

NEW ČEBYŠEV TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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Abstract. The aim of this paper is to establish new extension of the weighted Montgomery identity for functions of two independent variables, then obtain new Čebyšev type inequalities.

1. Introduction

In [4], Čebyšev proved for two absolutely continuous functions $f, g : [a, b] \rightarrow \mathbb{R}$ that

$$|T(f, g)| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty, \quad (1.1)$$

where the functional $T(f, g)$ is defined by

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right), \quad (1.2)$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a, b]$ defined as $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [a, b]} |f(t)|$.

In [6] the authors gives the following Montgomery identity

$$f(x) = \frac{1}{(b-a)} \int_a^b f(t) dt + \int_a^b P(x, t) f'(t) dt \quad (1.3)$$

where

$$P(x, t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x \\ \frac{t-b}{b-a}, & x < t \leq b \end{cases} \quad (1.4)$$

is the Peano kernel.

In the last years, many papers were devoted to the generalization of Čebyšev type inequalities, we can mention the works [1,3,5,7-11]. Find new inequalities in the multidimensional cases still an interesting problem. In [2,5], the authors proved the double integrals Montgomery identity:

$$f(x, y) = \frac{1}{(b-a)} \int_a^b f(t, y) dt + \frac{1}{(d-c)} \int_c^d f(x, s) ds \quad (1.5)$$

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$$\begin{aligned}
& - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) ds dt \\
& + \int_a^b \int_c^d P(x,t) Q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt,
\end{aligned}$$

where $f: I = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is differentiable, the derivative $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ is integrable on I , $P(x,t)$ is defined by (1.4) and the Peano kernel $Q(y,s)$ is defined by

$$Q(y,s) = \begin{cases} \frac{s-c}{d-c}, c \leq s \leq y \\ \frac{s-d}{d-c}, y < s \leq d \end{cases} \quad (1.6)$$

The main purpose of this work is to obtain new Čebyšev type inequalities using new extension of the double integrals weighted Montgomery identity.

This paper is organized as follows: In section 2, we establish new Čebyšev type inequalities similar to (1.1) for functions of two independent variables.

In the third section, we prove a new extension of the weighted Montgomery identity for double integrals then we used it to establish new Čebyšev type inequalities.

2. Čebyšev type inequalities for double integrals

THEOREM 1. *Let $f, g: I \rightarrow \mathbb{R}$ be differentiable functions such that their second derivatives $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ and $\frac{\partial^2 g(s,t)}{\partial s \partial t}$ are integrable on I . Then*

$$|T(f,g)| \leq \frac{49}{3600} (b-a)^2 (d-c)^2 \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_{\infty} \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_{\infty} \quad (2.1)$$

where

$$\begin{aligned}
T(f,g) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) g(x,y) dx dy \\
& - \frac{1}{(b-a)^2 (d-c)} \int_a^b \int_c^d g(x,y) \int_a^b f(t,y) dt dx dy \\
& - \frac{1}{(b-a)(d-c)^2} \int_a^b \int_c^d g(x,y) \int_c^d f(x,s) ds dx dy \\
& + \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d f(x,s) ds dx \int_c^d \int_a^b g(t,y) dt dy.
\end{aligned} \quad (2.2)$$

Proof. Let F, G, \tilde{F} and \tilde{G} be defined as follows

$$\begin{aligned}
F &= f(x,y) - \frac{1}{(b-a)} \int_a^b f(t,y) dt - \frac{1}{(d-c)} \int_c^d f(x,s) ds \\
& + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s) ds dt,
\end{aligned} \quad (2.3)$$

$$\begin{aligned}
 G &= g(x, y) - \frac{1}{(b-a)} \int_a^b g(t, y) dt - \frac{1}{(d-c)} \int_c^d g(x, s) ds \\
 &\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(t, s) ds dt, \\
 \tilde{F} &= \int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt,
 \end{aligned}
 \tag{2.4}$$

and

$$\tilde{G} = \int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt.$$

Since

$$FG = \tilde{F}\tilde{G}$$

integrating FG over I , multiplying the resultant equality by $\frac{1}{(b-a)(d-c)}$ it yields

$$\begin{aligned}
 &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dx dy \\
 &- \frac{1}{(b-a)^2(d-c)} \int_a^b \int_c^d g(x, y) \int_a^b f(t, y) dt dx dy \\
 &- \frac{1}{(b-a)(d-c)^2} \int_a^b \int_c^d g(x, y) \int_c^d f(x, s) ds dx dy \\
 &+ \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_c^d f(x, s) ds dx \int_c^d \int_a^b g(t, y) dt dy \\
 &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left(\int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) \\
 &\quad \times \left(\int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right) dx dy.
 \end{aligned}
 \tag{2.5}$$

Consequently

$$\begin{aligned}
 |T(f, g)| &\leq \frac{1}{(b-a)(d-c)} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty} \\
 &\quad \times \int_a^b \int_c^d \left(\int_a^b \int_c^d |P(x, t) Q(y, s)| ds dt \right)^2 dx dy \\
 &= \frac{1}{16(b-a)^3(d-c)^3} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty} \\
 &\quad \times \int_a^b \int_c^d \left([(x-a)^2 + (b-x)^2] [(y-c)^2 + (d-y)^2] \right)^2 dx dy \\
 &= \frac{49}{3600} (b-a)^2 (d-c)^2 \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty}. \quad \square
 \end{aligned}
 \tag{2.6}$$

THEOREM 2. *Under the hypothesis of Theorem 1, we have*

$$\begin{aligned}
 |T(f, g)| \leq & \frac{1}{8(b-a)^2(d-c)^2} \int_a^b \int_c^d \left[|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \right. \\
 & \left. + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty} \right] \\
 & \times \left[\left((x-a)^2 + (b-x)^2 \right) \left((y-c)^2 + (d-y)^2 \right) \right] dx dy.
 \end{aligned} \tag{2.7}$$

Proof. Applying identity (1.5) to the function g we get

$$\begin{aligned}
 g(x, y) = & \frac{1}{(b-a)} \int_a^b g(t, y) dt + \frac{1}{(d-c)} \int_c^d g(x, s) ds \\
 & + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(t, s) ds dt \\
 & + \int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt.
 \end{aligned} \tag{2.8}$$

Multiplying (1.5) by $\frac{1}{(b-a)(d-c)}g(x, y)$, (2.8) by $\frac{1}{(b-a)(d-c)}f(x, y)$, summing the resultant equalities, then integrating on I , we obtain

$$\begin{aligned}
 & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) g(x, y) dx dy \\
 = & \frac{1}{(b-a)^2(d-c)} \int_a^b \int_c^d g(x, y) \int_a^b f(t, y) dt dx dy \\
 & \frac{1}{(b-a)(d-c)^2} \int_a^b \int_c^d g(x, y) \int_c^d f(x, s) ds dx dy \\
 & - \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_c^d g(x, y) dx dy \int_a^b \int_c^d f(t, s) ds dt \\
 & + \frac{1}{2} \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d g(x, y) \left(\int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) dx dy \right. \\
 & \left. + \int_a^b \int_c^d f(x, y) \left(\int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right) dx dy \right],
 \end{aligned}$$

from that we deduce

$$\begin{aligned}
 T(f, g) = & \frac{1}{2} \frac{1}{(b-a)(d-c)} \\
 & \times \left[\int_a^b \int_c^d g(x, y) \left(\int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) dx dy \right. \\
 & \left. + \int_a^b \int_c^d f(x, y) \left(\int_a^b \int_c^d P(x, t) Q(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right) dx dy \right],
 \end{aligned} \tag{2.9}$$

consequently

$$\begin{aligned}
 |T(f, g)| \leq & \frac{1}{8(b-a)^2(d-c)^2} \int_a^b \int_c^d \left[|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \right. \\
 & \left. + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty} \right] \\
 & \times \left[\left((x-a)^2 + (b-x)^2 \right) \left((y-c)^2 + (d-y)^2 \right) \right] dx dy. \quad \square
 \end{aligned}$$

3. Čebyšev type inequalities for double weighted integrals

Let $w : [a, b] \rightarrow [0, +\infty[$ be a probability density function and set $W(t) = \int_a^t w(x) dx$ for $a \leq t \leq b$, so $W(a) = 0$ and $W(b) = 1$.

Let $\varphi : [c, d] \rightarrow [0, +\infty[$ a probability density function and set $\Psi(s) = \int_c^s \varphi(y) dy$ for $c \leq s \leq d$, so $\Psi(c) = 0$, $\Psi(d) = 1$.

Now we give a new extension of the weighted Montgomery identity for functions of two independent variables.

REMARK 1. If we take $p(t, s) = w(t) \varphi(s)$, $t \in [a, b]$, $s \in [c, d]$, in Theorem 1 from [9], then weighted Montgomery identity can be written as

$$\begin{aligned}
 f(x, y) = & \int_a^b w(t) f(t, y) dt + \int_c^d \varphi(s) f(x, s) ds \tag{3.1} \\
 & - \int_a^b w(t) \int_c^d \varphi(s) f(t, s) ds dt \\
 & + \int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt
 \end{aligned}$$

where $f, g : I \rightarrow \mathbb{R}$ are such that the partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ exist and are continuous on I , $P_w(x, t)$ and $Q_\varphi(y, s)$ are the Peano kernel defined by

$$P_w(x, t) = \begin{cases} W(t), & a \leq t \leq x \\ W(t) - 1, & x < t \leq b \end{cases} \text{ and } Q_\varphi(y, s) = \begin{cases} \Psi(s), & c \leq s \leq y \\ \Psi(s) - 1, & y < s \leq d \end{cases} .$$

To simplify the notation, for some given functions f and $g : I \rightarrow \mathbb{R}$, we denote

$$\begin{aligned}
 T(w, \varphi, f, g) = & \int_a^b \int_c^d w(x) \varphi(y) f(x, y) g(x, y) dx dy \tag{3.2} \\
 & - \int_a^b \int_c^d w(x) \varphi(y) g(x, y) \left(\int_a^b w(t) f(t, y) dt \right) dx dy \\
 & - \int_a^b \int_c^d w(x) \varphi(y) g(x, y) \left(\int_c^d \varphi(s) f(x, s) ds \right) dx dy
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_a^b w(x) \left(\int_c^d \varphi(s) f(x,s) ds \right) dx \right) \\
& \times \left(\int_c^d \varphi(y) \left(\int_a^b w(t) g(t,y) dt \right) dy \right)
\end{aligned}$$

Now we give new Čebyšev type inequalities for double integrals:

THEOREM 3. Let $f, g : I \rightarrow \mathbb{R}$ such that the partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ exist and are continuous on I . Then

$$|T(w, \varphi, f, g)| \leq \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_{\infty} \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_{\infty} \int_a^b \int_c^d w(x) \varphi(y) H^2(x,y) dx dy, \quad (3.3)$$

where

$$H(x,y) = \int_a^b \int_c^d |P_w(x,t) Q_{\varphi}(y,s)| ds dt$$

Proof. Noting that the identity (3.1) is satisfied for both f and g then we can write

$$\begin{aligned}
g(x,y) &= \int_a^b w(t) g(t,y) dt + \int_c^d \varphi(s) g(x,s) ds \\
&\quad - \int_a^b w(t) \int_c^d \varphi(s) g(t,s) ds dt \\
&\quad + \int_a^b \int_c^d P_w(x,t) Q_{\varphi}(y,s) \frac{\partial^2 g(t,s)}{\partial t \partial s} ds dt
\end{aligned} \quad (3.4)$$

Multiplying (3.1) by (3.4) we obtain

$$\begin{aligned}
& \left[f(x,y) - \int_a^b w(t) f(t,y) dt - \int_c^d \varphi(s) f(x,s) ds + \int_a^b w(t) \int_c^d \varphi(s) f(t,s) ds dt \right] \\
& \times \left[g(x,y) - \int_a^b w(t) g(t,y) dt - \int_c^d \varphi(s) g(x,s) ds + \int_a^b w(t) \int_c^d \varphi(s) g(t,s) ds dt \right] \\
& = \left[\int_a^b \int_c^d P_w(x,t) Q_{\varphi}(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \right] \left[\int_a^b \int_c^d P_w(x,t) Q_{\varphi}(y,s) \frac{\partial^2 g(t,s)}{\partial t \partial s} ds dt \right].
\end{aligned} \quad (3.5)$$

Consequently,

$$\begin{aligned}
& f(x,y) g(x,y) - f(x,y) \int_a^b w(t) g(t,y) dt \\
& - f(x,y) \int_c^d \varphi(s) g(x,s) ds + f(x,y) \int_a^b w(t) \int_c^d \varphi(s) g(t,s) ds dt \\
& - g(x,y) \int_a^b w(t) f(t,y) dt + \int_a^b w(t) f(t,y) dt \int_a^b w(t) g(t,y) dt
\end{aligned}$$

$$\begin{aligned}
 & + \int_a^b w(t) f(t, y) dt \int_c^d \varphi(s) g(x, s) ds \\
 & - \int_a^b w(t) f(t, y) dt \int_a^b w(t) \int_c^d \varphi(s) g(t, s) ds dt \\
 & - g(x, y) \int_c^d \varphi(s) f(x, s) ds + \int_c^d \varphi(s) f(x, s) ds \int_a^b w(t) g(t; y) dt \\
 & + \int_c^d \varphi(s) f(x, s) ds \int_c^d \varphi(s) g(x, s) ds \\
 & - \int_c^d \varphi(s) f(x, s) ds \int_a^b w(t) \int_c^d \varphi(s) g(t, s) \\
 & + g(x, y) \int_a^b w(t) \int_c^d \varphi(s) f(t, s) ds dt \\
 & - \int_a^b w(t) g(t; y) dt \int_a^b w(t) \int_c^d \varphi(s) f(t, s) ds dt \\
 & - \int_c^d \varphi(s) g(x, s) ds \int_a^b w(t) \int_c^d \varphi(s) f(t, s) ds dt \\
 & + \int_a^b w(t) \int_c^d \varphi(s) g(t, s) ds dt \int_a^b w(t) \int_c^d \varphi(s) f(t, s) ds dt \\
 & = \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right). \tag{3.6}
 \end{aligned}$$

Multiplying both sides of (3.6) by $w(x) \varphi(y)$, then integrating the resultant identity over I , we get:

$$\begin{aligned}
 T(w, \varphi, f, g) &= \int_a^b \int_c^d w(x) \varphi(y) \tag{3.7} \\
 & \left[\left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) \right. \\
 & \left. \times \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right) \right] dx dy,
 \end{aligned}$$

that implies

$$\begin{aligned}
 |T(w, \varphi, f, g)| &\leq \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_\infty \\
 & \times \int_a^b \int_c^d w(x) \varphi(y) \left(\int_a^b \int_c^d |P_w(x, t) Q_\varphi(y, s)| ds dt \right)^2 dx dy \\
 & = \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_\infty \int_a^b \int_c^d w(x) \varphi(y) H(x, y) dx dy.
 \end{aligned}$$

This completes the proof of Theorem 3. \square

THEOREM 4. *Under the hypothesis of Theorem 3, we have*

$$|T(w, \varphi, f, g)| \leq \frac{1}{2} \int_a^b \int_c^d w(x) \varphi(y) \left[|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_\infty \right] H(x, y) dx dy. \quad (3.8)$$

Proof. Multiplying (3.1) by $w(x) \varphi(y) g(x, y)$ and (3.4) by $w(x) \varphi(y) f(x, y)$, summing the resultant identities, then integrating over I , we obtain

$$\begin{aligned} & \int_a^b \int_c^d w(x) \varphi(y) f(x, y) g(x, y) dx dy \quad (3.9) \\ = & \frac{1}{2} \left[\int_a^b \int_c^d w(x) \varphi(y) g(x, y) \int_a^b w(t) f(t, y) dt dx dy \right. \\ & + \int_a^b \int_c^d w(x) \varphi(y) g(x, y) \int_c^d \varphi(s) f(x, s) ds dx dy \\ & - \int_a^b \int_c^d w(x) \varphi(y) g(x, y) dx dy \int_a^b \int_c^d w(t) \varphi(s) f(t, s) ds dt \\ & \quad \left. - \int_a^b \int_c^d w(x) \varphi(y) f(x, y) \int_a^b w(t) g(t, y) dt dx dy \right. \\ & + \int_a^b \int_c^d w(x) \varphi(y) f(x, y) \int_c^d \varphi(s) g(x, s) ds dx dy \\ & \quad \left. - \int_a^b \int_c^d w(x) \varphi(y) f(x, y) dx dy \int_a^b \int_c^d w(t) \varphi(s) g(t, s) ds dt \right] \\ & + \frac{1}{2} \left[\int_a^b \int_c^d w(x) \varphi(y) g(x, y) \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) dx dy \right. \\ & \left. + \int_a^b \int_c^d w(x) \varphi(y) f(x, y) \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right) dx dy \right]. \end{aligned}$$

So,

$$\begin{aligned} T(w, \varphi, f, g) = & \frac{1}{2} \left[\int_a^b \int_c^d w(x) \varphi(y) g(x, y) \right. \quad (3.10) \\ & \times \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \right) dx dy \\ & + \int_a^b \int_c^d w(x) \varphi(y) f(x, y) \\ & \left. \times \left(\int_a^b \int_c^d P_w(x, t) Q_\varphi(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt \right) dx dy \right]. \end{aligned}$$

From (3.10) and using module's properties, it yields

$$|T(w, \varphi, f, g)| \leq \frac{1}{2} \left[\int_a^b \int_c^d w(x) \varphi(y) |g(x, y)| \right.$$

$$\begin{aligned}
& \times \left(\int_a^b \int_c^d |P_w(x,t) Q_\varphi(y,s)| \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| ds dt \right) dx dy \\
& + \int_a^b \int_c^d w(x) \varphi(y) |f(x,y)| \\
& \times \left(\int_a^b \int_c^d |P_w(x,t) Q_\varphi(y,s)| \left| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right| ds dt \right) dx dy \Big] \\
& \leq \frac{1}{2} \int_a^b \int_c^d w(x) \varphi(y) \left[|g(x,y)| \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \right. \\
& \quad \left. + |f(x,y)| \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \right] \\
& \quad \left(\int_a^b \int_c^d |P_w(x,t) Q_\varphi(y,s)| ds dt \right) dx dy \\
& = \frac{1}{2} \int_a^b \int_c^d w(x) \varphi(y) \left[|g(x,y)| \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \right. \\
& \quad \left. + |f(x,y)| \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \right] H(x,y) dx dy
\end{aligned}$$

This completes the proof of Theorem 4. \square

REMARK 2. We can mention that results from Theorems 3 and 4 for uniform weighted functions are reduced to results of Theorems 1 and 2.

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