Journal of
Mathematical
Inequalities
Volume 5, Number 3 (2011), 435–450

$a(x)$–MONOTONIC FUNCTIONS AND THEIR INEQUALITIES

JOSIP PEČARIĆ AND KSENIJA SMOLJAK

(*Communicated by N. Elezović*)

Abstract. In this paper $a(x)$-monotonic functions are defined and some inequalities for them are derived. Related analogous of the Lagrange and the Cauchy mean value theorems are also derived and means of the Cauchy type are generated. Furthermore, it is shown that the monotonicity of the Stolarsky means can be proved using the notion of generalized monotonic functions.

1. Introduction

The following result is well known (it can be found in [2, p. 133–134]):
If the linear differential equation

$$u'(t) = a(t)u(t), \quad u(0) = c, \tag{1.1}$$

and the linear differential inequality

$$v'(t) \geqslant a(t)v(t), \quad v(0) = c, \tag{1.2}$$

are both valid for $0 \leqslant t \leqslant T$, then

$$v(t) \geqslant u(t), \quad 0 \leqslant t \leqslant T. \tag{1.3}$$

Differential inequality $y'(x) - a(x)y(x) \geqslant 0$ is sometimes used as a definition of generalized increasing functions. In this paper we shall observe an analoguous generalization which we will call $a(x)$-monotonic functions.

First, let us recall Stolarsky means:

$$\begin{aligned} E(x,y;s,p) &= \left\{ \frac{s(y^p - x^p)}{p(y^s - x^s)} \right\}^{\frac{1}{p-s}} \\ E(x,y;r,r) &= e^{-\frac{1}{r}} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{1/(x^r - y^r)} \\ E(x,y;r,0) = E(x,y;0,r) &= \left\{ \frac{y^r - x^r}{r(\ln y - \ln x)} \right\}^{1/r} \\ E(x,y;0,0) &= \sqrt{xy}. \end{aligned}$$

Mathematics subject classification (2010): Primary 26D10, Secondary 26D15.
Keywords and phrases: differential equation, $a(x)$-monotonic function, $a(x)$-monotonic inequality, exponential convexity, mean value theorems, Stolarsky means.

Paper JMI-05-38

where x and y are positive real numbers $x \neq y$, p and s are any real numbers. Stolarsky introduced this means in 1975 (see [13]).

In the following theorem we state monotonicity of Stolarsky means first proved by Stolarsky (see [13]).

THEOREM 1.1. *Let $r \leqslant s$, $l \leqslant p$, then the following inequality is valid*

$$E(x,y;r,l) \leqslant E(x,y;s,p) \tag{1.4}$$

that is, the mean $E(x,y;s,p)$ is monotonic.

Another proof, using a definition of monotonic functions, is given in [5]. In this paper we shall show that Theorem 1.1 can also be proved by using definitions of generalized monotonic functions.

REMARK 1.1. Necessary and sufficient conditions for (1.4) to be valid are given in [9].

The paper is organised as follows. In Section 2 we define $a(x)$-monotonic functions and give some inequalities for them. In Section 3 we derive analogues of the Lagrange mean value theorem and the Cauchy mean value theorem. In Section 4 we prove the exponential convexity of a function defined as the difference between the left-hand and the right-hand side of the inequality which defines $a(x)$-monotonic functions, give means of Cauchy type and prove Theorem 1.1 using generalized monotonic functions.

2. Properties of $a(x)$-monotonic functions and their inequalities

DEFINITION 2.1. Let f, a be real functions defined on interval $I \subseteq \mathbb{R}$ such that af is integrabile. Function f is called $a(x)$*-increasing* on interval I if for every $x,y \in I$

$$(y-x)(f(y)-f(x)) \geqslant (y-x)\int_x^y a(t)f(t)dt \tag{2.1}$$

holds.

Function f is called $a(x)$*-decreasing* if the inequality in (2.1) is reversed.

Function f is called $a(x)$*-monotonic* if it satisfies (2.1) or the reversed inequality.

If two functions are both $a(x)$-increasing, or both $a(x)$-decreasing, we say that they are $a(x)$*-monotonic in the same sense*.

REMARK 2.1. Notice that for $a(x) = 0$, f is monotonic.

REMARK 2.2. If $x \neq y$, (2.1) is equivallent to

$$\frac{f(y)-f(x)}{y-x} \geqslant \frac{1}{y-x}\int_x^y a(t)f(t)dt. \tag{2.2}$$

If f is $a(x)$-increasing, $-f$ is $a(x)$-decreasing. So we will only give properties of $a(x)$-increasing functions, because they are the same for $a(x)$-decreasing functions. Properties of $a(x)$-increasing functions:

1) Let f and g be $a(x)$-increasing functions. Then $f+g$ is $a(x)$-increasing.
   If f and g are $a(x)$-monotonic functions (withouth further specifications), we can't conclude that $f+g$ is $a(x)$-monotonic.

2) If f is $a(x)$-increasing function and λ is non-negative real number, then λf is $a(x)$-increasing function.

In applications we often use $a(x)$-monotonicity criteria given in the following theorem.

THEOREM 2.1. *If f' is a continuous function and af an integrabile function on interval I, f is $a(x)$-monotonic on interval I if and only if the function $f'(x)-a(x)f(x)$ is non-negative or non-positive on I. More precisely, f is $a(x)$-increasing function if and only if $f'(x)-a(x)f(x)\geqslant 0$; f is $a(x)$-decreasing function if and only if $f'(x)-a(x)f(x)\leqslant 0$.*

Proof. Let f be $a(x)$-increasing function, then for $x,y\in I$ such that $x\neq y$ we have

$$\frac{f(y)-f(x)}{y-x}\geqslant\frac{1}{y-x}\int_x^y a(t)f(t)dt.$$

Taking a limit when $y\to x$ we get $f'(x)\geqslant a(x)f(x)$.
Conversely, let $f'(x)\geqslant a(x)f(x)$. For $x,y\in I$ such that $x<y$ we have that

$$\int_x^y f'(t)dt\geqslant\int_x^y a(t)f(t)dt. \tag{2.3}$$

Since f' is continuous, we have that for every $[x,y]\subset I$, $\int_x^y f'(t)dt=f(y)-f(x)$. Furthermore, since $x<y$ we can multiply inequality (2.3) by $y-x$ and get

$$(y-x)(f(y)-f(x))\geqslant(y-x)\int_x^y a(t)f(t)dt.$$

Hence, f is $a(x)$-increasing function.
Using the same reasoning we get criteria for $a(x)$-decreasing functions. So the proof is completed. □

REMARK 2.3. Note that for f differentiabile and $a(x)$-monotonic we have:

(i) for $a(x)=\frac{1}{x}$, $\frac{f(x)}{x}$ is monotonic (this case is studied in [11]);

(ii) for $a(x)=\frac{a}{x}$, where a is some constant, $\frac{f(x)}{x^a}$ is monotonic;

(iii) for $a(x)=\frac{h'(x)}{h(x)}$, $\frac{f}{h}$ is monotonic (this case is studied in [6]).

Proof of this remarks follows from $a(x)$-monotonicity criteria in Theorem 2.1. For example, for $a(x)=\frac{1}{x}$ and $a(x)$-increasing function f, we have $f'(x)\geqslant\frac{f(x)}{x}$, so for $x>0$ we have $\left(\frac{f(x)}{x}\right)'=\frac{xf'(x)-f(x)}{x^2}\geqslant\frac{f(x)-f(x)}{x^2}=0$, hence $\frac{f(x)}{x}$ is increasing function for $x>0$.

THEOREM 2.2. *A function f is $a(x)$-increasing if and only if the function F defined by*

$$F(x)=f(x)-\int_{x_0}^{x}a(t)f(t)dt \tag{2.4}$$

is increasing.

Proof. Suppose that $y>x$. Then (2.1) is equivalent to

$$f(y)-f(x)\geqslant\int_{x}^{y}a(t)f(t)dt$$

i.e.

$$f(y)-f(x)\geqslant\int_{x_0}^{y}a(t)f(t)dt-\int_{x_0}^{x}a(t)f(t)dt$$

i.e.

$$f(y)-\int_{x_0}^{y}a(t)f(t)dt\geqslant f(x)-\int_{x_0}^{x}a(t)f(t)dt$$

i.e.

$$F(y)\geqslant F(x).$$

Since we have equivalence in each step, the proof is completed. □

We can apply the function F defined by (2.4) to inequalities for monotonic functions and get inequalities for $a(x)$-monotonic functions.

Here we give Steffensen's inequality for $a(x)$-monotonic functions.

COROLLARY 2.1. *Suppose that f is $a(x)$-increasing and g is integrabile on $[b,c]$ with $0\leqslant g\leqslant 1$ and $\lambda=\int_b^c g(x)dx$. Then we have*

$$\begin{aligned}\int_b^{b+\lambda}f(x)dx-\int_b^{b+\lambda}\int_{x_0}^{x}a(t)f(t)dt\,dx&\leqslant\int_b^c f(x)g(x)dx-\int_b^c\left(g(x)\int_{x_0}^{x}a(t)f(t)dt\right)dx\\&\leqslant\int_{c-\lambda}^{c}f(x)dx-\int_{c-\lambda}^{c}\int_{x_0}^{x}a(t)f(t)dt\,dx.\end{aligned} \tag{2.5}$$

The inequalities are reversed for f $a(x)$-decreasing.

Proof. Let the function F be defined by (2.4). Since, F is increasing we can apply Steffensen's inequality, hence

$$\int_b^{b+\lambda} F(x)dx \leqslant \int_b^c F(x)g(x)dx \leqslant \int_{c-\lambda}^c F(x)dx.$$

By elementary calculation we get (2.5). □

LEMMA 2.1. *Let f be a positive, differentiabile and $a(x)$-increasing function. Then functions G, H defined by*

$$G(x) = f(x) \cdot e^{-\int a(x)dx}, \tag{2.6}$$

$$H(x) = \ln f(x) - \int a(x)dx \tag{2.7}$$

are increasing.

Proof. Function f is $a(x)$-increasing, so from Theorem 2.1 we have $f'(x) - a(x)f(x) \geqslant 0$. Since f is positive, we get $\frac{f'(x)}{f(x)} - a(x) \geqslant 0$. From (2.6) we get

$$G'(x) = f'(x) \cdot e^{-\int a(x)dx} - a(x)f(x)e^{-\int a(x)dx} \geqslant 0.$$

Hence, G is increasing. From (2.7) we get

$$H'(x) = \frac{f'(x)}{f(x)} - a(x) \geqslant 0.$$

Hence, H is increasing. □

REMARK 2.4. Using functions G and H defined by (2.6) and (2.7) we can get new $a(x)$-monotonic inequalities.

THEOREM 2.3. *Let $u(t) = ce^{\int_0^t a(t)dt}$ for $0 \leqslant t \leqslant T$. Let v satisfy (1.2) with $a(x) \geqslant 0$ for $0 \leqslant x \leqslant T$ and let $v(0) = c$. Then $v(x) - u(x)$ is an increasing function for $0 \leqslant x \leqslant T$.*

Proof. Notice that u is the solution of the differential equation $u'(t) - a(t)u(t) = 0$ with $u(0) = c$, so u satisfies (1.1). Hence (1.3) is valid. Since v satisfies (1.2), from Theorem 2.1 we have that v is $a(x)$-monotonic function. So

$$(y-x)(v(y)-v(x)) \geqslant (y-x)\int_x^y a(t)v(t)dt. \tag{2.8}$$

For $x, y \in [0,T]$ such that $x < y$ we can divide (2.8) by $y-x$ and then apply (1.3). We get

$$v(y) - v(x) \geqslant \int_x^y a(t)v(t)dt \geqslant \int_x^y a(t)u(t)dt = \int_x^y u'(t)dt = u(y) - u(x). \tag{2.9}$$

Hence,

$$v(y) - u(y) \geqslant v(x) - u(x).$$

So the proof is completed. □

Now we give Steffensen's inequality for function $v(x) - u(x)$:

COROLLARY 2.2. *Let functions u and v be such that conditions of Theorem 2.3 are satisfied. Let g be an integrabile function on $[0,T]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_0^T g(t)dt$. Then we have*

$$\int_0^{\lambda} (v(t) - u(t))dt \leqslant \int_0^T (v(t) - u(t))g(t)dt \leqslant \int_{T-\lambda}^{T} (v(t) - u(t))dt. \tag{2.10}$$

Proof. From Theorem 2.3 we have that $v(x) - u(x)$ is an increasing function, so we can apply Steffensen's inequality and get (2.10). □

3. Mean value theorems

LEMMA 3.1. *Let I be an open interval. Let a be an integrabile and $h \in C^1(I)$ be such that $h' - ah$ is bounded by integrabile functions M and m, that is, $m(x) \leqslant h'(x) - a(x)h(x) \leqslant M(x)$, for every $x \in I$. Then functions Φ_1, Φ_2 defined by*

$$\Phi_1(x) = R_1(x) - h(x),$$

$$\Phi_2(x) = h(x) - R_2(x),$$

where

$$R_1(x) = e^{\int a(x)dx} \int M(x)\, e^{-\int a(x)dx} dx, \tag{3.1}$$

$$R_2(x) = e^{\int a(x)dx} \int m(x)\, e^{-\int a(x)dx} dx, \tag{3.2}$$

are $a(x)$-increasing.

Proof. Since $h'(x) - a(x)h(x) \leqslant M(x)$ and $R_1'(x) - a(x)R_1(x) = M(x)$ we have,

$$\Phi_1'(x) - a(x)\Phi_1(x) = R_1'(x) - a(x)R_1(x) - (h'(x) - a(x)h(x)) \geqslant 0.$$

So Φ_1 is $a(x)$-increasing function. In the same way, since $h'(x) - a(x)h(x) \geqslant m(x)$ we have,

$$\Phi_2'(x) - a(x)\Phi_2(x) = h'(x) - a(x)h(x) - (R_2'(x) - a(x)R_2(x)) \geqslant 0.$$

So Φ_2 is $a(x)$-increasing function. □

Now we will state and prove the Lagrange-type mean value theorem. This theorem is a consequence of the Cauchy mean value theorem but we will prove it by using $a(x)$-increasing functions from Lemma 3.1.

THEOREM 3.1. *Let a, h' be continuous and g be a positive and continuous function on $[x,y]\subseteq\mathbb{R}$. Then there exists $\eta\in[x,y]$ such that*

$$\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)h(t)dt=\frac{h'(\eta)-a(\eta)h(\eta)}{g(\eta)}\cdot\frac{1}{y-x}\int_x^y g(t)dt. \tag{3.3}$$

Proof. Since $\frac{h'-ah}{g}$ is continuous on $[x,y]$, there exists

$$m=\min_{t\in[x,y]}\left(\frac{h'(t)-a(t)h(t)}{g(t)}\right) \text{ and } M=\max_{t\in[x,y]}\left(\frac{h'(t)-a(t)h(t)}{g(t)}\right).$$

Applying (2.2) on functions Φ_1 and Φ_2 from Lemma 3.1, with $M(x)=Mg(x)$, $m(x)=mg(x)$, the following inequalities hold:

$$\frac{\Phi_1(y)-\Phi_1(x)}{y-x}\geqslant\frac{1}{y-x}\int_x^y a(t)\Phi_1(t)dt,$$
$$\frac{\Phi_2(y)-\Phi_2(x)}{y-x}\geqslant\frac{1}{y-x}\int_x^y a(t)\Phi_2(t)dt.$$

It follows,

$$\begin{aligned}m&\left(\frac{R_3(y)-R_3(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)R_3(t)dt\right)\\ &\leqslant\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)h(t)dt\\ &\leqslant M\left(\frac{R_3(y)-R_3(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)R_3(t)dt\right),\end{aligned}$$

where

$$R_3(x)=e^{\int a(x)dx}\int g(x)\,e^{-\int a(x)dx}dx.$$

Therefore, there exists $\eta\in[x,y]$ such that

$$\begin{aligned}&\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)h(t)dt\\ &\quad=\frac{h'(\eta)-a(\eta)h(\eta)}{g(\eta)}\left(\frac{R_3(y)-R_3(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)R_3(t)dt\right).\end{aligned}$$

holds. Since $R_3'(x)-a(x)R_3(x)=g(x)$, we have $a(x)R_3(x)=R_3'(x)-g(x)$. So

$$\begin{aligned}&\frac{R_3(y)-R_3(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)R_3(t)dt\\ &\quad=\frac{R_3(y)-R_3(x)}{y-x}-\frac{1}{y-x}\int_x^y (R_3'(t)-g(t))dt\\ &\quad=\frac{1}{y-x}\int_x^y g(t)dt.\end{aligned}$$

Hence, there exists $\eta\in[x,y]$ such that (3.3) holds. □

REMARK 3.1. For $a(x)=0$ Theorem 3.1 gives Cauchy mean value theorem, that is,

$$\frac{h'(\eta)}{g(\eta)}=\frac{h(y)-h(x)}{\int_x^y g(t)dt}.$$

Additionally, taking $g\equiv 1$ we get Lagrange mean value theorem, that is,

$$h'(\eta)=\frac{h(y)-h(x)}{y-x}.$$

REMARK 3.2. For $a(x)=\frac{1}{x}$ Theorem 3.1 gives

$$\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y \frac{h(t)}{t}dt=\frac{\eta h'(\eta)-h(\eta)}{\eta g(\eta)}\cdot\frac{1}{y-x}\int_x^y g(t)dt.$$

Furthermore, taking $g\equiv 1$ we get

$$\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y \frac{h(t)}{t}dt=h'(\eta)-\frac{h(\eta)}{\eta}.$$

REMARK 3.3. For $a(x)=\frac{k'(x)}{k(x)}$ Theorem 3.1 gives

$$\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y \frac{k'(t)}{k(t)}h(t)dt=\frac{h'(\eta)k(\eta)-k'(\eta)h(\eta)}{k(\eta)g(\eta)}\cdot\frac{1}{y-x}\int_x^y g(t)dt.$$

Furthermore, taking $g=h$ we get

$$\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y \frac{k'(t)}{k(t)}h(t)dt=\left(\frac{h'(\eta)}{h(\eta)}-\frac{k'(\eta)}{k(\eta)}\right)\int_x^y h(t)dt.$$

THEOREM 3.2. *Let I be an interval in $\mathbb{R}$ and let $x,y\in I$ be such that $x\neq y$. Let $f,h\in C^1(I)$ and let a be a continuous function such that*

$$h(y)-h(x)-\int_x^y a(t)h(t)dt\neq 0. \tag{3.4}$$

Then there exists $\eta\in[x,y]$ such that

$$\frac{f'(\eta)-a(\eta)f(\eta)}{h'(\eta)-a(\eta)h(\eta)}=\frac{f(y)-f(x)-\int_x^y a(t)f(t)dt}{h(y)-h(x)-\int_x^y a(t)h(t)dt}. \tag{3.5}$$

Proof. Let

$$c_1=\frac{h(y)-h(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)h(t)dt,$$

$$c_2=\frac{f(y)-f(x)}{y-x}-\frac{1}{y-x}\int_x^y a(t)f(t)dt.$$

Now we apply (3.3) to the function $c_1 f - c_2 h$. The following equality holds:

$$c_1\left[\frac{f(y)-f(x)}{y-x} - \frac{1}{y-x}\int_x^y a(t)f(t)dt\right] - c_2\left[\frac{h(y)-h(x)}{y-x} - \frac{1}{y-x}\int_x^y a(t)h(t)dt\right]$$
$$= \frac{c_1 f'(\xi) - c_2 h'(\xi) - a(\xi)\left(c_1 f(\xi) - c_2 h(\xi)\right)}{g(\xi)} \cdot \left[\frac{1}{y-x}\int_x^y g(t)dt\right]. \tag{3.6}$$

It is easy to see that the left-hand side of (3.6) is equal to 0, so the right-hand side should also be equal to 0. From (3.4) we get that the right-hand side in (3.3) is not equal to 0, so the part in square brackets on the right-hand side of (3.6) is not equal to 0. For the right-hand side in (3.6) to be equal to 0 it follows that $c_1 f'(\xi) - c_2 h'(\xi) - a(\xi)\left(c_1 f(\xi) - c_2 h(\xi)\right) = 0$. After a short calculation, it is easy to see that (3.5) follows from $c_1(f'(\xi) - a(\xi)f(\xi)) - c_2(h'(\xi) - a(\xi)h(\xi)) = 0$, so the proof is completed. □

REMARK 3.4. Theorem 3.2 is equivalent to Cauchy mean value theorem.

4. Exponential convexity and means of Cauchy type

First we recall some basic facts about convexity, log-convexity and log-convexity in the Jensen sense (see e.g. [4], [10], [12]).

DEFINITION 4.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is *convex in the Jensen sense* on an interval I if for each $a, b \in I$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}$$

holds.

We recall that for a continuous function f, convexity and convexity in the Jensen sense are equivalent properties.

DEFINITION 4.2. A positive function $f : I \to (0,\infty)$ is said to be *logarithmically convex* if $\log f$ is convex function on I. For such function f, we shortly say f is log-convex. A positive function $f : I \to (0,\infty)$ is *log-convex in the Jensen sense* if for each $a, b \in I$

$$f^2\left(\frac{a+b}{2}\right) \leqslant f(a)f(b)$$

holds, i.e., if $\log f$ is convex in the Jensen sense.

The following result on log-convex functions is given in [12].

LEMMA 4.1. *Let positive function* $f: I \to (0,\infty)$ *be log-convex and let* $a_1, a_2, b_1, b_2 \in I$ *be such that* $a_1 \leqslant b_1$, $a_2 \leqslant b_2$ *and* $a_1 \neq a_2$, $b_1 \neq b_2$. *Then the following inequality is valid*

$$\left[\frac{f(a_2)}{f(a_1)}\right]^{\frac{1}{a_2-a_1}} \leqslant \left[\frac{f(b_2)}{f(b_1)}\right]^{\frac{1}{b_2-b_1}}.$$

Next we recall some basic facts about exponential convexity (see e.g. [3], [8], [7]).

DEFINITION 4.3. A function $h:(a,b) \to \mathbb{R}$ is *exponentially convex* if it is continuous and

$$\sum_{i,j=1}^{n} t_i t_j h\left(x_i + x_j\right) \geqslant 0,$$

holds for all $n \in \mathbb{N}$ and all choices $t_i, x_i \in \mathbb{R}$, $i = 1,\ldots,n$ such that $x_i + x_j \in (a,b)$, $1 \leqslant i,j \leqslant n$.

The following lemma gives characterization of exponential convexity (see [1], [8]).

LEMMA 4.2. *Let* $h:(a,b) \to \mathbb{R}$. *The following statements are equivalent:*

(i) h *is exponentially convex,*

(ii) h *is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j h\left(\frac{x_i + x_j}{2}\right) \geqslant 0, \tag{4.1}$$

for every $n \in \mathbb{N}$, $t_i \in \mathbb{R}$ *and every* $x_i \in (a,b)$, $1 \leqslant i \leqslant n$.

REMARK 4.1. Condition (4.1) is equivalent with positive semi-definitness of matrices

$$\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}, \tag{4.2}$$

for all $n \in \mathbb{N}$.

REMARK 4.2. Note that for $n = 2$ from (4.2) we get

$$h(x_1)h(x_2) - h^2\left(\frac{x_1 + x_2}{2}\right) \geqslant 0,$$

hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

LEMMA 4.3. *Let* $p \in \mathbb{R}$. *Then the function* φ_p *defined by*

$$\varphi_p(x) = e^{\int a(x)dx} \int x^{p-1} e^{-\int a(x)dx} dx \tag{4.3}$$

is $a(x)$*-increasing for* $x > 0$.

Proof. Since $\varphi_p'(x) - a(x)\varphi_p(x) = x^{p-1} \geqslant 0$, $x > 0$, therefore $\varphi_p(x)$ is $a(x)$-increasing function for $x > 0$. □

LEMMA 4.4. *Let $p \in \mathbb{R}$. Then the function ψ_p defined by*

$$\psi_p(x) = e^{\int a(x)dx} \int e^{px} e^{-\int a(x)dx} dx \tag{4.4}$$

is $a(x)$-increasing function for $x \in \mathbb{R}$.

Proof. Since $\psi_p'(x) - a(x)\psi_p(x) = e^{px} \geqslant 0$, therefore $\psi_p(x)$ is $a(x)$-increasing function for $x \in \mathbb{R}$. □

LEMMA 4.5. *Let $p \in \mathbb{R}$ and let the function φ_p be defined by (4.3) for $x, y > 0$, $x \neq y$. Then*

$$\frac{\varphi_p(y) - \varphi_p(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)\varphi_p(t)dt = \begin{cases} \frac{1}{y-x} \cdot \frac{y^p - x^p}{p}, & p \neq 0; \\ \frac{\ln y - \ln x}{y-x}, & p = 0. \end{cases} \tag{4.5}$$

Proof. Since $\varphi_p'(x) - a(x)\varphi_p(x) = x^{p-1}$, we have $a(x)\varphi_p(x) = \varphi_p'(x) - x^{p-1}$, so

$$\begin{aligned} \frac{\varphi_p(y) - \varphi_p(x)}{y - x} - \frac{1}{y - x} \int_x^y a(t)\varphi_p(t)dt &= \frac{\varphi_p(y) - \varphi_p(x)}{y - x} - \frac{1}{y - x} \int_x^y (\varphi_p'(t) - t^{p-1})dt \\ &= \begin{cases} \frac{1}{y-x} \cdot \frac{y^p - x^p}{p}, & p \neq 0; \\ \frac{\ln y - \ln x}{y-x}, & p = 0. \end{cases} \end{aligned}$$ □

Let us define the right-hand side in (4.5) as

$$\xi(p) = \begin{cases} \frac{1}{y-x} \cdot \frac{y^p - x^p}{p}, & p \neq 0; \\ \frac{\ln y - \ln x}{y-x}, & p = 0. \end{cases} \tag{4.6}$$

Obviously, we have that $\xi(p) > 0$ for all $p \in \mathbb{R}$.

In the following theorem we explore some properties of the mapping $p \to \xi(p)$.

THEOREM 4.1. *Let $p \in \mathbb{R}$ and let the function ξ be defined by (4.6) for $x, y > 0$, $x \neq y$. Then*

(i) the function $p \to \xi(p)$ is continuous on $\mathbb{R}$,

(ii) for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}$, $p_{ij} = \frac{p_i + p_j}{2}$, $i, j = 1, \ldots, n$, the matrix $\left[\xi\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^n$ is a positive semi-definite matrix. In particular

$$\det\left[\xi\left(\frac{p_i + p_j}{2}\right)\right]_{i,j=1}^n \geqslant 0;$$

(iii) the function $p \to \xi(p)$ *is exponentially convex,*

(iv) the function $p \to \xi(p)$ *is log-convex.*

Proof.

(i) In order to prove that the function $p \to \xi(p)$ is continuous on $\mathbb{R}$, we only need to verify that $\lim_{p\to 0}\xi(p) = \xi(0)$ which is obtained by a simple calculation. Hence, ξ is continuous on $\mathbb{R}$.

(ii) Let $n \in \mathbb{N}$, $t_i \in \mathbb{R}$, $p_i \in \mathbb{R}$, $i = 1,2,\dots,n$. Denote $p_{ij} = \frac{p_i+p_j}{2}$.
Let φ_p be defined by (4.3). Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$f(x) = \sum_{i,j=1}^{n} t_i t_j \varphi_{p_{ij}}(x).$$

Then

$$\begin{aligned} f'(x) - a(x)f(x) &= \sum_{i,j=1}^{n} t_i t_j \varphi'_{p_{ij}}(x) - a(x)\sum_{i,j=1}^{n} t_i t_j \varphi_{p_{ij}}(x) \\ &= \sum_{i,j=1}^{n} t_i t_j (\varphi'_{p_{ij}}(x) - a(x)\varphi_{p_{ij}}(x)) = \sum_{i,j=1}^{n} t_i t_j x^{p_{ij}-1} \\ &= \left(\sum_{i=1}^{n} t_i x^{(p_{ij}-1)/2}\right)^2 \geqslant 0 \end{aligned}$$

Hence, f is $a(x)$-increasing function.
Now we can apply (2.2) to the function f defined above, and obtain

$$\sum_{i,j=1}^{n} t_i t_j \left(\frac{\varphi_{p_{ij}}(y) - \varphi_{p_{ij}}(x)}{y-x} - \frac{1}{y-x}\int_x^y a(t)\varphi_{p_{ij}}(t)dt \right) \geqslant 0.$$

Now, from (4.5) it follows that

$$\sum_{i,j=1}^{n} t_i t_j \xi(p_{ij}) \geqslant 0.$$

Therefore, the matrix $\left[\xi(\frac{p_i+p_j}{2})\right]_{i,j=1}^{n}$ is positive semi-definite.

(iii) Follows form (i), (ii) and Lemma 4.2.

(iv) Follows from (iii) and Remark 4.2. □

LEMMA 4.6. *Let* $p \in \mathbb{R}$ *and let the function* ψ_p *be defined by* (4.4) *for* $x \neq y$. *Then*

$$\frac{\psi_p(y) - \psi_p(x)}{y-x} - \frac{1}{y-x}\int_x^y a(t)\psi_p(t)dt = \begin{cases} \frac{1}{y-x}\cdot\frac{e^{py}-e^{px}}{p}, & p \neq 0; \\ 1, & p = 0. \end{cases} \tag{4.7}$$

Proof. Similar to the proof of Lemma 4.5. □

Let us define the right-hand side in (4.7) as

$$\zeta(p)=\begin{cases}\frac{1}{y-x}\cdot\frac{e^{py}-e^{px}}{p},\ p\neq 0;\\ 1,\ p=0.\end{cases} \tag{4.8}$$

Obviously, we have that $\zeta(p)>0$ for all $p\in\mathbb{R}$.

In the following theorem we explore some properties of the mapping $p\to\zeta(p)$.

THEOREM 4.2. *Let $p\in\mathbb{R}$ and let the function ζ be defined by (4.8) for $x\neq y$. Then*

(i) the function $p\to\zeta(p)$ is continuous on $\mathbb{R}$,

(ii) for every $n\in\mathbb{N}$ and $p_i\in\mathbb{R}$, $p_{ij}=\frac{p_i+p_j}{2}$, $i,j=1,\dots,n$, the matrix $\left[\zeta\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^n$ is a positive semi-definite matrix. In particular

$$\det\left[\zeta\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^n\geqslant 0;$$

(iii) the function $p\to\zeta(p)$ is exponentially convex,

(iv) the function $p\to\zeta(p)$ is log-convex.

Proof. Similar to the proof of Theorem 4.1. □

Theorem 3.2 enables us to define various types of means, because if the function $\frac{f'-af}{h'-ah}$ has inverse, from (3.5) we have

$$\eta=\left(\frac{f'-af}{h'-ah}\right)^{-1}\left(\frac{f(y)-f(x)-\int_x^y a(t)f(t)dt}{h(y)-h(x)-\int_x^y a(t)h(t)dt}\right),\quad \eta\in[x,y]$$

which means that η is a mean of numbers x and y.

First, let us observe differential equations $f'(\eta)-a(\eta)f(\eta)=\eta^{p-1}$ and $h'(\eta)-a(\eta)h(\eta)=\eta^{s-1}$ for $ps(p-s)\neq 0$. Then from (3.5) we get

$$\eta=\left(\frac{f(y)-f(x)-\int_x^y a(t)f(t)dt}{h(y)-h(x)-\int_x^y a(t)h(t)dt}\right)^{\frac{1}{p-s}}.$$

From $f'(t)-a(t)f(t)=t^{p-1}$ we have $a(t)f(t)=f'(t)-t^{p-1}$, so

$$f(y)-f(x)-\int_x^y a(t)f(t)dt=\frac{y^p-x^p}{p}.$$

In the same way we get,

$$h(y)-h(x)-\int_x^y a(t)h(t)dt=\frac{y^s-x^s}{s}.$$

Hence,

$$\eta=\left\{\frac{s(y^p-x^p)}{p(y^s-x^s)}\right\}^{\frac{1}{p-s}}.$$

Moreover, we have Stolarsky mean

$$E(x,y;s,p)=\left\{\frac{s(y^p-x^p)}{p(y^s-x^s)}\right\}^{\frac{1}{p-s}}$$

where x and y are positive real numbers $x\neq y$, $s\neq p$, $s,p\neq 0$. All continuos extensions of Stolarsky means are known and given in the Introduction.
Furthermore,

$$E(x,y;s,p)=\left(\frac{\xi(p)}{\xi(s)}\right)^{\frac{1}{p-s}}$$

where ξ is defined by (4.6).

REMARK 4.3. Now we can give another proof of Theorem 1.1 by using definition of $a(x)$-monotonic functions. Since ξ defined by (4.6) is a log-convex function, we can apply Lemma 4.1 and get

$$\left(\frac{\xi(l)}{\xi(r)}\right)^{\frac{1}{l-r}}\leqslant\left(\frac{\xi(p)}{\xi(s)}\right)^{\frac{1}{p-s}} \tag{4.9}$$

hence, we get (1.4).

Now, let us observe differential equations $f'(\eta)-a(\eta)f(\eta)=e^{p\eta}$ and $h'(\eta)-a(\eta)h(\eta)=e^{s\eta}$. Then from (3.5) we get

$$e^{(p-s)\eta}=\left(\frac{f(y)-f(x)-\int_x^y a(t)f(t)dt}{h(y)-h(x)-\int_x^y a(t)h(t)dt}\right)$$

From $f'(t)-a(t)f(t)=e^{pt}$ we have $a(t)f(t)=f'(t)-e^{pt}$, so

$$f(y)-f(x)-\int_x^y a(t)f(t)dt=\frac{e^{py}-e^{px}}{p}.$$

In the same way we get,

$$h(y)-h(x)-\int_x^y a(t)h(t)dt=\frac{e^{sy}-e^{sx}}{s}.$$

So,

$$\eta=\ln\left\{\frac{s(e^{py}-e^{px})}{p(e^{sy}-e^{sx})}\right\}^{\frac{1}{p-s}}$$

i.e.

$$\eta=\ln\left(\frac{\zeta(p)}{\zeta(s)}\right)^{\frac{1}{p-s}}.$$

Making substitutions $e^y \to y$, $e^x \to x$ and then $\ln \frac{s(y^p - x^p)}{p(y^s - x^s)} \to \frac{s(y^p - x^p)}{p(y^s - x^s)}$ we consider the following expression

$$E(x,y;s,p) = \left\{ \frac{s(y^p - x^p)}{p(y^s - x^s)} \right\}^{\frac{1}{p-s}}.$$

Hence, again we get Stolarsky mean.

REMARK 4.4. From Remark 2.3 (iii) we have that for $a(x) = \frac{h'(x)}{h(x)}$, function f/h is monotonic. Special case, when f/h is an increasing function is studied in [6]. Using generalization of Steffensen's inequality for f/h increasing given in [12, p. 192] linear functional L is defined as the difference between the left-hand and the right-hand side of Steffensen's inequality by

$$L(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt.$$

Let $a(x) = \frac{k'(x)}{k(x)}$, $g(x) = k(x)$, $m(x) = mk(x)$ and $M(x) = Mk(x)$. Then R_1 defined by (3.1) is equal to $Mxk(x)$ and R_2 defined by (3.2) is equal to $mxk(x)$. Hence functions Φ_1 and Φ_2 defined in Lemma 3.1 are

$$\Phi_1 = Mxk(x) - h(x), \quad \Phi_2 = h(x) - mxk(x)$$

which are exactly the functions used in the proof of the Lagrange-type mean value theorem given in [6]. Moreover, using linear functional L instead of the difference between the left-hand and the right-hand side of inequality (2.2), from Theorem 3.1 we can obtain Lagrange-type mean value theorem given in [6]. In the same way, from Theorem 3.2, we can obtain the Cauchy-type mean value theorem given in [6].

Acknowledgements. The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant $117-1170889-0888$.

REFERENCES

[1] N. I. AKHIEZER, *The classical moment problem and some related questions in analysis*, Olover & Boyd Ltd, Edinburgh And London, 1965.
[2] E. F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer-Verlag, Berlin, 1961.
[3] S. N. BERNSTEIN, *Sur les fonctions absolument monotones*, Acta Math. **52** (1929), 1–66.
[4] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge, 1967.
[5] J. JAKŠETIĆ, J. PEČARIĆ AND ATIQ UR REHMAN, *On Stolarsky and Related Means*, Math. Inequal. Appl., **13**, 4 (2010), 899–909.
[6] K. KRULIĆ, J. PEČARIĆ, AND K. SMOLJAK, *New generalized Steffensen means*, Collect. Math. to appear
[7] D. S. MITRINOVIĆ, J. E. PEČARIĆ, *On some Inequalities for Monotone Functions*, Boll. Unione. Mat. Ital. **7**, 5-B (1991), 407–416.

[8] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and new Inequalities in analysis*, Kluwer Academic Publishers, The Netherlands, 1993.

[9] ZS. PALES, *Inequalities for differences of powers*, J. Math. Anal. Appl. 131 (1988), 271–281

[10] C. NICULESCU AND L.-E. PERSSON, *Convex functions and their applications. A contemporary approach*, CMC Books in Mathematics, Springer, New York, 2006.

[11] J. PEČARIĆ AND ATIQ UR REHMAN, *On Logarithmic Convexity for Power Sums and Related Results*, J. Inequal. Appl., vol. 2008, ArticleID 389410, 9 pages

[12] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Academic Press, San Diego, 1992.

[13] K. B. STOLARSKY, *Generalization of logarithmic mean*, Math. Mag. 48 (1975), 87–92.

Josip Pečarrić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10000 Zagreb, Croatia
e-mail: pecaric@element.hr

Ksenija Smoljak
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10000 Zagreb, Croatia
e-mail: ksmoljak@ttf.hr