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NEW ČEBYŠEV TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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Abstract. The aim of this paper is to establish new extension of the weighted Montgomery identity for functions of two independent variables, then obtain new Čebyšev type inequalities.

1. Introduction

In [4], Čebyšev proved for two absolutely continuous functions $f,g:[a,b]\to\mathbb{R}$ that

$$|T(f,g)| \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.1}$$

where the functional $T(f,g)$ is defined by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

and $\|\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess\sup\limits_{t\in[a,b]}|f(t)|$.

In [6] the authors gives the following Montgomery identity

$$f(x) = \frac{1}{(b-a)}\int_a^b f(t)\,dt + \int_a^b P(x,t)\,f'(t)\,dt \tag{1.3}$$

where

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, a \leqslant t \leqslant x \\ \frac{t-b}{b-a}, x < t \leqslant b \end{cases} \tag{1.4}$$

is the Peano kernel.

In the last years, many papers were devoted to the generalization of Čebyšev type inequalities, we can mention the works [1,3,5,7-11]. Find new inequalities in the multidimensional cases still an interesting problem. In [2,5], the authors proved the double integrals Montgomery identity:

$$f(x,y) = \frac{1}{(b-a)}\int_a^b f(t,y)\,dt + \frac{1}{(d-c)}\int_c^d f(x,s)\,ds \tag{1.5}$$

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$$-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt$$
$$+\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt,$$

where $f: I=[a,b]\times[c,d]\to\mathbb{R}$ is differentiable, the derivative $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ is integrable on I, $P(x,t)$ is defined by (1.4) and the Peano kernel $Q(y,s)$ is defined by

$$Q(y,s)=\begin{cases}\frac{s-c}{d-c}, c\leqslant s\leqslant y\\ \frac{s-d}{d-c}, y<s\leqslant d\end{cases} \tag{1.6}$$

The main purpose of this work is to obtain new Čebyšev type inequalities using new extension of the double integrals weighted Montgomery identity.

This paper is organized as follows: In section 2, we establish new Čebyšev type inequalities similar to (1.1) for functions of two independent variables.

In the third section, we prove a new extension of the weighted Montgomery identity for double integrals then we used it to establish new Čebyšev type inequalities.

2. Čebyšev type inequalities for double integrals

THEOREM 1. *Let $f,g: I\to\mathbb{R}$ be differentiable functions such that their second derivatives $\frac{\partial^2 f(s,t)}{\partial s\partial t}$ and $\frac{\partial^2 g(s,t)}{\partial s\partial t}$ are integrable on I. Then*

$$|T(f,g)|\leqslant\frac{49}{3600}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \tag{2.1}$$

where

$$\begin{aligned}T(f,g)=&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dxdy \\ &-\frac{1}{(b-a)^2(d-c)}\int_a^b\int_c^d g(x,y)\int_a^b f(t,y)\,dtdxdy \\ &-\frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d g(x,y)\int_c^d f(x,s)\,dsdxdy \\ &+\frac{1}{(b-a)^2(d-c)^2}\int_a^b\int_c^d f(x,s)\,dsdx\int_c^d\int_a^b g(t,y)\,dtdy.\end{aligned} \tag{2.2}$$

Proof. Let F, G, $\widetilde{F}$ and $\widetilde{G}$ be defined as follows

$$\begin{aligned}F=&f(x,y)-\frac{1}{(b-a)}\int_a^b f(t,y)\,dt-\frac{1}{(d-c)}\int_c^d f(x,s)\,ds \\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt,\end{aligned} \tag{2.3}$$

$$
\begin{aligned}
G= & g(x,y)-\frac{1}{(b-a)}\int_a^b g(t,y)\,dt-\frac{1}{(d-c)}\int_c^d g(x,s)\,ds \\
& +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(t,s)\,dsdt, \\
\widetilde{F}= & \int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt,
\end{aligned}
\tag{2.4}
$$

and

$$
\widetilde{G}=\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt.
$$

Since

$$
FG=\widetilde{F}\widetilde{G}
$$

integrating FG over I, multiplying the resultant equality by $\frac{1}{(b-a)(d-c)}$ it yields

$$
\begin{aligned}
& \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dxdy \\
& -\frac{1}{(b-a)^2(d-c)}\int_a^b\int_c^d g(x,y)\int_a^b f(t,y)\,dtdxdy \\
& -\frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d g(x,y)\int_c^d f(x,s)\,dsdxdy \\
& +\frac{1}{(b-a)^2(d-c)^2}\int_a^b\int_c^d f(x,s)\,ds\,dx\int_c^d\int_a^b g(t,y)\,dtdy \\
= & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left(\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right) \\
& \times\left(\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right)\qquad dxdy.
\end{aligned}
\tag{2.5}
$$

Consequently

$$
\begin{aligned}
|T(f,g)|\leqslant & \frac{1}{(b-a)(d-c)}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \\
& \times\int_a^b\int_c^d\left(\int_a^b\int_c^d |P(x,t)\,Q(y,s)|\,dsdt\right)^2 dxdy \\
= & \frac{1}{16(b-a)^3(d-c)^3}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \\
& \times\int_a^b\int_c^d\left(\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]\right)^2 dxdy \\
= & \frac{49}{3600}(b-a)^2(d-c)^2\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty. \qquad \square
\end{aligned}
\tag{2.6}
$$

THEOREM 2. *Under the hypothesis of Theorem 1, we have*

$$|T(f,g)| \leqslant \frac{1}{8(b-a)^2(d-c)^2}\int_a^b\int_c^d \left[|g(x,y)|\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\right. \tag{2.7}$$
$$\left.+|f(x,y)|\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\right]$$
$$\times\left[\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right]dxdy.$$

Proof. Applying identity (1.5) to the function g we get

$$g(x,y) = \frac{1}{(b-a)}\int_a^b g(t,y)\,dt + \frac{1}{(d-c)}\int_c^d g(x,s)\,ds \tag{2.8}$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(t,s)\,dsdt$$
$$+\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt.$$

Multiplying (1.5) by $\frac{1}{(b-a)(d-c)}g(x,y)$, (2.8) by $\frac{1}{(b-a)(d-c)}f(x,y)$, summing the resultant equalities, then integrating on I, we obtain

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dxdy$$
$$=\frac{1}{(b-a)^2(d-c)}\int_a^b\int_c^d g(x,y)\int_a^b f(t,y)\,dtdxdy$$
$$\frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d g(x,y)\int_c^d f(x,s)\,ds\,dxdy$$
$$-\frac{1}{(b-a)^2(d-c)^2}\int_a^b\int_c^d g(x,y)\,dxdy\int_a^b\int_c^d f(t,s)\,dsdt$$
$$+\frac{1}{2}\frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d g(x,y)\left(\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right)dxdy\right.$$
$$\left.+\int_a^b\int_c^d f(x,y)\left(\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right)dxdy\right],$$

from that we deduce

$$T(f,g) = \frac{1}{2}\frac{1}{(b-a)(d-c)} \tag{2.9}$$
$$\times\left[\int_a^b\int_c^d g(x,y)\left(\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right)dxdy\right.$$
$$\left.+\int_a^b\int_c^d f(x,y)\left(\int_a^b\int_c^d P(x,t)\,Q(y,s)\,\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right)dxdy\right],$$

consequently

$$|T(f,g)| \leqslant \frac{1}{8(b-a)^2(d-c)^2} \int_a^b \int_c^d \left[|g(x,y)| \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \right.$$
$$\left. + |f(x,y)| \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \right]$$
$$\times \left[\left((x-a)^2 + (b-x)^2 \right) \left((y-c)^2 + (d-y)^2 \right) \right] dxdy. \qquad \square$$

3. Čebyšev type inequalities for double weighted integrals

Let $w : [a,b] \to [0,+\infty[$ be a probability density function and set $W(t) = \int\limits_a^t w(x)\,dx$ for $a \leqslant t \leqslant b$, so $W(a) = 0$ and $W(b) = 1$.

Let $\varphi : [c,d] \to [0,+\infty[$ a probability density function and set $\Psi(s) = \int\limits_c^s \varphi(y)\,dy$ for $c \leqslant s \leqslant d$, so $\Psi(c) = 0$, $\Psi(d) = 1$.

Now we give a new extension of the weighted Montgomery identity for functions of two independent variables.

REMARK 1. If we take $p(t,s) = w(t)\varphi(s)$, $t \in [a,b]$, $s \in [c,d]$, in Theorem 1 from [9], then weighted Montgomery identity can be written as

$$f(x,y) = \int_a^b w(t) f(t,y)\,dt + \int_c^d \varphi(s) f(x,s)\,ds \tag{3.1}$$
$$- \int_a^b w(t) \int_c^d \varphi(s) f(t,s)\,dsdt$$
$$+ \int_a^b \int_c^d P_w(x,t) Q_\varphi(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt$$

where $f,\ g : I \to \mathbb{R}$ are such that the partial derivatives $\frac{\partial f(s,t)}{\partial s}, \frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ exist and are continuous on I, $P_w(x,t)$ and $Q_\varphi(y,s)$ are the Peano kernel defined by

$$P_w(x,t) = \begin{cases} W(t), a \leqslant t \leqslant x \\ W(t) - 1, x < t \leqslant b \end{cases} \text{ and } Q_\varphi(y,s) = \begin{cases} \Psi(s), c \leqslant s \leqslant y \\ \Psi(s) - 1, y < s \leqslant d \end{cases}.$$

To simplify the notation, for some given functions f and $g : I \to \mathbb{R}$, we denote

$$T(w,\varphi,f,g) = \int_a^b \int_c^d w(x)\varphi(y) f(x,y) g(x,y)\,dxdy \tag{3.2}$$
$$- \int_a^b \int_c^d w(x)\varphi(y) g(x,y) \left(\int_a^b w(t) f(t,y)\,dt \right) dxdy$$
$$- \int_a^b \int_c^d w(x)\varphi(y) g(x,y) \left(\int_c^d \varphi(s) f(x,s)\,ds \right) dxdy$$

$$
\begin{aligned}
&+\left(\int_a^b w(x)\left(\int_c^d \varphi(s) f(x,s)\,ds\right)dx\right)\\
&\times\left(\int_c^d \varphi(y)\left(\int_a^b w(t) g(t,y)\,dt\right)dy\right)
\end{aligned}
$$

Now we give new Čebyšev type inequalities for double integrals:

THEOREM 3. *Let* $f,\ g: I\to\mathbb{R}$ *such that the partial derivatives* $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ *and* $\frac{\partial^2 f(s,t)}{\partial s\partial t}$ *exist and are continuous on* I. *Then*

$$
|T(w,\varphi,f,g)| \leqslant \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty \int_a^b\int_c^d w(x)\varphi(y) H^2(x,y)\,dxdy, \tag{3.3}
$$

where

$$
H(x,y)=\int_a^b\int_c^d \left|P_w(x,t) Q_\varphi(y,s)\right| dsdt
$$

Proof. Noting that the identity (3.1) is satisfied for both f and g then we can write

$$
\begin{aligned}
g(x,y) = &\int_a^b w(t) g(t,y)\,dt + \int_c^d \varphi(s) g(x,s)\,ds \\
&-\int_a^b w(t)\int_c^d \varphi(s) g(t,s)\,dsdt\\
&+\int_a^b\int_c^d P_w(x,t) Q_\varphi(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s} dsdt
\end{aligned} \tag{3.4}
$$

Multiplying (3.1) by (3.4) we obtain

$$
\begin{aligned}
&\left[f(x,y)-\int_a^b w(t) f(t,y)\,dt-\int_c^d \varphi(s) f(x,s)\,ds+\int_a^b w(t)\int_c^d \varphi(s) f(t,s)\,dsdt\right]\\
&\times\left[g(x,y)-\int_a^b w(t) g(t,y)\,dt-\int_c^d \varphi(s) g(x,s)\,ds+\int_a^b w(t)\int_c^d \varphi(s) g(t,s)\,dsdt\right]\\
&=\left[\int_a^b\int_c^d P_w(x,t) Q_\varphi(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right]\left[\int_a^b\int_c^d P_w(x,t) Q_\varphi(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right].
\end{aligned} \tag{3.5}
$$

Consequently,

$$
\begin{aligned}
&f(x,y) g(x,y) - f(x,y)\int_a^b w(t) g(t;y)\,dt\\
&-f(x,y)\int_c^d \varphi(s) g(x,s)\,ds + f(x,y)\int_a^b w(t)\int_c^d \varphi(s) g(t,s)\,dsdt\\
&-g(x,y)\int_a^b w(t) f(t,y)\,dt+\int_a^b w(t) f(t,y)\,dt\int_a^b w(t) g(t,y)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_a^b w(t) f(t,y)\,dt \int_c^d \varphi(s)\, g(x,s)\,ds \\
&-\int_a^b w(t) f(t,y)\,dt \int_a^b w(t) \int_c^d \varphi(s)\, g(t,s)\,dsdt \\
&-g(x,y)\int_c^d \varphi(s) f(x,s)\,ds + \int_c^d \varphi(s) f(x,s)\,ds \int_a^b w(t)\, g(t;y)\,dt \\
&+\int_c^d \varphi(s) f(x,s)\,ds \int_c^d \varphi(s)\, g(x,s)\,ds \\
&-\int_c^d \varphi(s) f(x,s)\,ds \int_a^b w(t) \int_c^d \varphi(s)\, g(t,s) \\
&+g(x,y)\int_a^b w(t) \int_c^d \varphi(s) f(t,s)\,dsdt \\
&-\int_a^b w(t)\, g(t;y)\,dt \int_a^b w(t) \int_c^d \varphi(s) f(t,s)\,dsdt \\
&-\int_c^d \varphi(s)\, g(x,s)\,ds \int_a^b w(t) \int_c^d \varphi(s) f(t,s)\,dsdt \\
&+\int_a^b w(t) \int_c^d \varphi(s)\, g(t,s)\,dsdt \int_a^b w(t) \int_c^d \varphi(s) f(t,s)\,dsdt \\
=&\left(\int_a^b \int_c^d P_w(x,t)\, Q_\varphi(y,s)\, \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt\right)\left(\int_a^b \int_c^d P_w(x,t)\, Q_\varphi(y,s)\, \frac{\partial^2 g(t,s)}{\partial t \partial s} dsdt\right).
\end{aligned}
\tag{3.6}
$$

Multiplying both sides of (3.6) by $w(x)\varphi(y)$, then integrating the resultant identity over I, we get:

$$
\begin{aligned}
T(w,\varphi,f,g) = \int_a^b \int_c^d w(x)\,\varphi(y) & \\
&\left[\left(\int_a^b \int_c^d P_w(x,t)\, Q_\varphi(y,s)\, \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt\right)\right. \\
&\left.\times\left(\int_a^b \int_c^d P_w(x,t)\, Q_\varphi(y,s)\, \frac{\partial^2 g(t,s)}{\partial t \partial s} dsdt\right)\right] dxdy,
\end{aligned}
\tag{3.7}
$$

that implies

$$
\begin{aligned}
|T(w,\varphi,f,g)| &\leqslant \left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty \left\|\frac{\partial^2 g(t,s)}{\partial t \partial s}\right\|_\infty \\
&\quad\times \int_a^b \int_c^d w(x)\,\varphi(y) \left(\int_a^b \int_c^d \left|P_w(x,t)\, Q_\varphi(y,s)\right| dsdt\right)^2 dxdy \\
&= \left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty \left\|\frac{\partial^2 g(t,s)}{\partial t \partial s}\right\|_\infty \int_a^b \int_c^d w(x)\,\varphi(y)\, H(x,y)\, dxdy.
\end{aligned}
$$

This completes the proof of Theorem 3. $\square$

THEOREM 4. *Under the hypothesis of Theorem 3, we have*

$$|T(w,\varphi,f,g)| \leqslant \frac{1}{2}\int_a^b \int_c^d w(x)\varphi(y)\left[|g(x,y)|\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty + |f(x,y)|\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\right] H(x,y)\,dxdy. \tag{3.8}$$

Proof. Multiplying (3.1) by $w(x)\varphi(y)g(x,y)$ and (3.4) by $w(x)\varphi(y)f(x,y)$, summing the resultant identities, then integrating over I, we obtain

$$\begin{aligned}
&\int_a^b \int_c^d w(x)\varphi(y)f(x,y)g(x,y)\,dxdy \qquad (3.9)\\
&= \frac{1}{2}\left[\int_a^b \int_c^d w(x)\varphi(y)g(x,y)\int_a^b w(t)f(t,y)\,dtdxdy\right.\\
&\quad + \int_a^b \int_c^d w(x)\varphi(y)g(x,y)\int_c^d \varphi(s)f(x,s)\,dsdxdy\\
&\quad - \int_a^b \int_c^d w(x)\varphi(y)g(x,y)\,dxdy\int_a^b \int_c^d w(t)\varphi(s)f(t,s)\,dsdt\\
&\quad \int_a^b \int_c^d w(x)\varphi(y)f(x,y)\int_a^b w(t)g(t,y)\,dtdxdy\\
&\quad + \int_a^b \int_c^d w(x)\varphi(y)f(x,y)\int_c^d \varphi(s)g(x,s)\,dsdxdy\\
&\quad \left. - \int_a^b \int_c^d w(x)\varphi(y)f(x,y)\,dxdy\int_a^b \int_c^d w(t)\varphi(s)g(t,s)\,dsdt\right]\\
&\quad + \frac{1}{2}\left[\int_a^b \int_c^d w(x)\varphi(y)g(x,y)\left(\int_a^b \int_c^d P_w(x,t)Q_\varphi(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right)dxdy\right.\\
&\quad \left. + \int_a^b \int_c^d w(x)\varphi(y)f(x,y)\left(\int_a^b \int_c^d P_w(x,t)Q_\varphi(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right)dxdy\right].
\end{aligned}$$

So,

$$\begin{aligned}
T(w,\varphi,f,g) = \frac{1}{2}&\left[\int_a^b \int_c^d w(x)\varphi(y)g(x,y)\right.\\
&\times\left(\int_a^b \int_c^d P_w(x,t)Q_\varphi(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right)dxdy\\
&+ \int_a^b \int_c^d w(x)\varphi(y)f(x,y)\\
&\left.\times\left(\int_a^b \int_c^d P_w(x,t)Q_\varphi(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right)dxdy\right].
\end{aligned} \tag{3.10}$$

From (3.10) and using module's properties, it yields

$$|T(w,\varphi,f,g)| \leqslant \frac{1}{2}\left[\int_a^b \int_c^d w(x)\varphi(y)|g(x,y)|\right.$$

$$
\begin{aligned}
&\times\left(\int_a^b\int_c^d \left|P_w(x,t)\,Q_\varphi(y,s)\right|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| dsdt\right) dxdy\\
&+\int_a^b\int_c^d w(x)\,\varphi(y)\,|f(x,y)|\\
&\times\left(\int_a^b\int_c^d \left|P_w(x,t)\,Q_\varphi(y,s)\right|\left|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right| dsdt\right) dxdy\Bigg]\\
\leqslant\ & \frac{1}{2}\int_a^b\int_c^d w(x)\,\varphi(y)\left[|g(x,y)|\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\right.\\
&\left.+|f(x,y)|\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\right]\\
&\left(\int_a^b\int_c^d \left|P_w(x,t)\,Q_\varphi(y,s)\right| dsdt\right) dxdy\\
=\ & \frac{1}{2}\int_a^b\int_c^d w(x)\,\varphi(y)\left[|g(x,y)|\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\right.\\
&\left.+|f(x,y)|\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\right] H(x,y)\,dxdy
\end{aligned}
$$

This completes the proof of Theorem 4. □

REMARK 2. We can mention that results from Theorems 3 and 4 for uniform weighted functions are reduced to results of Theorems 1 and 2.

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