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FOUR INEQUALITIES OF VOLKMANN TYPE

WŁODZIMIERZ FECHNER

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Abstract. We deal with four functional inequalities which are motivated by a result of A. Chaljub-Simon and P. Volkmann from 1994 and by several later results concerning the following two equations:

$$\max\{f(x+y), f(x-y)\} = f(x)+f(y) \quad \text{(for each } x,y)$$
$$\min\{f(x+y), f(x-y)\} = |f(x)-f(y)| \quad \text{(for each } x,y).$$

The purpose of the paper is to establish some basic properties of the inequalities discussed and to compare them with some well known classical functional inequalities, such as the inequality of subadditivity or the inequality of Jensen-quasiconvexity.

1. Introduction

In [6] we dealt with the following functional inequality:

$$|f(x)-f(y)| \leqslant f(x+y)+f(x-y)-f(x)-f(y) \leqslant \min\{f(x+y), f(x-y)\} \tag{1}$$

and with the corresponding functional equation:

$$|f(x)-f(y)| = f(x+y)+f(x-y)-f(x)-f(y) \tag{2}$$

for a real mapping f acting on an Abelian group G. In particular, using a generalization of the Hahn-Banach theorem due to R. Ger [7] we proved that each mapping $f\colon G \to \mathbb{R}$ which satisfies (1) for all $x,y \in G$ jointly with $f(0)=0$ is of the form

$$f(x) = \|A(x)\|, \quad x \in G,$$

where $A\colon G \to X$ is an additive mapping having values in a certain normed linear space X. Then, using this result, we derived the general solution of (2) as

$$f(x) = |A(x)| + c, \quad x \in G$$

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with an arbitrary additive mapping $A\colon G \to \mathbb{R}$ and a constant $c \in \mathbb{R}$. Later, P. Volkmann (private communication) observed that equation (2) for mapping $f\colon G \to \mathbb{R}$ vanishing at zero can be transformed equivalently to the following equation:

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y), \tag{3}$$

which was first discussed in A. Chaljub-Simon and P. Volkmann [4]. Recently a new proof and an interesting application of their result were obtained by T. Kochanek [12]. Some more results and comments can be found in a paper of W. Jarczyk and P. Volkmann [11].

The following equation:

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|, \tag{4}$$

which is in a sense complementary to equation (3) but not equivalent to it, was introduced in [4]. Some related results are due to R.M. Redheffer and P. Volkmann [13] and more recently the topic was studied by K. Baron and P. Volkmann [3] in more general settings.

Therefore, it is well-motivated to ask about solutions of the following functional inequalities:

$$\max\{f(x+y), f(x-y)\} \leqslant f(x) + f(y), \tag{A}$$

$$\max\{f(x+y), f(x-y)\} \geqslant f(x) + f(y), \tag{B}$$

$$\min\{f(x+y), f(x-y)\} \geqslant |f(x) - f(y)|, \tag{C}$$

$$\min\{f(x+y), f(x-y)\} \leqslant |f(x) - f(y)|. \tag{D}$$

All the above inequalities are discussed in the subsequent sections.

Recently, A. Gilányi [8] proposed to call a broad class of related functional inequalities *Volkmann type inequalities* due to outstanding contribution of Professor Peter Volkmann to this field. Therefore, we take the liberty to put this name into use and call inequalities (A)-(D) to be of Volkmann type.

A straightforward observation concerning these inequalities is the lack of symmetry between (A) and (B) and between (C) and (D), respectively. In fact, (A) can be rewritten as a system of two simultaneous inequalities:

$$\begin{cases} f(x+y) \leqslant f(x) + f(y); \\ f(x-y) \leqslant f(x) + f(y), \end{cases}$$

whereas inequality (B) can be written an alternative of two inequalities:

$$\underset{x,y}{\forall} [f(x+y) \leqslant f(x) + f(y) \quad \text{or} \quad f(x-y) \leqslant f(x) + f(y)].$$

Similarly, (C) is a system of two simultaneous inequalities and (D) is a respective alternative. Therefore, one may expect relatively broad classes of solutions of inequalities (B) and (D) and fewer solutions of (A) and (C).

2. Inequality (A)

LEMMA 2.1. *Assume that* $(G,+)$ *is an Abelian group and* $f\colon G \to \mathbb{R}$ *satisfies inequality* (A) *for all* $x,y \in G$. *Then:*

(a) $\max\{f(2x), f(0)\} \leqslant 2f(x)$ *for each* $x \in G$,

(b) f *is nonnegative,*

(c) f *is subadditive.*

Proof. Substitute $x=0$ and $y=0$ in (A) to get $f(0) \geqslant 0$. Then, substitution $y=x$ in (A) immediately implies assertion (a), which, jointly with the previous observation gives us

$$0 \leqslant f(0) \leqslant \max\{f(2x), f(0)\} \leqslant 2f(x),$$

and therefore (b) is proved. The last assertion is obvious. $\square$

Since each solution of (A) is a nonnegative subadditive mapping, one may ask whether the converse is also true. However, the following example provides a negative answer to this question.

EXAMPLE 2.2. Let us define $f_1\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$f_1(x) = \begin{cases} \frac{1}{2}, & x < 0, \\ x + \frac{1}{2}, & x \geqslant 0. \end{cases}$$

Clearly, f_1 is nonnegative and one can easily see that f_1 is subadditive. However, inequality (A) is not fulfilled; to see this it is enough to take $x=0$ and $y=-1$:

$$\max\{f_1(-1), f_1(1)\} = 2 > 1 = f_1(0) + f_1(-1).$$

Therefore, we need to seek for an additional condition under which we are able to obtain a full description of solutions of inequality (A).

THEOREM 2.3. *Assume that* $(G,+)$ *is an Abelian group and* $f\colon G \to \mathbb{R}$ *satisfies* $f(0)=0$. *Then* f *fulfills inequality* (A) *for all* $x,y \in G$ *if and only if* f *is even and subadditive mapping.*

Proof. If f satisfies (A) then clearly f is subadditive. Additionally, we have

$$f(x-y) \leqslant f(x) + f(y), \quad x,y \in G.$$

Substitute $x=0$ in the above inequality and apply the condition $f(0)=0$ to deduce the estimate

$$f(-y) \leqslant f(0) + f(y) = f(y), \quad y \in G.$$

Now, to obtain the evenness of f, it is enough to replace y by $-y$.

To justify the converse implication it suffices to observe that if f is even and subadditive then

$$f(x-y)=f(x+(-y))\leqslant f(x)+f(-y)=f(x)+f(y),$$

which, jointly with the subadditivity of f, implies inequality (A). $\square$

Now, we will show that the condition $f(0)=0$ in the foregoing theorem can be slightly relaxed.

THEOREM 2.4. *Assume that* $(G,+)$ *is an Abelian group and* $f\colon G\to\mathbb{R}$ *satisfies inequality* (A) *for all* $x,y\in G$. *If* f *vanishes at any point* $x_0\in G$ *then* f *vanishes at zero.*

Moreover, if the set

$$K=\{x\in G: f(x)=0\}$$

is nonempty then K *forms a subgroup of* G.

Proof. If $f(x_0)=0$ then, by Lemma 2.1 we deduce that

$$0\leqslant f(0)\leqslant \max\{f(2x_0),f(0)\}\leqslant 2f(x_0)=0,$$

thus $f(0)=0$.

Next, if $x,y\in K$ then clearly

$$0\leqslant \max\{f(x+y),f(x-y)\}\leqslant f(x)+f(y)=0;$$

thus, in particular, $f(x-y)=0$, i.e. $x-y\in K$. $\square$

In view of the foregoing statements, it remains to discuss solutions of (A) which do not have zeros. Our next easy example shows that there is a lot of freedom to construct such solutions of (A).

EXAMPLE 2.5. Let $(G,+)$ be an arbitrary Abelian group, let $M>0$ be arbitrarily fixed constant and let $f_2\colon G\to\mathbb{R}$ be an arbitrary function such that $f_2(G)\subset[M,2M]$. Then f_2 provides a solution of inequality (A). Indeed, for each $x,y\in G$ we have

$$\max\{f_2(x+y),f_2(x-y)\}\leqslant 2M\leqslant f_2(x)+f_2(y).$$

3. Inequality (B)

In this section we will focus on inequality (B) and on a related functional equation. Let us recall the notion of quasiconvexity and Jensen-quasiconvexity. The terminology is in accordance with the survey paper H.J. Greenberg and W.P. Pierskalla [10] and with the monograph A.W. Roberts and D.E. Varberg [14]. If $I \subset \mathbb{R}$ is a nonempty interval and $f\colon I \to \mathbb{R}$ is an arbitrary mapping, then f is called *quasiconvex* if it satisfies

$$f(u) \leqslant \max\{f(x), f(y)\}, \quad x,y,u \in I, x \leqslant u \leqslant y, \tag{5}$$

and f is *Jensen-quasiconvex* if it fulfills

$$f\left(\frac{x+y}{2}\right) \leqslant \max\{f(x), f(y)\}, \quad x,y \in I. \tag{6}$$

Clearly, if a map f is quasiconvex then it is Jensen-quasiconvex. In 2004 A. Gilányi, K. Nikodem and Zs. Páles [9] proved that in the class of upper semicontinuous functions quasiconvexity and Jensen-quasiconvexity are equivalent. On the other hand, it is well known that in general settings the quasiconvexity is an essentially weaker property than the classical convexity.

Note that inequality (6) can be transformed into

$$f(x) = f\left(\frac{x+y+x-y}{2}\right) \leqslant \max\{f(x+y), f(x-y)\}. \tag{7}$$

Therefore, it is reasonable for a real mapping defined on an arbitrary Abelian group to call it Jensen-quasiconvex if it solves inequality (7).

Now, we are able to provide some straightforward examples of solutions of inequality (B).

EXAMPLE 3.1. Let $(G,+)$ be an arbitrary Abelian group. Each superadditive mapping defined on G provides a solution of inequality (B). Further, each nonpositive Jensen-quasiconvex map on G satisfies (B). Moreover, if f solves (A) then $-f$ solves (B) (it is clear that the converse implication is not true: if f fulfills (B) then $-f$ needs not to fulfill (A)). Finally, $f_3\colon \mathbb{R} \to \mathbb{R}$ given by $f_3(x) = x^2$ satisfies inequality (B). Indeed,

$$\max\{(x+y)^2, (x-y)^2\} = x^2 + y^2 + 2|xy| \geqslant x^2 + y^2$$

for each $x,y \in \mathbb{R}$. Function f_3 serves as an example of convex (and thus Jensen-quasiconvex) solution of (B) which is not nonpositive. Moreover, f_3 is not superadditive.

In view of the foregoing examples we see that inequality (B) is relatively weak. Therefore, to obtain a reasonable description of solutions of (B) we need to impose some additional conditions.

Now, let us rewrite inequality (B) as an alternative:

$$\forall_{x,y} [\Phi_1(x,y) \text{ or } \Phi_2(x,y)], \tag{8}$$

where

$$\Phi_1(x,y) \equiv [f(x+y) - f(x) - f(y) \geqslant 0]$$

and

$$\Phi_2(x,y) \equiv [f(x-y) - f(x) - f(y) \geqslant 0].$$

Observe that in case of superadditive solutions of (B) always Φ_1 is fulfilled, Mapping $-f_2$ from Example 2.5 fulfills both $\Phi_1(x,y)$ and $\Phi_2(x,y)$ for each x,y, whereas for f_3 from Example 3.1 exactly one of the two possibilities holds, unless $xy = 0$.

In what follows we will examine the situation, where exactly one possibility $\Phi_1(x,y)$ or $\Phi_2(x,y)$ is true excluding the case of equality in one or both inequalities. Assume that $G = \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. In particular, for each fixed $x,y \in \mathbb{R}$ the map

$$[-1,1] \ni t \mapsto \varphi_{x,y}(t) := f(x+ty) - f(x) - f(y) \in \mathbb{R} \tag{9}$$

is continuous. Then, we may rewrite $\Phi_1(x,y)$ and $\Phi_2(x,y)$ as $\varphi_{x,y}(1) \geqslant 0$ and $\varphi_{x,y}(-1) \geqslant 0$, respectively. Clearly, if exactly one possibility Φ_1 or Φ_2 holds true then there exists a point $\xi_{x,y} \in [-1,1]$ such that $\varphi_{x,y}(\xi_{x,y}) = 0$; in other words, the following functional equation is fulfilled:

$$f(x + \xi_{x,y} \cdot y) = f(x) + f(y). \tag{10}$$

If we denote $x \circ y := x + \xi_{x,y} \cdot y$ for each $x,y \in \mathbb{R}$ then equation (10) takes the form

$$f(x \circ y) = f(x) + f(y), \quad x,y \in \mathbb{R}.$$

In other words f is a homomorphism.

If additionally f is invertible then we have the following representation of the operation $\circ$:

$$x \circ y = f^{-1}(f(x) + f(y)), \quad x,y \in \mathbb{R}.$$

But this is the general form of associative, continuous and cancellative operations on the real line or a real interval (see J. Aczél [1], J. Aczél [2, Chapter 6.2], R. Craigen and Z. Páles [5]).

It is straightforward to check that always 0 is the neutral element (not necessarily unique) of the operation $\circ$. Therefore, we can think of invertible elements with respect to this operation.

Since $\xi_{x,y} \in [-1,1]$ then

$$|x \circ y - x| \leqslant |y|, \quad x,y \in \mathbb{R}. \tag{11}$$

Inequality (B) can be rewritten as:

$$f(x \circ y) \leqslant \max\{f(x+y), f(x-y)\}, \quad x,y \in \mathbb{R},$$

If additionally f is strictly increasing then we have equivalently

$$x \circ y \leqslant \max\{x+y, x-y\} = x+|y|, \quad x,y \in \mathbb{R}.$$

Therefore, in this case inequality (B) is a consequence of relation (11). Consequently, in view of the above mentioned result of J. Aczél, each associative operation $\circ$ which is continuous, cancellative and satisfies (11) yields a continuous solution of inequality (B).

4. Inequality (C)

EXAMPLE 4.1. The mapping f_2 from Example 2.5 provides a solution of inequality (C). Indeed, for each $x,y \in G$ we have

$$\min\{f_2(x+y), f_2(x-y)\} \geqslant 2M \geqslant |f_2(x) - f_2(y)|.$$

Furthermore, each mapping which is constant and nonnegative solves (C).

In view of the foregoing example we need to impose an additional assumption that $f(0) = 0$.

THEOREM 4.2. *Assume that $(G,+)$ is an Abelian group and $f\colon G \to \mathbb{R}$ satisfies $f(0)=0$. Then f fulfills inequality* (C) *for all $x,y \in G$ if and only if f is even and subadditive mapping.*

Proof. To prove the "if" part substitute $x=0$ in (C) to obtain

$$\min\{f(y), f(-y)\} \geqslant |f(0) - f(y)| = |f(y)|,$$

for each $y \in G$. From this estimation we see that f is nonnegative and then that it is even. Having this we can easily calculate that for each $s,t \in G$ we have

$$\begin{aligned} f(s) = f(s+t-t) &\geqslant \min\{f((s+t)+t), f((s+t)-t)\} \\ &\geqslant |f(s+t) - f(t)| \geqslant \big||f(s+t)| - |f(t)|\big| \geqslant f(s+t) - f(t), \end{aligned}$$

thus f is subadditive.

To prove the converse implication it is enough to fix $x,y \in G$ and observe that from the subadditivity and from the evenness of f we get

$$f(x) - f(y) \leqslant f(x-y)$$

and

$$f(y) - f(x) \leqslant f(y-x) = f(x-y).$$

Thus

$$|f(x) - f(y)| \leqslant f(x-y)$$

and also

$$|f(x)-f(y)| = |f(x)-f(-y)| \leqslant f(x+y).$$

The last two estimates prove that f satisfies inequality (C). $\square$

Now, we may establish an analogue to Theorem 2.4.

THEOREM 4.3. *Assume that* $(G,+)$ *is an Abelian group and* $f\colon G\to\mathbb{R}$ *satisfies inequality* (C) *for all* $x,y\in G$. *If* f *vanishes at any point* $x_0\in G$ *then* f *vanishes at zero.*

Moreover, each point from the set

$$K=\{x\in G: f(x)=0\}$$

is a period of f.

Proof. From inequality (C) applied for $y=0$ we deduce that

$$0=f(x_0)=\min\{f(x_0),f(x_0)\}\geqslant |f(x_0)-f(0)| = |f(0)|,$$

thus $f(0)=0$.

Now, if $x\in K$ and $y\in G$ is arbitrary then

$$0=f(x)\geqslant \min\{f((x+y)+y),f((x+y)-y)\}\geqslant |f(x+y)-f(y)|\geqslant 0,$$

so $f(x+y)=f(y)$. $\square$

5. **Inequality** (D)

Similarly like in cases of (A) and (B) situation is much less satisfactory for inequality (D) than for (C). Therefore, we restrict ourselves to providing a few examples of solutions of (D).

Observe first, that by putting $x=y=0$ in (D) we see that each solution of this inequality satisfies $f(0)\leqslant 0$.

EXAMPLE 5.1. Let $(G,+)$ be an arbitrary Abelian group. Clearly, each nonpositive map on G satisfies (D). Next, observe that each superadditive mapping defined on G provides a solution of inequality (D). Indeed, for arbitrary $x,y\in G$ we have

$$f(x+y)\leqslant f(x)-f(-y)$$

and

$$f(x-y)\leqslant f(x)-f(y),$$

thus

$$\begin{aligned}\min\{f(x+y),f(x-y)\}&\leqslant \min\{f(x)-f(-y),f(x)-f(y)\}\\ &= f(x)-\max\{f(y),f(-y)\}\leqslant f(x)-f(y)\leqslant |f(x)-f(y)|.\end{aligned}$$

Further, if f solves (C) then $-f$ solves (D) but not conversely.

Finally, we will show that if $G=\mathbb{R}$ then $f=\sinh$ solves (D). Fix arbitrary $x,y\in\mathbb{R}$, we want to prove the inequality

$$\min\{\sinh(x+y),\sinh(x-y)\}\leqslant|\sinh(x)-\sinh(y)|.$$

Using some elementary identities for hyperbolic functions this inequality can be transformed equivalently into

$$\begin{aligned}\min\left\{\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x+y}{2}\right),\sinh\left(\frac{x-y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\right\}\\ \leqslant\left|\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)\right|.\end{aligned}$$

Now, if we denote $\alpha=\frac{x+y}{2}$ and $\beta=\frac{x-y}{2}$ and observe that it is enough to restrict ourselves to the case $\alpha>0$ and $\beta>0$ so that $\sinh\alpha>0$ and $\sinh\beta>0$ are positive then we arrive at

$$\min\{\sinh\alpha\cosh\alpha,\sinh\beta\cosh\beta\}\leqslant\cosh\alpha\sinh\beta.$$

Since both mappings sinh and cosh are increasing and nonnegative then both estimates $\sinh\beta<\sinh\alpha$ and $\cosh\alpha<\cosh\beta$ cannot hold simultaneously and consequently the foregoing inequality is true.

It is worth to note that $f=\sin$ does not satisfy inequality (D). Indeed, it suffices to take $x=\frac{\pi}{2}$ and $y=\frac{\pi}{6}$ and calculate that

$$\min\left\{\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right),\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right\}=\frac{\sqrt{3}}{2}$$

and

$$\left|\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{6}\right)\right|=\frac{1}{2}.$$

REMARK 5.2. We are aware that we are far from providing satisfactory descriptions of inequalities discussed, especially of (B) and (D). Therefore, it is an open problem whether it is possible to prove any more definite results.

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Włodzimierz Fechner
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: wlodzimierz.fechner@us.edu.pl, fechner@math.us.edu.pl