a(x)–MONOTONIC FUNCTIONS AND THEIR INEQUALITIES

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Abstract. In this paper a(x)-monotonic functions are defined and some inequalities for them are derived. Related analogous of the Lagrange and the Cauchy mean value theorems are also derived and means of the Cauchy type are generated. Furthermore, it is shown that the monotonicity of the Stolarsky means can be proved using the notion of generalized monotonic functions.

1. Introduction

The following result is well known (it can be found in [2, p. 133–134]):
If the linear differential equation
\[ u'(t) = a(t)u(t), \quad u(0) = c, \] (1.1)
and the linear differential inequality
\[ v'(t) \geq a(t)v(t), \quad v(0) = c, \] (1.2)
are both valid for \(0 \leq t \leq T\), then
\[ v(t) \geq u(t), \quad 0 \leq t \leq T. \] (1.3)

Differential inequality \( y'(x) - a(x)y(x) \geq 0 \) is sometimes used as a definition of generalized increasing functions. In this paper we shall observe an analogous generalization which we will call \( a(x) \)-monotonic functions.

First, let us recall Stolarsky means:
\[
E(x, y; s, p) = \left\{ \frac{s(y^p - x^p)}{p(y^s - x^s)} \right\}^{1/p-s} \\
E(x, y; r, r) = e^{-\frac{1}{r}} \left( \frac{x^r}{y^r} \right)^{1/(x^r - y^r)} \\
E(x, y; r, 0) = E(x, y; 0, r) = \left\{ \frac{y^r - x^r}{r(\ln y - \ln x)} \right\}^{1/r} \\
E(x, y; 0, 0) = \sqrt{xy}.
\]


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where $x$ and $y$ are positive real numbers $x \neq y$, $p$ and $s$ are any real numbers. Stolarsky introduced this means in 1975 (see [13]).

In the following theorem we state monotonicity of Stolarsky means first proved by Stolarsky (see [13]).

**Theorem 1.1.** Let $r \leq s$, $l \leq p$, then the following inequality is valid

$$E(x, y; r, l) \leq E(x, y; s, p)$$

that is, the mean $E(x, y; s, p)$ is monotonic.

Another proof, using a definition of monotonic functions, is given in [5]. In this paper we shall show that Theorem 1.1 can also be proved by using definitions of generalized monotonic functions.

**Remark 1.1.** Necessary and sufficient conditions for (1.4) to be valid are given in [9].

The paper is organised as follows. In Section 2 we define $a(x)$-monotonic functions and give some inequalities for them. In Section 3 we derive analogues of the Lagrange mean value theorem and the Cauchy mean value theorem. In Section 4 we prove the exponential convexity of a function defined as the difference between the left-hand and the right-hand side of the inequality which defines $a(x)$-monotonic functions, give means of Cauchy type and prove Theorem 1.1 using generalized monotonic functions.

**2. Properties of $a(x)$-monotonic functions and their inequalities**

**Definition 2.1.** Let $f$, $a$ be real functions defined on interval $I \subseteq \mathbb{R}$ such that $af$ is integrable. Function $f$ is called $a(x)$-increasing on interval $I$ if for every $x, y \in I$

$$ (y - x)(f(y) - f(x)) \geq (y - x) \int_x^y a(t)f(t)dt $$

holds.

Function $f$ is called $a(x)$-decreasing if the inequality in (2.1) is reversed.

Function $f$ is called $a(x)$-monotonic if it satisfies (2.1) or the reversed inequality.

If two functions are both $a(x)$-increasing, or both $a(x)$-decreasing, we say that they are $a(x)$-monotonic in the same sense.

**Remark 2.1.** Notice that for $a(x) = 0$, $f$ is monotonic.

**Remark 2.2.** If $x \neq y$, (2.1) is equivalent to

$$ \frac{f(y) - f(x)}{y - x} \geq \frac{1}{y - x} \int_x^y a(t)f(t)dt. $$

(2.2)
If \( f \) is \( a(x) \)-increasing, \(-f\) is \( a(x) \)-decreasing. So we will only give properties of \( a(x) \)-increasing functions, because they are the same for \( a(x) \)-decreasing functions. Properties of \( a(x) \)-increasing functions:

1) Let \( f \) and \( g \) be \( a(x) \)-increasing functions. Then \( f + g \) is \( a(x) \)-increasing.

If \( f \) and \( g \) are \( a(x) \)-monotonic functions (without further specifications), we can’t conclude that \( f + g \) is \( a(x) \)-monotonic.

2) If \( f \) is \( a(x) \)-increasing function and \( \lambda \) is non-negative real number, then \( \lambda f \) is \( a(x) \)-increasing function.

In applications we often use \( a(x) \)-monotonicity criteria given in the following theorem.

**Theorem 2.1.** If \( f' \) is a continuous function and \( af \) an integrable function on interval \( I \), \( f \) is \( a(x) \)-monotonic on interval \( I \) if and only if the function \( f'(x) - a(x)f(x) \) is non-negative or non-positive on \( I \). More precisely, \( f \) is \( a(x) \)-increasing function if and only if \( f'(x) - a(x)f(x) \geq 0 \); \( f \) is \( a(x) \)-decreasing function if and only if \( f'(x) - a(x)f(x) \leq 0 \).

**Proof.** Let \( f \) be \( a(x) \)-increasing function, then for \( x, y \in I \) such that \( x \neq y \) we have

\[
\frac{f(y) - f(x)}{y - x} \geq \frac{1}{y - x} \int_x^y a(t)f(t)dt.
\]

Taking a limit when \( y \to x \) we get \( f'(x) \geq a(x)f(x) \).

Conversely, let \( f'(x) \geq a(x)f(x) \). For \( x, y \in I \) such that \( x < y \) we have that

\[
\int_x^y f'(t)dt \geq \int_x^y a(t)f(t)dt.
\]

Since \( f' \) is continuous, we have that for every \([x, y] \subset I\), \( \int_x^y f'(t)dt = f(y) - f(x) \).

Furthermore, since \( x < y \) we can multiply inequality (2.3) by \( y - x \) and get

\[
(y - x)(f(y) - f(x)) \geq (y - x) \int_x^y a(t)f(t)dt.
\]

Hence, \( f \) is \( a(x) \)-increasing function.

Using the same reasoning we get criteria for \( a(x) \)-decreasing functions. So the proof is completed. \( \square \)

**Remark 2.3.** Note that for \( f \) differentiable and \( a(x) \)-monotonic we have:

(i) for \( a(x) = \frac{1}{x} \), \( f(x) \frac{x}{x} \) is monotonic (this case is studied in [11]);

(ii) for \( a(x) = \frac{a}{x} \), where \( a \) is some constant, \( f(x) \frac{x}{x^a} \) is monotonic;

(iii) for \( a(x) = \frac{h(x)}{n(x)} \), \( f \frac{h}{n} \) is monotonic (this case is studied in [6]).
Proof of this remarks follows from \(a(x)\)-monotonicity criteria in Theorem 2.1. For example, for \(a(x) = \frac{1}{x}\) and \(a(x)\)-increasing function \(f\), we have \(f'(x) \geq \frac{f(x)}{x}\), so for \(x > 0\) we have \(\left(f'(x) - \frac{f(x)}{x}\right) = \frac{xf'(x) - f(x)}{x^2} \geq \frac{f(x) - f(x)}{x^2} = 0\), hence \(\frac{f(x)}{x}\) is increasing function for \(x > 0\).

**Theorem 2.2.** A function \(f\) is a \(a(x)\)-increasing if and only if the function \(F\) defined by

\[
F(x) = f(x) - \int_{x_0}^{x} a(t)f(t)\,dt
\]

is increasing.

**Proof.** Suppose that \(y > x\). Then (2.1) is equivalent to

\[
f(y) - f(x) \geq \int_{x}^{y} a(t)f(t)\,dt
\]

i.e.

\[
f(y) - f(x) \geq \int_{x_0}^{y} a(t)f(t)\,dt - \int_{x_0}^{x} a(t)f(t)\,dt
\]

i.e.

\[
f(y) - \int_{x_0}^{y} a(t)f(t)\,dt \geq f(x) - \int_{x_0}^{x} a(t)f(t)\,dt
\]

i.e.

\[
F(y) \geq F(x).
\]

Since we have equivalence in each step, the proof is completed. \(\square\)

We can apply the function \(F\) defined by (2.4) to inequalities for monotonic functions and get inequalities for \(a(x)\)-monotonic functions.

Here we give Steffensen’s inequality for \(a(x)\)-monotonic functions.

**Corollary 2.1.** Suppose that \(f\) is a \(a(x)\)-increasing and \(g\) is integrable on \([b, c]\) with \(0 \leq g \leq 1\) and \(\lambda = \int_{b}^{c} g(x)\,dx\). Then we have

\[
\int_{b}^{b+\lambda} f(x)\,dx - \int_{b}^{b+\lambda} \int_{x_0}^{x} a(t)f(t)\,dt\,dx \leq \int_{b}^{c} f(x)g(x)\,dx - \int_{b}^{c} \left(g(x)\int_{x_0}^{x} a(t)f(t)\,dt\right)\,dx
\]

\[
\leq \int_{c-\lambda}^{c} f(x)\,dx - \int_{c-\lambda}^{c} \int_{x_0}^{x} a(t)f(t)\,dt\,dx.
\]

The inequalities are reversed for \(f\) \(a(x)\)-decreasing.
Proof. Let the function $F$ be defined by (2.4). Since, $F$ is increasing we can apply Steffensen’s inequality, hence

$$\int_b^{b+\lambda} F(x)dx \leq \int_b^c F(x)g(x)dx \leq \int_{c-\lambda}^c F(x)dx.$$ 

By elementary calculation we get (2.5). □

LEMMA 2.1. Let $f$ be a positive, differentiable and $a(x)$-increasing function. Then functions $G, H$ defined by

$$G(x) = f(x) \cdot e^{-\int a(x)dx}, \quad (2.6)$$

$$H(x) = \ln f(x) - \int a(x)dx \quad (2.7)$$

are increasing.

Proof. Function $f$ is $a(x)$-increasing, so from Theorem 2.1 we have $f'(x) - a(x)f(x) \geq 0$. Since $f$ is positive, we get $\frac{f'(x)}{f(x)} - a(x) \geq 0$. From (2.6) we get

$$G'(x) = f'(x) \cdot e^{-\int a(x)dx} - a(x)f(x)e^{-\int a(x)dx} \geq 0.$$ 

Hence, $G$ is increasing. From (2.7) we get

$$H'(x) = \frac{f'(x)}{f(x)} - a(x) \geq 0.$$ 

Hence, $H$ is increasing. □

REMARK 2.4. Using functions $G$ and $H$ defined by (2.6) and (2.7) we can get new $a(x)$-monotonic inequalities.

THEOREM 2.3. Let $u(t) = ce^{\int_0^t a(t)dt}$ for $0 \leq t \leq T$. Let $v$ satisfy (1.2) with $a(x) \geq 0$ for $0 \leq x \leq T$ and let $v(0) = c$. Then $v(x) - u(x)$ is an increasing function for $0 \leq x \leq T$.

Proof. Notice that $u$ is the solution of the differential equation $u'(t) - a(t)u(t) = 0$ with $u(0) = c$, so $u$ satisfies (1.1). Hence (1.3) is valid. Since $v$ satisfies (1.2), from Theorem 2.1 we have that $v$ is $a(x)$-monotonic function. So

$$\int_x^y (y - x)(v(y) - v(x)) \geq (y - x) \int_x^y a(t)v(t)dt. \quad (2.8)$$

For $x, y \in [0, T]$ such that $x < y$ we can divide (2.8) by $y - x$ and then apply (1.3). We get

$$v(y) - v(x) \geq \int_x^y a(t)v(t)dt \geq \int_x^y a(t)u(t)dt = \int_x^y u'(t)dt = u(y) - u(x). \quad (2.9)$$
Hence,
\[ v(y) - u(y) \geq v(x) - u(x). \]

So the proof is completed. \( \square \)

Now we give Steffensen’s inequality for function \( v(x) - u(x) \):

**COROLLARY 2.2.** Let functions \( u \) and \( v \) be such that conditions of Theorem 2.3 are satisfied. Let \( g \) be an integrable function on \([0, T]\) with \( 0 \leq g \leq 1 \) and \( \lambda = \int_0^T g(t)dt \). Then we have

\[
\lambda \int_0^T (v(t) - u(t))dt \leq \int_0^T (v(t) - u(t))g(t)dt \leq \int_0^T (v(t) - u(t))dt. \tag{2.10}
\]

**Proof.** From Theorem 2.3 we have that \( v(x) - u(x) \) is an increasing function, so we can apply Steffensen’s inequality and get (2.10). \( \square \)

3. Mean value theorems

**LEMMA 3.1.** Let \( I \) be an open interval. Let \( a \) be an integrable and \( h \in C^1(I) \) be such that \( h' - ah \) is bounded by integrable functions \( M \) and \( m \), that is, \( m(x) \leq h'(x) - a(x)h(x) \leq M(x) \), for every \( x \in I \). Then functions \( \Phi_1, \Phi_2 \) defined by

\[
\Phi_1(x) = R_1(x) - h(x), \quad \Phi_2(x) = h(x) - R_2(x),
\]

where

\[
R_1(x) = e^{\int a(x)dx} \int M(x)e^{-\int a(x)dx}dx, \tag{3.1}
\]

\[
R_2(x) = e^{\int a(x)dx} \int m(x)e^{-\int a(x)dx}dx, \tag{3.2}
\]

are \( a(x) \)-increasing.

**Proof.** Since \( h'(x) - a(x)h(x) \leq M(x) \) and \( R_1'(x) - a(x)R_1(x) = M(x) \) we have,

\[
\Phi_1'(x) - a(x)\Phi_1(x) = R_1'(x) - a(x)R_1(x) - (h'(x) - a(x)h(x)) \geq 0.
\]

So \( \Phi_1 \) is \( a(x) \)-increasing function. In the same way, since \( h'(x) - a(x)h(x) \geq m(x) \) we have,

\[
\Phi_2'(x) - a(x)\Phi_2(x) = h'(x) - a(x)h(x) - (R_2'(x) - a(x)R_2(x)) \geq 0.
\]

So \( \Phi_2 \) is \( a(x) \)-increasing function. \( \square \)

Now we will state and prove the Lagrange-type mean value theorem. This theorem is a consequence of the Cauchy mean value theorem but we will prove it by using \( a(x) \)-increasing functions from Lemma 3.1.
Therefore, there exists $\eta \in [x, y]$ such that
\[
\frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)h(t)dt = \frac{h'(\eta) - a(\eta)h(\eta)}{g(\eta)} \cdot \frac{1}{y - x} \int_x^y g(t)dt. \tag{3.3}
\]

**Proof.** Since $\frac{h' - ah}{g}$ is continuous on $[x, y]$, there exists
\[
m = \min_{t \in [x, y]} \left( \frac{h'(t) - a(t)h(t)}{g(t)} \right) \quad \text{and} \quad M = \max_{t \in [x, y]} \left( \frac{h'(t) - a(t)h(t)}{g(t)} \right).
\]

Applying (2.2) on functions $\Phi_1$ and $\Phi_2$ from Lemma 3.1, with $M(x) = Mg(x)$, $m(x) = mg(x)$, the following inequalities hold:
\[
\frac{\Phi_1(y) - \Phi_1(x)}{y - x} \geq \frac{1}{y - x} \int_x^y a(t)\Phi_1(t)dt,
\]
\[
\frac{\Phi_2(y) - \Phi_2(x)}{y - x} \geq \frac{1}{y - x} \int_x^y a(t)\Phi_2(t)dt.
\]

It follows,
\[
m \left( \frac{R_3(y) - R_3(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)R_3(t)dt \right)
\]
\[
\leq \frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)h(t)dt
\]
\[
\leq M \left( \frac{R_3(y) - R_3(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)R_3(t)dt \right),
\]

where
\[
R_3(x) = e^{\int a(x)dx} \int g(x) e^{-\int a(x)dx} dx.
\]

Therefore, there exists $\eta \in [x, y]$ such that
\[
\frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)h(t)dt
\]
\[
= \frac{h'(\eta) - a(\eta)h(\eta)}{g(\eta)} \left( \frac{R_3(y) - R_3(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)R_3(t)dt \right).
\]

holds. Since $R'_3(x) - a(x)R_3(x) = g(x)$, we have $a(x)R_3(x) = R'_3(x) - g(x)$. So
\[
\frac{R_3(y) - R_3(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)R_3(t)dt
\]
\[
= \frac{R_3(y) - R_3(x)}{y - x} - \frac{1}{y - x} \int_x^y (R'_3(t) - g(t))dt
\]
\[
= \frac{1}{y - x} \int_x^y g(t)dt.
\]

Hence, there exists $\eta \in [x, y]$ such that (3.3) holds. □
REMARK 3.1. For \( a(x) = 0 \) Theorem 3.1 gives Cauchy mean value theorem, that is,
\[
\frac{h'(\eta)}{g(\eta)} = \frac{h(y) - h(x)}{\int_x^y g(t)dt}.
\]
Additionally, taking \( g \equiv 1 \) we get Lagrange mean value theorem, that is,
\[
h'(\eta) = \frac{h(y) - h(x)}{y - x}.
\]

REMARK 3.2. For \( a(x) = \frac{1}{x} \) Theorem 3.1 gives
\[
\frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y \frac{h(t)}{t} dt = \frac{\eta h'(\eta) - h(\eta)}{\eta g(\eta)} \cdot \frac{1}{y - x} \int_x^y g(t)dt.
\]
Furthermore, taking \( g \equiv 1 \) we get
\[
\frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y \frac{h(t)}{t} dt = h'(\eta) - \frac{h(\eta)}{\eta}.
\]

REMARK 3.3. For \( a(x) = \frac{k'(x)}{k(x)} \) Theorem 3.1 gives
\[
\frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y \frac{k'(t)}{k(t)} h(t) dt = \frac{h'(\eta)k(\eta) - k'(\eta)h(\eta)}{k(\eta)g(\eta)} \cdot \frac{1}{y - x} \int_x^y g(t)dt.
\]
Furthermore, taking \( g = h \) we get
\[
\frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y \frac{k'(t)}{k(t)} h(t) dt = \left( \frac{h'(\eta)}{h(\eta)} - \frac{k'(\eta)}{k(\eta)} \right) \int_x^y h(t)dt.
\]

THEOREM 3.2. Let \( I \) be an interval in \( \mathbb{R} \) and let \( x, y \in I \) be such that \( x \neq y \). Let \( f, h \in C^1(I) \) and let \( a \) be a continuous function such that
\[
h(y) - h(x) - \int_x^y a(t)h(t)dt \neq 0. \tag{3.4}
\]
Then there exists \( \eta \in [x, y] \) such that
\[
\frac{f'(\eta) - a(\eta)f(\eta)}{h'(\eta) - a(\eta)h(\eta)} = \frac{f(y) - f(x) - \int_x^y a(t)f(t)dt}{h(y) - h(x) - \int_x^y a(t)h(t)dt}. \tag{3.5}
\]

Proof. Let
\[
c_1 = \frac{h(y) - h(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)h(t)dt,
\]
\[
c_2 = \frac{f(y) - f(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)f(t)dt.
\]
Now we apply (3.3) to the function $c_1 f - c_2 h$. The following equality holds:

$$c_1 \left[ \frac{f(y) - f(x)}{y - x} - \frac{1}{y-x} \int_x^y a(t)f(t) dt \right] - c_2 \left[ \frac{h(y) - h(x)}{y - x} - \frac{1}{y-x} \int_x^y a(t)h(t) dt \right]$$

$$= c_1 f'(\xi) - c_2 h'(\xi) - \frac{a(\xi)(c_1 f(\xi) - c_2 h(\xi))}{g(\xi)} \cdot \left[ \frac{1}{y-x} \int_x^y g(t) dt \right].$$

(3.6)

It is easy to see that the left-hand side of (3.6) is equal to 0, so the right-hand side should also be equal to 0. From (3.4) we get that the right-hand side in (3.3) is not equal to 0, so the part in square brackets on the right-hand side of (3.6) is not equal to 0. For the right-hand side in (3.6) to be equal to 0 it follows that $c_1 f'(\xi) - c_2 h'(\xi) - a(\xi)(c_1 f(\xi) - c_2 h(\xi)) = 0$. After a short calculation, it is easy to see that (3.5) follows from $c_1 (f'(\xi) - a(\xi)f(\xi)) - c_2 (h'(\xi) - a(\xi)h(\xi)) = 0$, so the proof is completed. □

**REMARK 3.4.** Theorem 3.2 is equivalent to Cauchy mean value theorem.

**4. Exponential convexity and means of Cauchy type**

First we recall some basic facts about convexity, log-convexity and log-convexity in the Jensen sense (see e.g. [4], [10], [12]).

**DEFINITION 4.1.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex in the Jensen sense on an interval $I$ if for each $a, b \in I$

$$f \left( \frac{a+b}{2} \right) \leq \frac{f(a) + f(b)}{2}$$

holds.

We recall that for a continuous function $f$, convexity and convexity in the Jensen sense are equivalent properties.

**DEFINITION 4.2.** A positive function $f : I \rightarrow (0, \infty)$ is said to be logarithmically convex if $\log f$ is convex function on $I$. For such function $f$, we shortly say $f$ is log-convex. A positive function $f : I \rightarrow (0, \infty)$ is log-convex in the Jensen sense if for each $a, b \in I$

$$f^2 \left( \frac{a+b}{2} \right) \leq f(a)f(b)$$

holds, i.e., if $\log f$ is convex in the Jensen sense.

The following result on log-convex functions is given in [12].
**Lemma 4.1.** Let positive function \( f : I \to (0, \infty) \) be log-convex and let \( a_1, a_2, b_1, b_2 \in I \) be such that \( a_1 \leq b_1, \ a_2 \leq b_2 \) and \( a_1 \neq a_2, \ b_1 \neq b_2 \). Then the following inequality is valid

\[
\left[ \frac{f(a_2)}{f(a_1)} \right]^{\frac{1}{a_2-a_1}} \leq \left[ \frac{f(b_2)}{f(b_1)} \right]^{\frac{1}{b_2-b_1}}.
\]

Next we recall some basic facts about exponential convexity (see e.g. [3], [8], [7]).

**Definition 4.3.** A function \( h : (a,b) \to \mathbb{R} \) is exponentially convex if it is continuous and

\[
\sum_{i,j=1}^{n} t_i t_j h(x_i + x_j) \geq 0,
\]

holds for all \( n \in \mathbb{N} \) and all choices \( t_i, x_i \in \mathbb{R}, \ i = 1, \ldots, n \) such that \( x_i + x_j \in (a,b), \ 1 \leq i, j \leq n \).

The following lemma gives characterization of exponential convexity (see [1], [8]).

**Lemma 4.2.** Let \( h : (a,b) \to \mathbb{R} \). The following statements are equivalent:

(i) \( h \) is exponentially convex,

(ii) \( h \) is continuous and

\[
\sum_{i,j=1}^{n} t_i t_j h \left( \frac{x_i + x_j}{2} \right) \geq 0,
\]

for every \( n \in \mathbb{N}, \ t_i \in \mathbb{R} \) and every \( x_i \in (a,b), \ 1 \leq i \leq n \).

**Remark 4.1.** Condition (4.1) is equivalent with positive semi-definiteness of matrices

\[
\left[ h \left( \frac{x_i + x_j}{2} \right) \right]_{i,j=1}^{n},
\]

for all \( n \in \mathbb{N} \).

**Remark 4.2.** Note that for \( n = 2 \) from (4.2) we get

\[
\frac{h(x_1) h(x_2) - h^2 \left( \frac{x_1 + x_2}{2} \right)}{2} \geq 0,
\]

hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

**Lemma 4.3.** Let \( p \in \mathbb{R} \). Then the function \( \varphi_p \) defined by

\[
\varphi_p(x) = e^{\int a(x)dx} \int x^{p-1} e^{-\int a(x)dx} dx
\]

is \( a(x) \)-increasing for \( x > 0 \).
Proof. Since \( \varphi'_p(x) - a(x)\varphi_p(x) = x^{p-1} \geq 0, \ x > 0 \), therefore \( \varphi_p(x) \) is \( a(x) \)-increasing function for \( x > 0 \). \( \square \)

**Lemma 4.4.** Let \( p \in \mathbb{R} \). Then the function \( \psi_p \) defined by

\[
\psi_p(x) = e^{\int a(x)dx} \int e^{px} e^{-\int a(x)dx} dx
\]  

(4.4)

is \( a(x) \)-increasing function for \( x \in \mathbb{R} \).

Proof. Since \( \psi'_p(x) - a(x)\psi_p(x) = e^{px} \geq 0 \), therefore \( \psi_p(x) \) is \( a(x) \)-increasing function for \( x \in \mathbb{R} \). \( \square \)

**Lemma 4.5.** Let \( p \in \mathbb{R} \) and let the function \( \varphi_p \) be defined by (4.3) for \( x, y > 0, \ x \neq y \). Then

\[
\frac{\varphi_p(y) - \varphi_p(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)\varphi_p(t)dt = \begin{cases} \frac{1}{y - x} \cdot \frac{y^p - x^p}{\ln y - \ln x}, & p \neq 0; \\ \frac{1}{y - x} \cdot \frac{p}{\ln y - \ln x}, & p = 0. \end{cases}
\]

(4.5)

Proof. Since \( \varphi'_p(x) - a(x)\varphi_p(x) = x^{p-1} \), we have \( a(x)\varphi_p(x) = \varphi'_p(x) - x^{p-1} \), so

\[
\frac{\varphi_p(y) - \varphi_p(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)\varphi_p(t)dt = \frac{\varphi_p(y) - \varphi_p(x)}{y - x} - \frac{1}{y - x} \int_x^y (\varphi'_p(t) - t^{p-1})dt
\]

\[
= \begin{cases} \frac{1}{y - x} \cdot \frac{y^p - x^p}{\ln y - \ln x}, & p \neq 0; \\ \frac{1}{y - x} \cdot \frac{p}{\ln y - \ln x}, & p = 0. \end{cases} \quad \square
\]

Let us define the right-hand side in (4.5) as

\[
\xi(p) = \begin{cases} \frac{1}{y - x} \cdot \frac{y^p - x^p}{\ln y - \ln x}, & p \neq 0; \\ \frac{1}{y - x} \cdot \frac{p}{\ln y - \ln x}, & p = 0. \end{cases}
\]

(4.6)

Obviously, we have that \( \xi(p) > 0 \) for all \( p \in \mathbb{R} \).

In the following theorem we explore some properties of the mapping \( p \to \xi(p) \).

**Theorem 4.1.** Let \( p \in \mathbb{R} \) and let the function \( \xi \) be defined by (4.6) for \( x, y > 0, \ x \neq y \). Then

(i) the function \( p \to \xi(p) \) is continuous on \( \mathbb{R} \),

(ii) for every \( n \in \mathbb{N} \) and \( p_i \in \mathbb{R} \), \( p_{ij} = \frac{p_i + p_j}{2}, \ i, j = 1, \ldots, n \), the matrix \( \left[ \xi \left( \frac{p_i + p_j}{2} \right) \right]_{i,j=1}^n \)

is a positive semi-definite matrix. In particular

\[
\det \left[ \xi \left( \frac{p_i + p_j}{2} \right) \right]_{i,j=1}^n \geq 0;
\]
(iii) the function $p \to \xi(p)$ is exponentially convex,

(iv) the function $p \to \xi(p)$ is log-convex.

**Proof.**

(i) In order to prove that the function $p \to \xi(p)$ is continuous on $\mathbb{R}$, we only need to verify that $\lim_{p \to 0} \xi(p) = \xi(0)$ which is obtained by a simple calculation. Hence, $\xi$ is continuous on $\mathbb{R}$.

(ii) Let $n \in \mathbb{N}$, $t_i \in \mathbb{R}$, $p_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Denote $p_{ij} = \frac{p_i + p_j}{2}$.

Let $\phi_p$ be defined by (4.3). Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$f(x) = \sum_{i,j=1}^{n} t_{ij} \phi_{p_{ij}}(x).$$

Then

$$f'(x) - a(x)f(x) = \sum_{i,j=1}^{n} t_{ij} \phi'_{p_{ij}}(x) - a(x) \sum_{i,j=1}^{n} t_{ij} \phi_{p_{ij}}(x)$$

$$= \sum_{i,j=1}^{n} t_{ij} (\phi'_{p_{ij}}(x) - a(x) \phi_{p_{ij}}(x)) = \sum_{i,j=1}^{n} t_{ij} x^{p_{ij} - 1}$$

$$= \left( \sum_{i=1}^{n} t_i x^{(p_{ij} - 1)/2} \right)^2 \geq 0$$

Hence, $f$ is an increasing function.

Now we can apply (2.2) to the function $f$ defined above, and obtain

$$\sum_{i,j=1}^{n} t_{ij} \left( \frac{\phi_{p_{ij}}(y) - \phi_{p_{ij}}(x)}{y-x} - \frac{1}{y-x} \int_{x}^{y} a(t) \phi_{p_{ij}}(t) dt \right) \geq 0.$$

Now, from (4.5) it follows that

$$\sum_{i,j=1}^{n} t_{ij} \xi(p_{ij}) \geq 0.$$

Therefore, the matrix $\left[ \xi\left(\frac{p_i + p_j}{2}\right) \right]_{i,j=1}^{n}$ is positive semi-definite.

(iii) Follows from (i), (ii) and Lemma 4.2.

(iv) Follows from (iii) and Remark 4.2. □

**Lemma 4.6.** Let $p \in \mathbb{R}$ and let the function $\psi_p$ be defined by (4.4) for $x \neq y$.

Then

$$\frac{\psi_p(y) - \psi_p(x)}{y-x} - \frac{1}{y-x} \int_{x}^{y} a(t) \psi_p(t) dt = \begin{cases} \frac{1}{y-x} \cdot \frac{e^{px} - e^{py}}{p}, & p \neq 0; \\ 1, & p = 0. \end{cases} \quad (4.7)$$
Proof. Similar to the proof of Lemma 4.5. □

Let us define the right-hand side in (4.7) as

$$
\zeta(p) = \begin{cases} 
\frac{1}{y-x} \cdot \frac{e^{py} - e^{px}}{p}, & p \neq 0; \\
1, & p = 0.
\end{cases}
$$

(4.8)

Obviously, we have that $$\zeta(p) > 0$$ for all $$p \in \mathbb{R}$$.

In the following theorem we explore some properties of the mapping $$p \to \zeta(p)$$.

**Theorem 4.2.** Let $$p \in \mathbb{R}$$ and let the function $$\zeta$$ be defined by (4.8) for $$x \neq y$$.

Then

(i) the function $$p \to \zeta(p)$$ is continuous on $$\mathbb{R}$$,

(ii) for every $$n \in \mathbb{N}$$ and $$p_i \in \mathbb{R}$$, $$p_{ij} = \frac{p_i + p_j}{2}$$, $$i, j = 1, \ldots, n$$, the matrix $$\left[ \zeta \left( \frac{p_i + p_j}{2} \right) \right]_{i,j=1}^n$$ is a positive semi-definite matrix. In particular

$$
\det \left[ \zeta \left( \frac{p_i + p_j}{2} \right) \right]_{i,j=1}^n \geq 0;
$$

(iii) the function $$p \to \zeta(p)$$ is exponentially convex,

(iv) the function $$p \to \zeta(p)$$ is log-convex.

**Proof.** Similar to the proof of Theorem 4.1. □

Theorem 3.2 enables us to define various types of means, because if the function $$\frac{f' - af}{h' - ah}$$ has inverse, from (3.5) we have

$$
\eta = \left( \frac{f' - af}{h' - ah} \right)^{-1} \left( \frac{f(y) - f(x) - \int_x^y a(t)f(t)dt}{h(y) - h(x) - \int_x^y a(t)h(t)dt} \right), \quad \eta \in [x, y]
$$

which means that $$\eta$$ is a mean of numbers $$x$$ and $$y$$.

First, let us observe differential equations $$f'(\eta) - a(\eta)f(\eta) = \eta^{p-1}$$ and $$h'(\eta) - a(\eta)h(\eta) = \eta^{s-1}$$ for $$ps(p-s) \neq 0$$. Then from (3.5) we get

$$
\eta = \left( \frac{f(y) - f(x) - \int_x^y a(t)f(t)dt}{h(y) - h(x) - \int_x^y a(t)h(t)dt} \right)^{\frac{1}{p-s}}.
$$

From $$f'(t) - a(t)f(t) = t^{p-1}$$ we have $$a(t)f(t) = f'(t) - t^{p-1}$$, so

$$
f(y) - f(x) - \int_x^y a(t)f(t)dt = \frac{y^p - x^p}{p}.
$$

In the same way we get,

$$
h(y) - h(x) - \int_x^y a(t)h(t)dt = \frac{y^s - x^s}{s}.
$$
Hence,
\[
\eta = \left\{ \frac{s(y^p - x^p)}{p(y^s - x^s)} \right\}^{\frac{1}{p-s}}.
\]

Moreover, we have Stolarsky mean
\[
E(x, y; s, p) = \left\{ \frac{s(y^p - x^p)}{p(y^s - x^s)} \right\}^{\frac{1}{p-s}}
\]
where \(x\) and \(y\) are positive real numbers \(x \neq y\), \(s \neq p\), \(s, p \neq 0\). All continuous extensions of Stolarsky means are known and given in the Introduction. Furthermore,
\[
E(x, y; s, p) = \left( \frac{\xi(p)}{\xi(s)} \right)^{\frac{1}{p-s}}
\]
where \(\xi\) is defined by (4.6).

**Remark 4.3.** Now we can give another proof of Theorem 1.1 by using definition of \(a(x)\)-monotonic functions. Since \(\xi\) defined by (4.6) is a log-convex function, we can apply Lemma 4.1 and get
\[
\left( \frac{\xi(l)}{\xi(r)} \right)^{\frac{1}{l-r}} \leq \left( \frac{\xi(p)}{\xi(s)} \right)^{\frac{1}{p-s}} \tag{4.9}
\]
hence, we get (1.4).

Now, let us observe differential equations \(f'(\eta) - a(\eta)f(\eta) = e^{p\eta}\) and \(h'(\eta) - a(\eta)h(\eta) = e^{s\eta}\). Then from (3.5) we get
\[
e^{(p-s)\eta} = \left( \frac{f(y) - f(x) - \int_x^y a(t)f(t)dt}{h(y) - h(x) - \int_x^y a(t)h(t)dt} \right)
\]
From \(f'(t) - a(t)f(t) = e^{pt}\) we have \(a(t)f(t) = f'(t) - e^{pt}\), so
\[
f(y) - f(x) - \int_x^y a(t)f(t)dt = \frac{e^{py} - e^{px}}{p}.
\]
In the same way we get,
\[
h(y) - h(x) - \int_x^y a(t)h(t)dt = \frac{e^{sy} - e^{sx}}{s}.
\]
So,
\[
\eta = \ln \left\{ \frac{s(e^{py} - e^{px})}{p(e^{sy} - e^{sx})} \right\}^{\frac{1}{p-s}}
\]
i.e.
\[
\eta = \ln \left( \frac{\xi(p)}{\xi(s)} \right)^{\frac{1}{p-s}}.
\]
Making substitutions $e^y \to y$, $e^x \to x$ and then $\ln \frac{s(y^p-x^p)}{p(y^s-x^s)} = \frac{1}{p-1}$ we consider the following expression

$$E(x,y;s,p) = \left\{ \frac{s(y^p-x^p)}{p(y^s-x^s)} \right\}^{\frac{1}{p-1}}.$$ 

Hence, again we get Stolarsky mean.

**Remark 4.4.** From Remark 2.3 (iii) we have that for $a(x) = \frac{h'(x)}{h(x)}$, function $f/h$ is monotonic. Special case, when $f/h$ is an increasing function is studied in [6]. Using generalization of Steffensen’s inequality for $f/h$ increasing given in [12, p. 192] linear functional $L$ is defined as the difference between the left-hand and the right-hand side of Steffensen’s inequality by

$$L(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt.$$ 

Let $a(x) = \frac{k'(x)}{k(x)}$, $g(x) = k(x)$, $m(x) = mk(x)$ and $M(x) = Mk(x)$. Then $R_1$ defined by (3.1) is equal to $Mk(x)$ and $R_2$ defined by (3.2) is equal to $mk(x)$. Hence functions $\Phi_1$ and $\Phi_2$ defined in Lemma 3.1 are

$$\Phi_1 = Mk(x) - h(x), \quad \Phi_2 = h(x) - mk(x)$$

which are exactly the functions used in the proof of the Lagrange-type mean value theorem given in [6]. Moreover, using linear functional $L$ instead of the difference between the left-hand and the right-hand side of inequality (2.2), from Theorem 3.1 we can obtain Lagrange-type mean value theorem given in [6]. In the same way, from Theorem 3.2, we can obtain the Cauchy-type mean value theorem given in [6].

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