for $x \in \Delta\overline{\mathscr{H}}$, which implies

$$\|x\|_\Sigma^2 = \left(A(I+A)^{-1}x,x\right)_0 \tag{1.2}$$

for $x \in \Delta\overline{\mathscr{H}}$.

For regular Hilbert couples $\overline{\mathscr{H}}$ and $\overline{\mathscr{K}}$, we denote by $\mathscr{B}\left(\overline{\mathscr{H}},\overline{\mathscr{K}}\right)$ the space of all bounded linear operators $T\colon \Sigma\overline{\mathscr{H}} \to \Sigma\overline{\mathscr{K}}$ such that the restriction of T on $\mathscr{H}_j$ is bounded from $\mathscr{H}_j$ to $\mathscr{K}_j$ $(j = 0,1)$. The operator norm of T when it is restricted on $\mathscr{H}_j$ $(j = 0,1)$ or $\Delta\overline{\mathscr{H}}$ is denoted by $\|T\|_j$ or $\|T\|_\Delta$ respectively. We simply write $\mathscr{B}\left(\overline{\mathscr{H}}\right) = \mathscr{B}\left(\overline{\mathscr{H}},\overline{\mathscr{H}}\right)$. A regular interpolation method F is said to be quadratic if, for each regular Hilbert couple $\overline{\mathscr{H}}$, the interpolation space $F\left(\overline{\mathscr{H}}\right)$ is a Hilbert space. Let $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a quasi-concave function in the sense that

$$\varphi(t) \leqslant C\varphi(s)\big(1 \vee (t/s)\big)$$

for a positive constant C, and for all $s,t > 0$ [16]. Each quasi-concave function $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ determines a quadratic form on $\mathscr{H}_0$ by

$$\|x\|_\varphi^2 = (x,x)_\varphi = \big(\varphi(A)x,x\big)_0 \qquad \text{for } x \in \Delta\overline{\mathscr{H}}.$$

The corresponding Hilbert space is denoted by $\overline{\mathscr{H}}_\varphi$, which is a regular interpolation space for the couple $\overline{\mathscr{H}}$. Thus, the function φ determines a regular quadratic interpolation method.

A function $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is said to be a Pick function [9, 10], if φ can be represented in the form

$$\varphi(\lambda) = \int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda}\, d\mu(t) \qquad \text{for} \quad \lambda > 0, \tag{1.3}$$

where μ is a positive Radon measure on the compactified halfline $[0,\infty]$. In this case, if $x \in \overline{\mathscr{H}}_\varphi$, then

$$\|x\|_\varphi^2 = \int_{[0,\infty]} \left(1 + \frac{1}{t}\right) K_2(t,x)^2\, d\mu(t) \tag{1.4}$$

by (1.1) and (1.3). Moreover, φ is a Pick function iff $\overline{\mathscr{H}}_\varphi$ is an exact interpolation space of $\overline{\mathscr{H}}$ in the sense that

$$\|T\|_\varphi \leqslant \|T\|_0 \vee \|T\|_1 \tag{1.5}$$

for all $T \in \mathscr{B}\left(\overline{\mathscr{H}}, \overline{\mathscr{K}}\right)$, where $\|T\|_\varphi$ is the operator norm of $T\colon \overline{\mathscr{H}}_\varphi \to \overline{\mathscr{K}}_\varphi$. For instance, $\Delta(\lambda) = 1 + \lambda$, $j(\lambda) = \lambda^j$ $(j = 0,1)$, and $\theta(\lambda) = \lambda^\theta$ $(0 < \theta < 1)$ determine the exact interpolation spaces $\Delta\overline{\mathscr{H}}$, $\overline{\mathscr{H}}_j = \mathscr{H}_j$ $(j = 0,1)$, and $\overline{\mathscr{H}}_\theta$ $(0 < \theta < 1)$, respectively.

2. Interpolation type and reiteration

In this section, we first formulate an inequality concerning the interpolation norms of bounded linear operators, which improves the estimate in (1.5); and then obtain a reiteration result for the quadratic interpolation methods. Let $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a quasi-concave function. We denote

$$\varphi^*(t) = 1/\varphi(1/t) \quad \text{and} \quad \bar{\varphi}(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)} \qquad \text{for } t > 0.$$

A corresponding homogeneous function of two variables (again denoted by φ) is defined by

$$\varphi(t_0, t_1) = t_0 \varphi(t_1/t_0) \qquad \text{for } t_0,\, t_1 > 0.$$

PROPOSITION 2.1. *If φ is a Pick function, then*

$$\|T\|_\varphi \leqslant \bar{\varphi}\left(\|T\|_0^2, \|T\|_1^2\right)^{1/2} \tag{2.1}$$

for all $T \in \mathscr{B}\left(\overline{\mathscr{H}}, \overline{\mathscr{K}}\right)$.

Proof. For $T \in \mathscr{B}(\overline{\mathscr{H}}, \overline{\mathscr{K}})$, we set $M = \|T\|_1 / \|T\|_0$. If $x \in \overline{\mathscr{H}}_\varphi$ for which $x = x_0 + x_1$ with $x_j \in \mathscr{H}_j$ $(j = 0,1)$, then we have

$$K_2(t, Tx)^2 \leqslant \|Tx_0\|_0^2 + t\|Tx_1\|_1^2 \leqslant \|T\|_0^2 \left(\|x_0\|_0^2 + tM^2 \|x_1\|_1^2 \right),$$

and hence $K_2(t, Tx)^2 \leqslant \|T\|_0^2 K_2(tM^2, x)^2$. Assume that φ is given as in (1.3). By (1.1) and (1.4), we have

$$\begin{aligned}
\|Tx\|_\varphi^2 &= \int_0^\infty \left(1 + \frac{1}{t}\right) K_2(t, Tx)^2 \, d\mu(t) \\
&\leqslant \|T\|_0^2 \int_0^\infty \left(1 + \frac{1}{t}\right) K_2(tM^2, x)^2 \, d\mu(t) \\
&= \|T\|_0^2 \int_0^\infty \left(\left(1 + \frac{1}{t}\right) \int_{[0,\infty]} \frac{t\lambda M^2}{1 + t\lambda M^2} \, d(E_\lambda x, x)_0 \right) d\mu(t) \\
&= \|T\|_0^2 \int_{[0,\infty]} \left(\int_0^\infty \frac{(1+t)\lambda M^2}{1 + t\lambda M^2} \, d\mu(t) \right) d(E_\lambda x, x)_0 \\
&= \|T\|_0^2 \int_{[0,\infty]} \varphi(\lambda M^2) \, d(E_\lambda x, x)_0.
\end{aligned}$$

This, together with the inequality $\varphi(\lambda M^2) \leqslant \varphi(\lambda) \cdot \bar{\varphi}(M^2)$, implies that

$$\begin{aligned}
\|Tx\|_\varphi^2 &\leqslant \|T\|_0^2 \bar{\varphi}\left(\|T\|_1^2 / \|T\|_0^2 \right) \int_{[0,\infty]} \varphi(\lambda) \, d(E_\lambda x, x)_0 \\
&= \|T\|_0^2 \bar{\varphi}\left(\|T\|_1^2 / \|T\|_0^2 \right) \|x\|_\varphi^2.
\end{aligned}$$

Therefore,

$$\|T\|_\varphi \leqslant \bar{\varphi}\left(\|T\|_0^2, \|T\|_1^2 \right)^{1/2},$$

which completes the proof. $\square$

REMARKS. (i) If $\varphi \colon \mathbf{R}^+ \to \mathbf{R}^+$ is a concave function, then by [5, Lemma 1], there exists a Pick function ϕ such that $\phi \leqslant \varphi \leqslant 2\phi$. For $T \in \mathscr{B}(\overline{\mathscr{H}}, \overline{\mathscr{K}})$, the inequality in (2.1) becomes

$$\|T\|_\varphi \leqslant \sqrt{2}\, \bar{\varphi}\left(\|T\|_0^2, \|T\|_1^2 \right)^{1/2}. \tag{2.2}$$

This inequality improves the following result

$$\|T\|_\varphi \leqslant \sqrt{2}\left(\|T\|_0 \vee \|T\|_1 \right)$$

given in [14]. In general, if φ is quasi-concave, then

$$\|T\|_\varphi \leqslant C\, \bar{\varphi}\left(\|T\|_0^2, \|T\|_1^2 \right)^{1/2} \tag{2.3}$$

for some positive constant C.

(ii) Let $T = (T_1, \ldots, T_n)$, where $T_i \in \mathscr{B}(\overline{\mathscr{H}})$ and $T_i T_k = T_k T_i$ on $\Delta\overline{\mathscr{H}}$ for $1 \leqslant i,k \leqslant n$. We define $\|T\|_j$ $(j = 0,1)$ by

$$\|T\|_j = \sup\left\{ \left(\|T_1 x\|_j^2 + \cdots + \|T_n x\|_j^2\right)^{1/2} \,\middle|\, x \in \mathscr{H}_j \text{ and } \|x\|_j = 1 \right\},$$

and define $\|T\|_\varphi$ similarly. The inequalities in (2.2) and (2.3) are even valid for this kind of n-tuple T of commuting operators.

PROPOSITION 2.2. *Let ψ be a Pick fuction, let $\varphi_0, \varphi_1 \colon \mathbf{R}^+ \to \mathbf{R}^+$ be quasi-concave functions, and let $\varphi = \psi(\varphi_0, \varphi_1)$. Then*

$$\overline{\mathscr{H}}_\varphi = \left(\overline{\mathscr{H}}_{\varphi_0}, \overline{\mathscr{H}}_{\varphi_1}\right)_\psi$$

with equal norms. Furthermore, if φ_0 and φ_1 are Pick functions, then $\overline{\mathscr{H}}_\varphi$ is an exact interpolation space for $\overline{\mathscr{H}}$ and hence φ is also a Pick function.

Proof. First we assume that

$$\psi(\lambda) = \int_0^\infty \frac{(1+t)\lambda}{1+t\lambda}\, d\mu(t)$$

for $\lambda > 0$. Then

$$\varphi(\lambda) = \varphi_0(\lambda)\psi\left(\varphi_1(\lambda)\big/\varphi_0(\lambda)\right) = \int_0^\infty \frac{(1+t)\varphi_1(\lambda)}{1+t\varphi_1(\lambda)\big/\varphi_0(\lambda)}\, d\mu(t). \tag{2.4}$$

Let now $\mathscr{K}_j = \overline{\mathscr{H}}_{\varphi_j}$ $(j = 0,1)$, and let $\overline{\mathscr{K}} = (\mathscr{K}_0, \mathscr{K}_1)$. Then $\overline{\mathscr{K}}$ is a regular Hilbert couple, and $\Delta\overline{\mathscr{H}}$ is dense in $\Delta\overline{\mathscr{K}} = \overline{\mathscr{H}}_{\varphi_0} \cap \overline{\mathscr{H}}_{\varphi_1}$. Let

$$B = \int_{[0,\infty]} \lambda\, dE_\lambda^B$$

be the positive, injective, densely defined linear operator on $\mathscr{K}_0$ for which

$$(x,y)_{\mathscr{K}_1} = (Bx,y)_{\mathscr{K}_0} \qquad \text{for} \quad x,y \in \Delta\overline{\mathscr{K}}.$$

For each $x \in \Delta\overline{\mathscr{H}}$, we have on one hand

$$\|x\|^2_{\varphi_1(\overline{\mathscr{H}})} = (Bx,x)_{\mathscr{K}_0} = \left(\varphi_0(A)Bx,x\right)_{\mathscr{H}_0},$$

and on the other hand $\|x\|^2_{\varphi_1(\overline{\mathscr{H}})} = \left(\varphi_1(A)x,x\right)_{\mathscr{H}_0}$. This implies that

$$\varphi_1(A)x = \varphi_0(A)Bx,$$

and hence

$$Bx = \varphi_0(A)^{-1}\varphi_1(A)x = \varphi_1(A)\varphi_0(A)^{-1}x.$$

Observe that

$$\begin{aligned} K_2\big(t,x;\overline{\mathscr{K}}\big)^2 &= \int_{[0,\infty]} \frac{t\lambda}{1+t\lambda}\, d\big(E_\lambda^B x,x\big)_{\mathscr{K}_0} = \left(\frac{tB}{1+tB}x,x\right)_{\mathscr{K}_0} \\ &= \left(\frac{t\varphi_1(A)\varphi_0(A)^{-1}}{1+t\varphi_1(A)\varphi_0(A)^{-1}}\varphi_0(A)x,x\right)_{\mathscr{H}_0} \\ &= \int_{[0,\infty]} \frac{t\varphi_1(\lambda)}{1+t\varphi_1(\lambda)/\varphi_0(\lambda)}\, d\big(E_\lambda x,x\big)_{\mathscr{H}_0}. \end{aligned}$$

It turns out that

$$\begin{aligned} \|x\|^2_{\psi(\overline{\mathscr{K}})} &= \int_0^\infty \Big(1+\frac{1}{t}\Big) K_2\big(t,x;\overline{\mathscr{K}}\big)^2\, d\mu(t) \\ &= \int_0^\infty \Big(1+\frac{1}{t}\Big)\left(\int_{[0,\infty]} \frac{t\varphi_1(\lambda)}{1+t\varphi_1(\lambda)/\varphi_0(\lambda)}\, d\big(E_\lambda x,x\big)_{\mathscr{H}_0}\right) d\mu(t) \\ &= \int_{[0,\infty]} \left(\int_0^\infty \frac{(1+t)\varphi_1(\lambda)}{1+t\varphi_1(\lambda)/\varphi_0(\lambda)}\, d\mu(t)\right) d\big(E_\lambda x,x\big)_{\mathscr{H}_0}, \end{aligned}$$

and hence

$$\|x\|^2_{\psi(\overline{\mathscr{K}})} = \int_{[0,\infty]} \varphi(\lambda)\, d\big(E_\lambda x,x\big)_{\mathscr{H}_0} = \|x\|^2_{\varphi(\overline{\mathscr{H}})}$$

by (2.4). That is, $\overline{\mathscr{H}}_\varphi = \big(\overline{\mathscr{H}}_{\varphi_0},\overline{\mathscr{H}}_{\varphi_1}\big)_\psi$ with equal norms. □

3. On quasi-power function parameters

In this section, we investigate the quadratic interpolation space $\overline{\mathscr{H}}_\varphi$ associated with a quasi-power function. A function $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is said to be of quasi-power if $\exists C>0$ and $0<\alpha<1$, for which

$$\varphi(\lambda t) \leqslant C\big(t^\alpha \vee t^{1-\alpha}\big)\varphi(\lambda) \tag{3.1}$$

for all $\lambda, t > 0$. It is clear that a quasi-power function is always quasi-concave. Two quasi-concave functions φ, $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ are said to be equivalent, denoted by $\varphi \approx \psi$, if there exist positive constants a, b such that

$$a\psi(t) \leqslant \varphi(t) \leqslant b\psi(t) \qquad \text{for} \quad t>0.$$

Observe that if φ is a quasi-power function, and if $\psi \approx \varphi$, then ψ is also a quasi-power function.

LEMMA 3.1. *Let φ be a quasi-power function satisfying (3.1), and let*

$$\begin{aligned} \varphi_K(\lambda) &= \int_0^\infty \frac{\lambda t}{1+\lambda t}\varphi(1/t)\,\frac{dt}{t}, \\ \varphi_J(\lambda) &= \left(\int_0^\infty \frac{1}{1+\lambda t}\varphi^*(t)\,\frac{dt}{t}\right)^{-1}. \end{aligned}$$

Then $\varphi_K \approx \varphi_J \approx \varphi$. Consequently, φ_K and φ_J are quasi-power functions, and

$$\overline{\mathscr{H}}_{\varphi_K} = \overline{\mathscr{H}}_{\varphi_J} = \overline{\mathscr{H}}_{\varphi}.$$

with equivalent norms.

Proof. Let $\lambda > 0$. For function φ_K, we have

$$\begin{aligned}\varphi_K(\lambda) &= \int_0^\infty \frac{1}{1+\lambda t}\varphi(1/t)\,dt = \int_0^\infty \frac{t}{1+t}\varphi(\lambda/t)\,\frac{dt}{t} \\ &\leqslant C\varphi(\lambda)\int_0^\infty \frac{t^{-\alpha}\vee t^{\alpha-1}}{1+t}\,dt \leqslant \frac{2C}{\alpha\wedge(1-\alpha)}\varphi(\lambda),\end{aligned}$$

and similarly,

$$\varphi_K(\lambda) \geqslant \frac{2}{C\big(\alpha\vee(1-\alpha)\big)}\varphi(\lambda).$$

Now we consider function φ_J. Observe that

$$\begin{aligned}&\int_0^\infty \frac{1}{1+\lambda t}\varphi(1/t)^{-1}\,\frac{dt}{t} = \int_0^\infty \frac{1}{1+t}\varphi(\lambda/t)^{-1}\,\frac{dt}{t} \\ &\leqslant C\varphi(\lambda)^{-1}\int_0^\infty \frac{t^{-\alpha}\vee t^{\alpha-1}}{1+t}\,dt \leqslant \frac{2C}{\alpha\wedge(1-\alpha)}\varphi(\lambda)^{-1}.\end{aligned}$$

This gives that

$$\varphi_J(\lambda) \geqslant \frac{\alpha\wedge(1-\alpha)}{2C}\varphi(\lambda).$$

Similarly, we can obtain

$$\varphi_J(\lambda) \leqslant \frac{C\big(\alpha\wedge(1-\alpha)\big)}{2}\varphi(\lambda).$$

Therefore, $\varphi_K \approx \varphi_J \approx \varphi$. Consequently, φ_K and φ_J are quasi-power functions, and

$$\overline{\mathscr{H}}_{\varphi_K} = \overline{\mathscr{H}}_{\varphi_J} = \overline{\mathscr{H}}_{\varphi}.$$

with equivalent norms. $\square$

Let φ be a quasi-power function. We define Hilbert spaces $K_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)$ and $J_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)$ as below: $K_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)$ consists of all $x \in \Sigma\overline{\mathscr{H}}$ for which

$$\|x\|_{K_{2;\varphi,2}} = \left(\int_0^\infty \varphi(1/t)K_2(t,x)^2\,\frac{dt}{t}\right)^{1/2} < \infty;$$

and $J_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)$ consists of all $x \in \Sigma\overline{\mathscr{H}}$ such that there exists a strongly measurable function $u\colon \mathbf{R}^+ \to \Delta\overline{\mathscr{H}}$ for which $x = \int_0^\infty u(t)\,dt/t$ in $\Sigma\overline{\mathscr{H}}$ and

$$\int_0^\infty \varphi(1/t)J_2\big(t,u(t)\big)^2\,\frac{dt}{t} < \infty$$

with the norm

$$\|x\|_{J_{2;\varphi,2}} = \inf_u \left\{ \left(\int_0^\infty \varphi(1/t) J_2(t,u(t))^2 \frac{dt}{t} \right)^{1/2} \right\}.$$

Observe that both $K_{2;\varphi,2}(\overline{\mathscr{H}})$ and $J_{2;\varphi,2}(\overline{\mathscr{H}})$ are regular quadratic interpolation spaces for $\overline{\mathscr{H}}$. It is known by [4, Ex. 5.1] that, for the power function $\theta(t) = t^\theta$, $0 < \theta < 1$,

$$K_{2,\theta,2}(\overline{\mathscr{H}}) = J_{2,\theta,2}(\overline{\mathscr{H}}) = \overline{\mathscr{H}}_\theta \tag{3.2}$$

with the propotional norms

$$\|x\|_{K_{2,\theta,2}} = \|x\|_{J_{2,\theta,2}} = \sqrt{\frac{\sin \pi\theta}{\pi}} \|x\|_\theta. \tag{3.3}$$

We can show among other things that (3.2) is even valid for quasi-power functions with isomorphic norms.

PROPOSITION 3.1. *Let φ be a quasi-power function satisfying (3.1).*
(i) $K_{2;\varphi,2}(\overline{\mathscr{H}}) = J_{2;\varphi,2}(\overline{\mathscr{H}}) = \overline{\mathscr{H}}_\varphi$ *with equivalent norms.*
(ii) *If we define $\psi(t) = t^{-\alpha/(1-2\alpha)} \varphi_K(t^{1/(1-2\alpha)})$, then ψ is of quasi-power, for which*

$$\overline{\mathscr{H}}_\varphi = (\overline{\mathscr{H}}_\alpha, \overline{\mathscr{H}}_{1-\alpha})_\psi$$

with equivalent norms.

Proof. (i) According to Lemma 3.1, it is enough to show that

$$K_{2;\varphi,2}(\overline{\mathscr{H}}) = \overline{\mathscr{H}}_{\varphi_K} \quad \text{and} \quad J_{2;\varphi,2}(\overline{\mathscr{H}}) = \overline{\mathscr{H}}_{\varphi_J}. \tag{3.4}$$

Let $x \in \Delta\overline{\mathscr{H}}$. First we have

$$\begin{aligned} \int_0^\infty \varphi(1/t) K_2(t,x)^2 \frac{dt}{t} &= \int_0^\infty \varphi(1/t) \left(\int_{[0,\infty]} \frac{\lambda t}{1+\lambda t} d(E_\lambda x, x)_0 \right) \frac{dt}{t} \\ &= \int_{[0,\infty]} \varphi_K(\lambda)\, d(E_\lambda x, x)_0 = (\varphi_K(A)x, x)_0 \end{aligned}$$

by (1.1). This implies that $\|x\|_{K_{2;\varphi,2}} = \|x\|_{\varphi_K}$. Next, let

$$d\nu(t) = (1+t)^{-1} \varphi(1/t)^{-1}\, dt/t.$$

Then ν is a positive radon measure over $\mathbf{R}^+$, and

$$\varphi_J(\lambda)^{-1} = \int_0^\infty \frac{1+t}{1+\lambda t} d\nu(t).$$

If $x = \int_0^\infty u(t)\, dt/t$ is the canonical decomposition, and if we set

$$w(t) = (1+t)\varphi(1/t)u(t),$$

then $x = \int_0^\infty w(t)\,d\nu(t)$ and

$$\int_0^\infty \varphi(1/t) J_2\big(t,u(t)\big)^2 \frac{dt}{t} = \int_0^\infty \frac{1}{1+t} J_2\big(t,w(t)\big)^2 \,d\nu(t).$$

Furthermore, by a version of the Foias-Lions theorem [4, Th. FL], we have

$$\|x\|_{J_{2;\varphi,2}} = \|x\|_{\varphi_J}.$$

Therefore, the identities in (3.4) hold with equal norms. Making appeal to the regularity of $\overline{\mathscr{H}}$ concludes the proof of part (i).

(ii) By applying Proposition 2.2 on Pick functions φ_K, $\varphi_0(t) = t^\alpha$, and $\varphi_1(t) = t^{1-\alpha}$, we obtain $\varphi_K = \psi\big(\varphi_0,\varphi_1\big)$, and hence $\overline{\mathscr{H}}_{\varphi_K} = \big(\overline{\mathscr{H}}_\alpha, \overline{\mathscr{H}}_{1-\alpha}\big)_\psi$ with equal norms. Combining this with Lemma 3.1, we have

$$\overline{\mathscr{H}}_\varphi = \big(\overline{\mathscr{H}}_\alpha, \overline{\mathscr{H}}_{1-\alpha}\big)_\psi$$

with equivalent norms. $\square$

EXAMPLE. Given a regular Hilbert couple $\overline{\mathscr{H}}$, let $\mathscr{P}\big(\mathbf{S},\Delta\overline{\mathscr{H}}\big)$ be the set of all polynomials on the strip $\mathbf{S} = \big\{z \in \mathbf{C} \,\big|\, 0 \leqslant \operatorname{Re} z \leqslant 1\big\}$ with coefficients in $\Delta\overline{\mathscr{H}}$. We denote by $\mathscr{H}^2_\theta\big(\mathbf{S},\overline{\mathscr{H}}\big)$ the Hilbert space completion of $\mathscr{P}\big(\mathbf{S},\Delta\overline{\mathscr{H}}\big)$ with the norm

$$\|f\|_{\mathscr{H}^2} = \Big(\int_{-\infty}^\infty \big\|f(it)\big\|_0^2 P_0(\theta,t)\,dt + \int_{-\infty}^\infty \|f(1+it)\|_1^2 P_1(\theta,t)\,dt\Big)^{1/2},$$

where

$$P_j(s+i\tau,t) = \frac{\exp(-\pi(t-\tau))\sin\pi s}{\sin^2\pi s + \big(\cos\pi s - \exp(ij\pi - \pi(t-\tau))\big)^2} \qquad (j=0,1)$$

are the Poisson kernels for the strip $\mathbf{S}$. For $0<\theta<1$ and $n = 0,1,2,\ldots$, we may define the complex interpolation space $C_{\theta(n)}\big(\overline{\mathscr{H}}\big)$ with the n-th derivative at θ by

$$C_{\theta(n)}\big(\overline{\mathscr{H}}\big) = \left\{ x \in \Sigma\overline{X} \,\middle|\, x = \frac{1}{n!} f^{(n)}(\theta),\ f \in \mathscr{H}^2\big(\mathbf{S},\overline{X}\big) \right\}$$

with the norm $\|x\|_{C_{\theta(n)}} = \inf\Big\{ \|f\|_{\mathscr{H}^2} \,\Big|\, x = \frac{1}{n!} f^{(n)}(\theta) \Big\}$. We simply write

$$C_\theta\big(\overline{\mathscr{H}}\big) = C_{\theta(0)}\big(\overline{\mathscr{H}}\big),$$

which is equivalent to the classical complex interpolation space as given in [7, Sec. 4.1]. It is known that

$$C_\theta\big(\overline{\mathscr{H}}\big) = \overline{\mathscr{H}}_\theta \tag{3.5}$$

isomorphically. Let now

$$\varphi(t) = t^\theta \left(1 + \frac{\theta(1-\theta)}{n} |\log t|\right)^{-n}$$

for $t>0$. Then φ is a quasi-power function. By combining the equivalence in [11, Cor. 2.1 & Ex. 4.5], and the reiteration in [11, Rmk. 5.8 (iii)] and [12, Th. 5.4 (i)], with Proposition 3.1 (i) and (3.5), we obtain

$$K_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)=J_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)=\overline{\mathscr{H}}_{\varphi}=C_{\theta(n)}\big(\overline{\mathscr{H}}\big)$$

isomorphically.

4. On some operator inequalities

Because of the equivalence given in Proposition 3.1 (i), we may study the commutator estimates arising from the real interpolation methods for Hilbert spaces. Let $c>1$ be a constant. For $x\in\Sigma\overline{\mathscr{H}}$, the decomposition $x=x_0(t)+x_1(t),\ t>0$, is (c-) almost optimal if

$$K_2(t,x)\leqslant\Big(\big\|x_0(t)\big\|_0^2+t\big\|x_1(t)\big\|_1^2\Big)^{1/2}\leqslant c\,K_2(t,x).$$

An almost optimal projection is an operator $D(t)\colon\Sigma\overline{\mathscr{H}}\longrightarrow\mathscr{H}_0$ defined by

$$D(t)x=D\big(t,\overline{\mathscr{H}}\big)x=x_0(t)$$

for some almost optimal decomposition. The corresponding quasi-logarithmic operator $\Omega_{\overline{\mathscr{H}}}$ is defined by

$$\Omega_{\overline{\mathscr{H}}}(x)=\int_0^\infty\big(D(t)-I\cdot\chi_{(1,\infty)}(t)\big)x\,\frac{dt}{t}\tag{4.1}$$

for $x\in\Sigma\overline{\mathscr{H}}$. We refer to [15] for further details. However, if $x\in\mathscr{D}om(A)$, the domain of A, then x has the optimal decomposition which is given by

$$x_0(t)=tA\big(I+tA\big)^{-1}(x)\quad\text{and}\quad x_1(t)=\big(I+tA\big)^{-1}(x)\tag{4.2}$$

for $t>0$. That is, $x=x_0(t)+x_1(t)$ and $K_2(t,x)^2=\big\|x_0(t)\big\|_0^2+t\big\|x_1(t)\big\|_1^2$ for $t>0$ [3].

LEMMA 4.1. $\Omega=\log A$.

Proof. Let F and G be operator functions given by

$$F(t)=\log\big(I+tA\big)\quad\text{and}\quad G(t)=\big(\log t\big)I-\log\big(I+tA\big)=\log\big(t(I+tA)^{-1}\big),$$

respectively, for $t>0$. It is easy to see that

$$F'(t)=D(t)/t\quad\text{and}\quad G'(t)=\big(I-D(t)\big)/t.$$

By integration, we obtain

$$\begin{aligned}\Omega&=\big(F(1)-F(0)\big)-\big(G(\infty)-G(1)\big)\\&=\log\big(I+A\big)-\log\big(A^{-1}\big)+\log\big((I+A)^{-1}\big)=\log A.\quad\square\end{aligned}$$

PROPOSITION 4.1. *If φ is a quasi-power Pick function, then*

$$\left\|T\Omega_{\overline{\mathscr{H}}}-\Omega_{\overline{\mathscr{H}}}T\right\|_{K_{2,\varphi,2},J_{2,\varphi,2}}\leqslant 2\bar{\varphi}\left(\|T\|_0^2,\|T\|_1^2\right)^{1/2}$$

for all $T\in\mathscr{B}(\overline{\mathscr{H}})$. More generally, if φ is a quasi-power function, then there exists a constant γ depending on φ such that

$$\left\|(\log A)T-T(\log A)\right\|_{\varphi}\leqslant\gamma\,\bar{\varphi}\left(\|T\|_0^2,\|T\|_1^2\right)^{1/2}$$

for all $T\in\mathscr{B}(\overline{\mathscr{H}})$.

Proof. For $x\in\mathscr{D}om(A)$ and for $T\in\mathscr{B}(\overline{\mathscr{H}})$, let $x=x_0(t)+x_1(t)$ and $Tx=(Tx)_0(t)+(Tx)_1(t)$, $t>0$, be the corresponding optimal decompositions given in (4.2). Then, by (4.1),

$$\Omega_{\mathscr{H}}(x)=\int_0^1 x_0(x)\frac{dt}{t}-\int_1^\infty x_1(t)\frac{dt}{t},$$

and

$$\left(T\Omega_{\overline{\mathscr{H}}}-\Omega_{\overline{\mathscr{H}}}T\right)(x)=\int_0^1\left(Tx_0(t)-(Tx)_0(t)\right)\frac{dt}{t}-\int_1^\infty\left(Tx_1(t)-(Tx)_1(t)\right)\frac{dt}{t}.$$

For $t>0$, let $y_j(t)=Tx_j(t)-(Tx)_j(t)$ $(j=0,1)$, and let $M=\|T\|_1/\|T\|_0$. Then $y_0(t)+y_1(t)=Tx-Tx=0$ and hence

$$\left(T\Omega_{\overline{\mathscr{H}}}-\Omega_{\overline{\mathscr{H}}}T\right)(x)=\int_0^\infty y_0(t)\frac{dt}{t}=\int_0^\infty y_0\left(tM^2\right)\frac{dt}{t}.$$

This implies that

$$\begin{aligned}
\left\|\left(T\Omega_{\overline{\mathscr{H}}}-\Omega_{\overline{\mathscr{H}}}T\right)(x)\right\|_{J_{2,\varphi,2}}^2&\leqslant\int_0^\infty\varphi(1/t)J_2\left(t,y_0\left(tM^2\right)\right)\frac{dt}{t}\\
&=\int_0^\infty\left(\varphi(1/t)\left\|y_0\left(tM^2\right)\right\|_0^2+t\left\|y_1\left(tM^2\right)\right\|_1^2\right)\frac{dt}{t}\\
&\leqslant 4\|T\|_0^2\int_0^\infty\varphi(1/t)\left(\left\|x_0\left(tM^2\right)\right\|_0^2+tM^2\left\|x_1\left(tM^2\right)\right\|_1^2\right)\frac{dt}{t}\\
&=4\|T\|_0^2\int_{[0,\infty)}\varphi\left(M^2/t\right)K_2(t,x)^2\frac{dt}{t}\\
&\leqslant 4\bar{\varphi}\left(\|T\|_0^2,\|T\|_1^2\right)\|x\|_{K_{2,\varphi,2}}^2.
\end{aligned}$$

This, combined with the regularity of $\overline{\mathscr{H}}$, implies

$$\left\|T\Omega_{\overline{\mathscr{H}}}-\Omega_{\overline{\mathscr{H}}}T\right\|_{K_{2,\varphi,2},J_{2,\varphi,2}}\leqslant 2\bar{\varphi}\left(\|T\|_0^2,\|T\|_1^2\right)^{1/2}.$$

More generally, assume that φ is a quasi-power function. As a consequence of the above estimate, Lemma 4.1, and the equivalences given by Lemma 3.1 and [5, Lemma 1], we obtain

$$\left\|(\log A)T - T(\log A)\right\|_{\varphi} \leqslant \gamma\,\bar{\varphi}\left(\|T\|_0^2, \|T\|_1^2\right)^{1/2}$$

for a constant γ depending on φ. $\square$

REMARK. In [13], we studied the similar estimates for the power functions $\theta(t) = t^{\theta}$, $0 < \theta < 1$, by using the complex interpolation and the corresponding derivation operators. According to [13, Prop. 1.4], if we assume that $A \in \mathscr{B}(\mathscr{H}_0)$ is invertible, then

$$\sqrt{\frac{\sin \pi\theta}{\pi}}\left\|(\log A)T - T(\log A)\right\|_{\theta} \tag{4.3}$$

$$\leqslant 2\|T\|_0^{1-\theta}\|T\|_1^{\theta}\left(\sqrt{1+\left(\frac{2\sin\pi\theta}{\pi}\log\frac{\|T\|_1}{\|T\|_0}\right)^2} + \frac{2\sin\pi\theta}{\pi}\log\frac{\|T\|_1}{\|T\|_0}\right)^{1/2}$$

for all $T \in \mathscr{B}(\overline{\mathscr{H}})$. We apply now Proposition 4.1 and (3.3) on these functions without assumming the boundedness and invertibility for A. Therefore, the estimate given in (4.3) can be improved as follows

$$\sqrt{\frac{\sin \pi\theta}{\pi}}\left\|(\log A)T - T(\log A)\right\|_{\theta} \leqslant 2\|T\|_0^{1-\theta}\|T\|_1^{\theta}$$

for all $T \in \mathscr{B}(\overline{\mathscr{H}})$.

Let φ be a Pick function, and let $T \in \mathscr{B}(\overline{\mathscr{H}})$. If T is invertible on $\overline{\mathscr{H}}_{\varphi}$, then we write T_{φ}^{-1} as the inverse of T on $\overline{\mathscr{H}}_{\varphi}$.

PROPOSITION 4.2. *Assume that $T \in \mathscr{B}(\overline{\mathscr{H}})$. If T is invertible on both $\mathscr{H}_0$ and $\mathscr{H}_1$, then T is invertible on $\overline{\mathscr{H}}_{\varphi}$ for all Pick functions φ, and all inverses T_{φ}^{-1} agree on $\Delta\overline{\mathscr{H}}$. In addition,*

$$\|T^{-1}\|_{\varphi} \leqslant \bar{\varphi}\left(\|T^{-1}\|_0^2, \|T_{\varphi}^{-1}\|_1^2\right)^{1/2}.$$

Proof. Observe first that, by the assumption, T is a bounded linear bijection of $\Sigma\overline{\mathscr{H}}$ onto itself. In fact, for any $y = y_0 + y_1 \in \Sigma\overline{\mathscr{H}}$, we can find $x_j \in \mathscr{H}_j$ with $Tx_j = y_j$ and $\|x_j\|_j \leqslant C\|y_j\|_j$ $(j = 0,1)$ for some positive constant C. Let $x = x_0 + x_1 \in \Sigma\overline{\mathscr{H}}$. Then $y = Tx$ and $\|x\|_{\Sigma} \leqslant C\|y\|_{\Sigma}$. It implies that T is an isomorphism on $\Sigma\overline{\mathscr{H}}$ by the open mapping theorem, and hence $T^{-1} \in \mathscr{B}(\overline{\mathscr{H}})$. Consequently, T is invertible on $\overline{\mathscr{H}}_{\varphi}$ for all Pick functions φ, and all inverses T_{φ}^{-1} agree on $\Delta\overline{\mathscr{H}}$. The estimate

$$\|T_{\varphi}^{-1}\|_{\varphi} \leqslant \bar{\varphi}\left(\|T^{-1}\|_0^2, \|T^{-1}\|_1^2\right)^{1/2}$$

follows from Proposition 2.1. $\square$

REMARKS. For a Hilbert space $\mathscr{H}$ and for $T \in \mathscr{B}(\mathscr{H})$, let $\sigma(T,\mathscr{H})$ be the spectrum of T on $\mathscr{H}$, and let $r(T,\mathscr{H})$ be the spectral radius of T on $\mathscr{H}$. Observe that

$$r(T,\mathscr{H}) = \lim_{n\to 0}\left\|T^n\right\|_{\mathscr{H},\mathscr{H}}^{1/n}. \tag{4.4}$$

(i) By Proposition 4.2, we have

$$\sigma\big(T,\Delta\overline{\mathscr{H}}\big) \subseteq \sigma\big(T,\mathscr{H}_0\big) \cup \sigma\big(T,\mathscr{H}_1\big). \tag{4.5}$$

According to [1, Th. 2.3 (i)], the union of any two of the following three sets

$$\sigma\big(T,\mathscr{H}_0\big) \cup \sigma\big(T,\mathscr{H}_1\big), \quad \sigma\big(T,\Delta\overline{\mathscr{H}}\big), \quad \sigma\big(T,\Sigma\overline{\mathscr{H}}\big)$$

contains the third. Thus,

$$\sigma\big(T,\Sigma\overline{\mathscr{H}}\big) \subseteq \sigma\big(T,\mathscr{H}_0\big) \cup \sigma\big(T,\mathscr{H}_1\big).$$

Furthermore, if φ is a Pick function, then by (4.5) and Proposition 4.2 again

$$\sigma\big(T,\overline{\mathscr{H}}_\varphi\big) \subseteq \sigma\big(T,\mathscr{H}_0\big) \cup \sigma\big(T,\mathscr{H}_1\big).$$

By combining (4.4) and (2.1), we obtain

$$r\big(T,\overline{\mathscr{H}}_\varphi\big) \leqslant \bar{\varphi}\Big(r\big(T,\mathscr{H}_0\big)^2, r\big(T,\mathscr{H}_1\big)^2\Big)^{1/2}.$$

(ii) For a compact subset $\mathbf{K}$ of $\mathbf{C}$, the capacity of $\mathbf{K}$ is defined by

$$\operatorname{Cap}\mathbf{K} = \inf_p \max_{z\in\mathbf{K}} |p(z)|^{1/\deg p},$$

where the infimum is taken over all polynomials p with the leading coefficient equal to 1. Assume that φ is a quasi-power function satisfying (3.1). By Proposition 3.1 (i) and (4.5), we have

$$\sigma\big(T,\overline{\mathscr{H}}_\varphi\big) \subseteq \sigma\big(T,\overline{\mathscr{H}}_\alpha\big) \cup \sigma\big(T,\overline{\mathscr{H}}_{1-\alpha}\big).$$

This, together with [2, Cor. 7], gives that

$$\begin{aligned}&\operatorname{Cap}\sigma\big(T,\overline{\mathscr{H}}_\varphi\big)\\ &\leqslant \Big(\operatorname{Cap}\sigma\big(T,\mathscr{H}_0\big)^{1-\alpha}\operatorname{Cap}\sigma\big(T,\mathscr{H}_1\big)^{\alpha}\Big) \vee \Big(\operatorname{Cap}\sigma\big(T,\mathscr{H}_0\big)^{\alpha}\operatorname{Cap}\sigma\big(T,\mathscr{H}_1\big)^{1-\alpha}\Big).\end{aligned}$$

We conclude this section by an estimate of the measure of noncompactness for bounded linear operators under quadratic interpolation. For Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, let $U_{\mathscr{H}}$ and $U_{\mathscr{K}}$ be the open unit balls of $\mathscr{H}$ and $\mathscr{K}$, and let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$. We define the (ball) measure of noncompactness of operator T by

$$\begin{aligned}\chi\big(T\colon \mathscr{H} \to \mathscr{K}\big) = \inf\Big\{ \eta > 0 \,\Big|\, & T(U_{\mathscr{H}}) \subseteq \cup_{i=1}^k \{y_i + \eta U_{\mathscr{K}}\} \text{ for}\\ & \text{some } y_i \in \mathscr{K} \text{ with } 1 \leqslant i \leqslant k < \infty\Big\}.\end{aligned}$$

Observe that $\chi\big(T\colon \mathscr{H} \to \mathscr{K}\big) = 0$ if and only if T is compact. Moreover, by [6, Th. 14.3.1],

$$\chi\big(T\colon \mathscr{H} \to \mathscr{K}\big) = \inf\Big\{ \|T+S\|_{\mathscr{H},\mathscr{K}} \,\Big|\, S \text{ is a compact operator from } \mathscr{H} \text{ to } \mathscr{K} \Big\},$$

the essential norm of T. For $T \in \mathscr{B}\big(\overline{\mathscr{H}},\overline{\mathscr{K}}\big)$, let $\chi_j(T) = \chi\big(T\colon \mathscr{H}_j \to \mathscr{K}_j\big)$ $(j = 0,1)$, and let $\chi_\varphi(T) = \chi\big(T\colon \mathscr{H}_\varphi \to \mathscr{K}_\varphi\big)$. If φ a quasi-power function, we may apply [8, Cor. 5.2] on the space $K_{2;\varphi,2}\big(\overline{\mathscr{H}}\big)$, which is equivalent to $\overline{\mathscr{H}}_\varphi$ by Proposition 3.1 (i), and obtain the following result.

PROPOSITION 4.3. *Let φ be a quasi-power function, and let $T \in \mathscr{B}\big(\overline{\mathscr{H}},\overline{\mathscr{K}}\big)$. Then*

$$\chi_\varphi(T) \leqslant c\,\bar{\varphi}\big(\chi_0(T)^2,\chi_1(T)^2\big)^{1/2}.$$

Consequently, if $T\colon \mathscr{H}_0 \to \mathscr{K}_0$ or $T\colon \mathscr{H}_1 \to \mathscr{K}_1$ is compact, then $T\colon \overline{\mathscr{H}}_\varphi \to \overline{\mathscr{K}}_\varphi$ is also compact.

REMARK. For a Hilbert space $\mathscr{H}$, and for $T \in \mathscr{B}\big(\mathscr{H}\big)$, let $r_e(T,\mathscr{H})$ be the essential spectral radius of T on $\mathscr{H}$. Recall that

$$r_e(T,\mathscr{H}) = \lim_{n\to\infty} \chi\big(T^n\colon \mathscr{H} \to \mathscr{H}\big)^{1/n}.$$

Combining this with Proposition 4.3, we obtain

$$r_e\big(T,\overline{\mathscr{H}}_\varphi\big) \leqslant \bar{\varphi}\Big(r_e\big(T,\mathscr{H}_0\big)^2, r_e\big(T,\mathscr{H}_1\big)^2\Big)^{1/2}$$

for any quasi-power function φ.

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