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OPTIMAL CONVEX COMBINATION BOUNDS OF SEIFFERT AND GEOMETRIC MEANS FOR THE ARITHMETIC MEAN

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Abstract. We find the greatest value α and the least value β such that the double inequality $\alpha T(a,b)+(1-\alpha)G(a,b) < A(a,b) < \beta T(a,b)+(1-\beta)G(a,b)$ holds for all $a,b>0$ with $a \neq b$. Here $T(a,b)$, $G(a,b)$, and $A(a,b)$ denote the Seiffert, geometric, and arithmetic means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a \neq b$ the Seiffert mean $T(a,b)$ was introduced by Seiffert [17] as follows:

$$T(a,b)=\frac{a-b}{2\arctan(\frac{a-b}{a+b})}. \tag{1.1}$$

Recently, the inequalities and monotonicity properties for the Seiffert mean $T(a,b)$ have attracted the attention of some researchers [7, 8, 10, 18]. We cite [1–6, 9, 11–16] as comprehensive references for inequalities in general.

Let $A(a,b)=(a+b)/2$, and $G(a,b)=\sqrt{ab}$ be the arithmetic, and geometric means of two positive real numbers a and b, respectively. Then it is well-known and elementary that $G(a,b)<A(a,b)$ for all $a,b>0$ with $a \neq b$.

Seiffert [17] proved that

$$T(a,b)>A(a,b)$$

for all $a,b>0$ with $a \neq b$.

Hästö [8] proved that $\frac{T(1,x)}{A_p(1,x)}$ is increasing in $(0,\infty)$ if $p \leqslant 1$, where $A_p(a,b)=(\frac{a^p+b^p}{2})^{\frac{1}{p}}$ $(p \neq 0)$ and $A_0(a,b)=\sqrt{ab}$ is the p-th power mean of two positive numbers a and b.

Chu, Wang and Qiu [7] found the greatest value $p=\log 3/\log(\pi/2) \cong 2.4328$ and the least value $q=5/2$ such that

$$H_p(a,b)<T(a,b)<H_q(a,b)$$

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holds for all $a,b>0$ with $a\neq b$. Here $H_p(a,b)=\left(\frac{a^p+(ab)^{\frac{p}{2}}+b^p}{3}\right)^{\frac{1}{p}}$ $(p\neq 0)$ and $H_0(a,b)=\sqrt{ab}$ is the p-th power-type Heron mean of two positive numbers a and b, and $H_p(a,b)$ is strictly increasing with respect to $p\in\mathbb{R}$ for fixed a and b with $a\neq b$.

In [18], the authors presented that

$$T(a,b)<L_{1/3}(a,b)$$

for all $a,b>0$ with $a\neq b$. Here, $L_p(a,b)=(a^{p+1}+b^{p+1})/(a^p+b^p)$ denotes the p-th Lehmer mean of two positive numbers a and b.

In [10], the authors found the greatest values $\alpha_1=\frac{2}{9}$ and $\alpha_2=\frac{1}{\pi}$, and the least values $\beta_1=\frac{1}{\pi}$ and $\beta_2=\frac{5}{12}$ such that inequalities

$$\alpha_1 C(a,b)+(1-\alpha_1)G(a,b)<P(a,b)<\beta_1 C(a,b)+(1-\beta_1)G(a,b)$$

and

$$\alpha_2 C(a,b)+(1-\alpha_2)H(a,b)<P(a,b)<\beta_2 C(a,b)+(1-\beta_2)H(a,b)$$

hold for all $a,b>0$ with $a\neq b$. Here, $C(a,b)=\frac{a^2+b^2}{a+b}$ and $H(a,b)=\frac{2ab}{a+b}$ are the contraharmonic and harmonic means of two positive numbers a and b, respectively.

The main purpose of this paper is to answer the question: what are the greatest value $\alpha\in(0,1)$ and the least value $\beta\in(0,1)$ such that the double inequality

$$\alpha T(a,b)+(1-\alpha)G(a,b)<A(a,b)<\beta T(a,b)+(1-\beta)G(a,b)$$

holds for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to establish our result we need a lemma, which we present in this section.

LEMMA 2.1. *Let $g(t)=(1-\alpha)t^6+\alpha(2-\alpha)t^5+(1-\alpha)(1-4\alpha)t^4-2(3\alpha^2-4\alpha+2)t^3+(1-\alpha)(1-4\alpha)t^2+\alpha(2-\alpha)t+(1-\alpha)$. If $\alpha=4/\pi=1.27\ldots$, then there exists $\lambda\in(1,\infty)$ such that $g(t)>0$ for $t\in(1,\lambda)$ and $g(t)<0$ for $t\in(\lambda,\infty)$.*

Proof. Simple computations lead to

$$g(1)=0, \tag{2.1}$$

$$\lim_{t\to+\infty} g(t)=-\infty, \tag{2.2}$$

$$\begin{aligned} g'(t)=\ &6(1-\alpha)t^5+5\alpha(2-\alpha)t^4+4(1-\alpha)(1-4\alpha)t^3\\ &-6(3\alpha^2-4\alpha+2)t^2+2(1-\alpha)(1-4\alpha)t+\alpha(2-\alpha), \end{aligned}$$

$$g'(1)=0, \tag{2.3}$$

$$\lim_{t\to+\infty} g'(t)=-\infty, \tag{2.4}$$

$$g''(t) = 30(1-\alpha)t^4 + 20\alpha(2-\alpha)t^3 + 12(1-\alpha)(1-4\alpha)t^2 \\ -12(3\alpha^2-4\alpha+2)t + 2(1-\alpha)(1-4\alpha),$$

$$g''(1) = 20 - 12\alpha = \tfrac{4}{\pi}(5\pi - 12) > 0, \tag{2.5}$$

$$\lim_{t\to+\infty} g''(t) = -\infty, \tag{2.6}$$

$$g'''(t) = 120(1-\alpha)t^3 + 60\alpha(2-\alpha)t^2 + 24(1-\alpha)(1-4\alpha)t - 12(3\alpha^2-4\alpha+2),$$

$$g'''(1) = 120 - 72\alpha = \tfrac{24}{\pi}(5\pi - 12) > 0, \tag{2.7}$$

$$\lim_{t\to+\infty} g'''(t) = -\infty, \tag{2.8}$$

$$g^{(4)}(t) = 360(1-\alpha)t^2 + 120\alpha(2-\alpha)t + 24(1-\alpha)(1-4\alpha),$$

$$g^{(4)}(1) = 24(16 - 10\alpha - \alpha^2) = \tfrac{192}{\pi^2}(2\pi^2 - 5\pi - 2) > 0, \tag{2.9}$$

$$\lim_{t\to+\infty} g^{(4)}(t) = -\infty, \tag{2.10}$$

$$g^{(5)}(t) = 720(1-\alpha)t + 120\alpha(2-\alpha) \leqslant 720(1-\alpha) + 120\alpha(2-\alpha) \\ = -\tfrac{240}{\pi^2}(8 + 8\pi - 3\pi^2) < 0 \tag{2.11}$$

for $t \geqslant 1$.

Inequality (2.11) implies that $g^{(4)}(t)$ is strictly decreasing in $[1,\infty)$, then from (2.9) and (2.10) we know that there exists $\lambda_1 > 1$ such that $g^{(4)}(t) > 0$ for $t \in [1,\lambda_1)$ and $g^{(4)}(t) < 0$ for $t \in (\lambda_1,\infty)$. Therefore, $g'''(t)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,\infty)$.

From (2.7) and (2.8) together with the piecewise monotonicity of $g'''(t)$ we clearly see that there exists $\lambda_2 > 1$ such that $g''(t)$ is strictly increasing in $[1,\lambda_2]$ and strictly decreasing in $[\lambda_2,\infty)$.

Inequality(2.5) and equation (2.6) together with the piecewise monotonicity of $g''(t)$ lead to that there exists $\lambda_3 > 1$ such that $g'(t)$ is strictly increasing in $[1,\lambda_3]$ and strictly decreasing in $[\lambda_3,\infty)$.

From (2.3) and (2.4) together with the piecewise monotonicity of $g'(t)$ we know that there exists $\lambda_4 > 1$ such that $g(t)$ is strictly increasing in $[1,\lambda_4]$ and strictly decreasing in $[\lambda_4,\infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the piecewise monotonicity of $g(t)$. □

3. Main results

THEOREM 3.1. *Inequality*

$$\frac{3}{5}T(a,b) + \frac{2}{5}G(a,b) < A(a,b) < \frac{\pi}{4}T(a,b) + \left(1 - \frac{\pi}{4}\right)G(a,b) \tag{3.1}$$

holds for all $a,b > 0$ *with* $a \neq b$, *and* $\alpha = \frac{3}{5}$ *and* $\beta = \frac{\pi}{4}$ *are the best possible parameters such that inequality* $\alpha T(a,b) + (1-\alpha)G(a,b) < A(a,b) < \beta T(a,b) + (1-\beta)G(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$.

Proof. Without loss of generality, we assume $a>b$. We first prove that $\frac{3}{5}T(a,b)+\frac{2}{5}G(a,b)<A(a,b)$. Let $t=\sqrt{\frac{a}{b}}>1$, then (1.1) leads to

$$\begin{aligned} &A(a,b)-\left[\frac{3}{5}T(a,b)+\frac{2}{5}G(a,b)\right] \\ &=\frac{b(5t^2-4t+5)}{10\arctan\left(\frac{t^2-1}{t^2+1}\right)}\left[\arctan\left(\frac{t^2-1}{t^2+1}\right)-\frac{3(t^2-1)}{5t^2-4t+5}\right]. \end{aligned} \tag{3.2}$$

Let

$$f(t)=\arctan\left(\frac{t^2-1}{t^2+1}\right)-\frac{3(t^2-1)}{5t^2-4t+5}, \tag{3.3}$$

then simple computations lead to

$$f(1)=0, \tag{3.4}$$

$$f'(t)=\frac{2f_1(t)}{(t^4+1)(5t^2-4t+5)^2}, \tag{3.5}$$

where

$$f_1(t)=6t^6-5t^5-34t^4+66t^3-34t^2-5t+6=(t-1)^4(6t^2+19t+6)>0 \tag{3.6}$$

for $t>1$.

Therefore, $\frac{3}{5}T(a,b)+\frac{2}{5}G(a,b)<A(a,b)$ follows from (3.2)-(3.6).

Next, we prove that $A(a,b)<\frac{\pi}{4}T(a,b)+(1-\frac{\pi}{4})G(a,b)$. Let $\alpha=\frac{4}{\pi}=1.27\ldots$ and $t=\sqrt{\frac{a}{b}}>1$, then (1.1) leads to

$$\begin{aligned} &\frac{\pi}{4}T(a,b)+\left(1-\frac{\pi}{4}\right)G(a,b)-A(a,b) \\ &=\frac{b(\alpha t^2+2(1-\alpha)t+\alpha)}{2\alpha\arctan\left(\frac{t^2-1}{t^2+1}\right)}\left[\frac{t^2-1}{\alpha(t-1)^2+2t}-\arctan\left(\frac{t^2-1}{t^2+1}\right)\right]. \end{aligned} \tag{3.7}$$

Let

$$F(t)=\frac{t^2-1}{\alpha(t-1)^2+2t}-\arctan\left(\frac{t^2-1}{t^2+1}\right), \tag{3.8}$$

the simple computations lead to

$$F(1)=\lim_{t\to+\infty}F(t)=0, \tag{3.9}$$

$$F'(t)=\frac{2g(t)}{(t^4+1)[\alpha(t-1)^2+2t]^2}, \tag{3.10}$$

where $g(t)$ is defined as in Lemma 2.1.

From (3.10) and Lemma 2.1 we clearly see that there exists $\lambda>1$ such that $F(t)$ is strictly increasing in $[1,\lambda]$ and strictly decreasing in $[\lambda,\infty)$.

Equation (3.9) and the piecewise monotonicity of $F(t)$ imply that

$$F(t) > 0 \tag{3.11}$$

for $t > 1$.

Therefore, $A(a,b) < \frac{\pi}{4}T(a,b) + (1-\frac{\pi}{4})G(a,b)$ follows from (3.7) and (3.8) together with (3.11).

Finally, we prove that $\alpha = \frac{3}{5}$ and $\beta = \frac{\pi}{4}$ are the best possible parameters such that inequality $\alpha T(a,b) + (1-\alpha)G(a,b) < A(a,b) < \beta T(a,b) + (1-\beta)G(a,b)$ holds for all $a,b > 0$ and $a \neq b$.

For any $\varepsilon > 0$ and $x > 0$, one has

$$\begin{aligned}
&\lim_{x\to+\infty} \frac{(\frac{\pi}{4}-\varepsilon)T(1,x) + (1+\varepsilon-\frac{\pi}{4})G(1,x)}{A(1,x)} \\
&= \lim_{x\to\infty} \frac{(\frac{\pi}{4}-\varepsilon)\frac{x-1}{2\arctan(\frac{x-1}{x+1})} + (1+\varepsilon-\frac{\pi}{4})\sqrt{x}}{\frac{x+1}{2}} \\
&= 1 - \frac{4}{\pi}\varepsilon < 1,
\end{aligned} \tag{3.12}$$

$$\left(\frac{3}{5}+\varepsilon\right)T(1+x,1) + \left(\frac{2}{5}-\varepsilon\right)G(1+x,1) - A(1+x,1) = \frac{J(x)}{2\arctan\left(\frac{x}{x+2}\right)}, \tag{3.13}$$

where

$$J(x) = \left(\frac{3}{5}+\varepsilon\right)x + 2\left[\left(\frac{2}{5}-\varepsilon\right)(1+x)^{\frac{1}{2}} - \left(1+\frac{x}{2}\right)\right]\arctan\left(\frac{x}{x+2}\right). \tag{3.14}$$

Letting $x \to 0$ and making use of the Taylor expansion we get

$$\begin{aligned}
J(x) = &\left(\frac{3}{5}+\varepsilon\right)x - \left[\left(\frac{3}{5}+\varepsilon\right) + \left(\frac{3}{10}+\frac{\varepsilon}{2}\right)x + \left(\frac{1}{20}-\frac{\varepsilon}{8}\right)x^2\right. \\
&\left.-\left(\frac{1}{40}-\frac{\varepsilon}{16}\right)x^3 + o(x^3)\right]\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)\right) \\
= &\frac{5}{24}\varepsilon x^3 + o(x^3).
\end{aligned} \tag{3.15}$$

Inequality (3.12) and equation (3.13)–(3.15) imply that for any $\varepsilon > 0$ there exist $X = X(\varepsilon) > 1$ and $\delta = \delta(\varepsilon) > 0$, such that $A(1,x) > (\frac{\pi}{4}-\varepsilon)T(1,x) + (1+\varepsilon-\frac{\pi}{4})G(1,x)$ for $x \in (X,\infty)$ and $(\frac{3}{5}+\varepsilon)T(1+x,1) + (\frac{2}{5}-\varepsilon)G(1+x,1) > A(1+x,1)$ for $x \in (0,\delta)$. $\square$

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