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UNCERTAINTY PRINCIPLE IN TERMS OF ENTROPY FOR THE SPHERICAL MEAN OPERATOR

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Abstract. The logarithmic uncertainty principle in terms of entropy is proved for the spherical mean operator.

1. Introduction

Uncertainty principles play an important role in harmonic analysis, they state that a function f and its Fourier transform $\widehat{f}$ cannot be at the same time simultaneously and sharply localized. That is, it is impossible for a non zero function and its Fourier transform to be simultaneously small. Many mathematical formulations of this general fact can be found in [11, 12].

For a probability density function f on $\mathbb{R}^n$, the entropy of f is defined according to Shanon [26] by

$$E(f) = -\int_{\mathbb{R}^n} \ln(f(x))f(x)\,dx. \tag{1.1}$$

The entropy $E(f)$ is closely related to quantum mechanics [4], and to the covariance matrix $\mathscr{V}(f)$ [26], and constitutes one of the important ways to measure the concentration of f, since $E(f)$ tends to be negative whenever f has sharp peaks, and conversely, a slow decay of f tends to make $E(f)$ positive. The measure of the uncertainty of a probability function f consists then in relating the entropy of $|f|^2$ with that of $|\widehat{f}|^2$. A first result has been given by Hirshman [16] who established a weak version of the uncertainty principle in terms of entropy by showing that for every square integrable function f on $\mathbb{R}^n$ with respect to the Lebesgue measure satisfying $\|f\|_2 = 1$, we have

$$E(|f|^2) + E(|\widehat{f}|^2) \geqslant 0, \tag{1.2}$$

where $\widehat{f}$ denotes the classical Fourier transform of f.

The relation (1.2) has been improved later by Beckner [2, 3] who proved a stronger inequality by showing that for every square integrable function f on $\mathbb{R}^n$ with respect to the Lebesgue measure satisfying $\|f\|_2 = 1$, we have

$$E(|f|^2) + E(|\widehat{f}|^2) \geqslant n\ln\left(\frac{e}{2}\right). \tag{1.3}$$

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Inequality (1.3) constitutes in fact a very powerful result which implies in particular the well known Heisenberg-Pauli-Weyl uncertainty principle.

The spherical mean operator $\mathscr{R}$ is defined, for a function f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, see [24], such that

$$\mathscr{R}(f)(r,x)=\int_{S^n} f(r\eta,x+r\xi)d\sigma_n(\eta,\xi), \quad (r,x)\in\mathbb{R}\times\mathbb{R}^n,$$

where S^n is the unit sphere of $\mathbb{R}^{n+1}$ and $d\sigma_n$ is the surface measure on S^n normalized to have total measure one.

The spherical mean operator $\mathscr{R}$ has many important physical applications, namely in image processing of so-called synthetic aperture radar (SAR) data [14, 15, 27, 28], or to the linearized inverse scattering problem in acoustics [8].

The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined for every measurable function f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, and integrable with respect to the measure $r^n dr\otimes dx$, by

$$\forall (s,y)\in\Gamma_{n+1},\quad \mathscr{F}(f)(s,y)=\int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\mathscr{R}\big(\cos(s.)e^{-i\langle y|.\rangle}\big)(r,x)\frac{r^n dr\otimes dx}{(\pi)^{\frac{n}{2}}2^{n-\frac{1}{2}}\Gamma(\frac{n+1}{2})}, \tag{1.4}$$

where

$$\Gamma_{n+1}=\mathbb{R}\times\mathbb{R}^n\cup\big\{(ir,x),\ (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ |r|\leqslant\|x\|\big\}. \tag{1.5}$$

Many harmonic analysis result related to the spherical mean operator and its Fourier transform have already been proved namely by Dziri, Jlassi, Nessibi, Rachdi and Trimèche [6, 17, 24, 25] or also by Peng and Zhao [20, 30]. Recently, Baccar, Omri and Rachdi [5] have studied the generalized Fock spaces associated with the spherical mean operator $\mathscr{R}$, and Msehli and Rachdi [22, 23] have established several uncertainty principles for the Fourier transform $\mathscr{F}$.

The aim of this work is to prove the logarithmic uncertainty principle in terms of entropy for the spherical mean operator, more precisely we shall show the following result:

THEOREM 1.1. *For every measurable function f on $\mathbb{R}_+\times\mathbb{R}^n$ satisfying*

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|^2 d\nu_{n+1}(r,x)=1,$$

we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\ln\big(|f(r,x)|^2\big)|f(r,x)|^2 d\nu_{n+1}(r,x) + \int_{\Gamma_{n+1,+}}\ln\big(|\mathscr{F}(f)(s,y)|^2\big)|\mathscr{F}(f)(s,y)|^2 d\gamma_{n+1}(s,y)\leqslant(2n+1)(\ln(2)-1), \tag{1.6}$$

where $d\nu_{n+1}$ *is the measure defined on* $\mathbb{R}_+ \times \mathbb{R}^n$ *by*

$$d\nu_{n+1}(r,x) = \frac{r^n}{(\pi)^{\frac{n}{2}} 2^{n-\frac{1}{2}} \Gamma(\frac{n+1}{2})} dr \otimes dx, \tag{1.7}$$

$\Gamma_{n+1,+}$ *is the subset of* Γ_{n+1} *defined by*

$$\Gamma_{n+1,+} = \mathbb{R}_+ \times \mathbb{R}^n \cup \big\{(ir,x),\ (r,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ r \leqslant \|x\|\big\}, \tag{1.8}$$

$\mathscr{F}(f)$ *is the Forurier transform associated with the spherical mean operator given by relation (*1.4*), and* $d\gamma_{n+1}$ *is a spectral measure that we shall define later, see relation (*2.10*).*

2. The spherical mean operator

In [24, 25] Nessibi, Rachdi and Trimèche showed that for every $(s,y) \in \mathbb{C} \times \mathbb{C}^n$, the function $\varphi_{s,y}$ defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$\varphi_{s,y}(r,x) = \mathscr{R}\left(\cos(s.)e^{-i\langle y|.\rangle}\right)(r,x), \tag{2.1}$$

is the unique infinitely differentiable function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, satisfying the following system

$$\begin{cases} \dfrac{\partial u}{\partial x_j}(r,x_1,...,x_n) = -iy_j u(r,x_1,...,x_n), & 1 \leqslant j \leqslant n, \\ \ell_{\frac{n-1}{2},r} u(r,x_1,...,x_n) - \Delta_x u(r,x_1,...,x_n) = -s^2 u(r,x_1,...,x_n), & \\ u(0,...,0) = 1, & \\ \dfrac{\partial u}{\partial r}(0,x_1,...,x_n) = 0. & \end{cases}$$

where $\ell_{(n-1)/2,r}$ is the Bessel operator of index $(n-1)/2$ defined by

$$\ell_{\frac{n-1}{2},r} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r}, \tag{2.2}$$

and Δ_x denotes as usual the Laplacian operator defined by

$$\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{2.3}$$

The authors proved also that the eigenfunction $\varphi_{s,y}$ defined by relation (2.1), is explicitly given by

$$\forall (r,x) \in \mathbb{R} \times \mathbb{R}^n, \quad \varphi_{s,y}(r,x) = e^{-i\langle y|x\rangle} j_{\frac{n-1}{2}}(r\sqrt{s^2 + \|y\|^2}), \tag{2.4}$$

where $j_{(n-1)/2}$ is the modified Bessel function defined by

$$j_{\frac{n-1}{2}}(s)=2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)\frac{J_{\frac{n-1}{2}}(s)}{s^{\frac{n-1}{2}}}=\Gamma\left(\frac{n+1}{2}\right)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\frac{n+1}{2}+k)}\left(\frac{s}{2}\right)^{2k}, \tag{2.5}$$

and $J_{(n-1)/2}$ is the Bessel function of the first kind and index $(n-1)/2$ [1, 19, 29].

Thus, the function $\varphi_{s,y}$ is bounded on $\mathbb{R}\times\mathbb{R}^n$, if and only if (s,y) belongs to the set Γ_{n+1} defined by relation (1.5), and in this case

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ \left|\varphi_{s,y}(r,x)\right|\leqslant 1. \tag{2.6}$$

In the following we shall define the Fourier transform associated with the spherical mean operator and we will recall some of its properties. For this we denote by

• $L^p(d\nu_{n+1})$, $p\in[1,+\infty]$ the Lebesgue space of measurable functions f on $\mathbb{R}_+\times\mathbb{R}^n$, satisfying

$$\begin{aligned}\|f\|_{p,\nu_{n+1}}&=\left(\int_0^{+\infty}\int_{\mathbb{R}^n}|f(r,x)|^p\,d\nu_{n+1}(r,x)\right)^{\frac{1}{p}}<+\infty, \text{ if } p\in[1,+\infty[;\\ \|f\|_{\infty,\nu_{n+1}}&=\text{ess sup}_{(r,x)\in\mathbb{R}_+\times\mathbb{R}^n}|f(r,x)|<+\infty, \qquad \text{if } p=+\infty.\end{aligned} \tag{2.7}$$

• $\mathscr{B}_{\Gamma_{n+1,+}}$ the σ-algebra defined on $\Gamma_{n+1,+}$ by

$$\mathscr{B}_{\Gamma_{n+1,+}}=\left\{\theta^{-1}(B)\ ,\ B\in\mathscr{B}_{Bor}(\mathbb{R}_+\times\mathbb{R}^n)\right\}, \tag{2.8}$$

where θ is the bijective function defined on the set $\Gamma_{n+1,+}$ by

$$\theta(s,y)=(\sqrt{s^2+||y||^2},y). \tag{2.9}$$

• $d\gamma_{n+1}$ the measure defined on $\mathscr{B}_{\Gamma_{n+1,+}}$ by

$$\forall\, B\in\mathscr{B}_{\Gamma_{n+1,+}},\quad \gamma_{n+1}(B)=\nu_{n+1}(\theta(B)). \tag{2.10}$$

• $L^p(d\gamma_{n+1})$, $p\in[1,+\infty]$ the space of measurable functions f on $\Gamma_{n+1,+}$, satisfying

$$\begin{aligned}\|f\|_{p,\gamma_{n+1}}&=\left(\int_{\Gamma_{n+1,+}}|f(s,y)|^p\,d\gamma_{n+1}(s,y)\right)^{\frac{1}{p}}<+\infty, \text{ if } p\in[1,+\infty[;\\ \|f\|_{\infty,\gamma_{n+1}}&=\text{ess sup}_{(s,y)\in\Gamma_{n+1,+}}|f(s,y)|<+\infty, \qquad \text{if } p=+\infty.\end{aligned} \tag{2.11}$$

• $\mathscr{C}_{e,0}(\mathbb{R}^{n+1})$ the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, satisfying $\lim\limits_{\|(r,x)\|\to+\infty}f(r,x)=0$.

• $S_e\big(\mathbb{R}^{n+1}\big)$ the subspace of the Schwartz class formed by functions, which are even with respect to the first variable.

• $d\tau_n$ the measure defined on $\mathbb{R}_+$ by

$$d\tau_n(r) = \frac{r^n}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}dr. \tag{2.12}$$

Then we have the following useful properties:

PROPOSITION 2.1. *i) For every nonnegative measurable function f on $\Gamma_{n+1,+}$, we have*

$$\int_{\Gamma_{n+1,+}} g(s,y)d\gamma_{n+1}(s,y) = \frac{1}{(\pi)^{\frac{n}{2}}2^{n-\frac{1}{2}}\Gamma(\frac{n+1}{2})}\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} g(s,y)(s^2+||y||^2)^{\frac{n-1}{2}}sdsdy$$
$$+\int_{\mathbb{R}^n}\int_0^{||y||} g(is,y)(||y||^2-s^2)^{\frac{n-1}{2}}sdsdy\Big).$$

ii) For every measurable function f on $\mathbb{R}_+\times\mathbb{R}^n$, the function

$$B(f) = f\circ\theta. \tag{2.13}$$

is measurable on $\Gamma_{n+1,+}$. Furthermore, if f is nonnegative or integrable on $\mathbb{R}_+\times\mathbb{R}^n$, then

$$\int_{\Gamma_{n+1,+}} B(f)(s,y)d\gamma_{n+1}(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)d\nu_{n+1}(r,x). \tag{2.14}$$

DEFINITION 2.2. The Fourier transform associated with the spherical mean operator $\mathscr{R}$ is defined on $L^1(d\nu_{n+1})$ by

$$\forall (s,y)\in\Gamma_{n+1},\quad \mathscr{F}(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)\varphi_{s,y}(r,x)d\nu_{n+1}(r,x). \tag{2.15}$$

The Fourier transform $\mathscr{F}$ satisfies

$$\forall (s,y)\in\Gamma_{n+1},\quad \mathscr{F}(f)(s,y) = B\circ\widetilde{\mathscr{F}}(f)(s,y), \tag{2.16}$$

where B is the mapping defined by relation (2.13) and $\widetilde{\mathscr{F}}$ is the transform defined on $L^1(d\nu_{n+1})$ by

$$\forall (s,y)\in\mathbb{R}\times\mathbb{R}^n,\ \widetilde{\mathscr{F}}(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)j_{\frac{n-1}{2}}(rs)e^{-i\langle y|x\rangle}d\nu_{n+1}(r,x). \tag{2.17}$$

On the other hand it is well known [24, 25] that the Fourier transform $\mathscr{F}$ associated with the spherical mean operator is a linear bounded operator from $L^1(d\nu_{n+1})$ into $\mathscr{C}_{e,0}(\mathbb{R}^{n+1})$ and that for every $f\in L^1(d\nu_{n+1})$, we have

$$\|\mathscr{F}(f)\|_{\infty,\gamma_{n+1}} \leqslant \|f\|_{1,\nu_{n+1}}. \tag{2.18}$$

Furthermore, the Fourier transform $\mathscr{F}$ satisfies the following inversion formula and Plancherel theorem:

THEOREM 2.3. (Inversion formula) *Let* $f \in L^1(d\nu_{n+1})$ *such that* $\mathscr{F}(f) \in L^1(d\gamma_{n+1})$, *then for almost every* $(r,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, *we have*

$$f(r,x) = \int_{\Gamma_{n+1,+}} \mathscr{F}(f)(s,y)\overline{\varphi_{s,y}(r,x)}d\gamma_{n+1}(s,y). \tag{2.19}$$

THEOREM 2.4. (Plancherel) *The Fourier transform* $\mathscr{F}$ *can be extended to an isometric isomorphism from* $L^2(d\nu_{n+1})$ *onto* $L^2(d\gamma_{n+1})$.

DEFINITION 2.5. For every $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^n$ the generalized shift operator $\mathscr{T}_{(s,y)}$ associated with the spherical mean operator is defined on $L^p(d\nu_{n+1})$, see [24, 25], such that for every $(r,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\mathscr{T}_{(s,y)}(f)(r,x) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta},x+y)(\sin\theta)^{n-1}d\theta. \tag{2.20}$$

Furthermore, for every $(s,y) \in]0,+\infty[\times\mathbb{R}^n$, and by a standard change of variables, the generalized shift operator may be expressed as an integral operator by

$$\forall (r,x) \in]0,+\infty[\times\mathbb{R}^n, \quad \mathscr{T}_{(s,y)}(f)(r,x) = \int_0^{+\infty} f(t,x+y)\mathscr{W}_n(r,s,t)d\tau_n(t), \tag{2.21}$$

with kernel

$$\mathscr{W}_n(r,s,t) = \frac{\Gamma(\frac{n+1}{2})^2}{2^{\frac{n-3}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}\frac{\left((r+s)^2-t^2\right)^{\frac{n}{2}-1}\left(t^2-(r-s)^2\right)^{\frac{n}{2}-1}}{(rst)^{n-1}}\chi_{]|r-s|,r+s[}(t), \tag{2.22}$$

where $\chi_{]|r-s|,r+s[}$ is the characteristic function defined on $\mathbb{R}_+$ by

$$\chi_{]|r-s|,r+s[}(t) = \begin{cases} 1, \text{ if } |r-s| < t < r+s; \\ 0, \text{ if } t \leqslant |r-s| \text{ or } t \geqslant r+s. \end{cases} \tag{2.23}$$

The kernel $\mathscr{W}_n$ satisfies then the following properties:

- For every $r,s,t > 0$ we have $\mathscr{W}_n(r,s,t) = \mathscr{W}_n(s,r,t) = \mathscr{W}_n(t,s,r) = \mathscr{W}_n(r,t,s)$.
- For every $r,s > 0$ and for every positive integer n, we have

$$\int_0^{+\infty} \mathscr{W}_n(r,s,t)d\tau_n(t) = 1. \tag{2.24}$$

Thus, for every $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^n$ the generalized shift operator $\mathscr{T}_{(s,y)}$ is a continuous linear operator from $L^p(d\nu_{n+1})$, $p \in [1,+\infty]$, into itself, satisfying for every $f \in L^p(d\nu_{n+1})$

$$\|\mathscr{T}_{(s,y)}(f)\|_{p,\nu_{n+1}} \leqslant \|f\|_{p,\nu_{n+1}}. \tag{2.25}$$

From relations (2.21) and (2.24), one can deduce the following useful property:

PROPOSITION 2.6. *For every $f \in L^1(d\nu_{n+1})$ and for every $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^n$, the function $\mathscr{T}_{(s,y)}(f)$ belongs to $L^1(d\nu_{n+1})$, and we have*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \mathscr{T}_{(s,y)}(f)(r,x)d\nu_{n+1}(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n} f(r,x)d\nu_{n+1}(r,x). \tag{2.26}$$

DEFINITION 2.7. The generalized convolution product associated with the spherical mean operator is defined for two measurable functions f,g on $\mathbb{R}_+ \times \mathbb{R}^n$ by

$$f\#g(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} \mathscr{T}_{(s,-y)}(\check{f})(r,x)g(r,x)d\nu_{n+1}(r,x), \tag{2.27}$$

whenever this integral is well defined, where $\check{f}(r,x) = f(r,-x)$.

Then we have the following properties:

PROPOSITION 2.8. *i) For every $f \in L^1(d\nu_{n+1})$ and for every $g \in L^p(d\nu_{n+1})$, $p \in [1,+\infty[$, the function $f\#g$ belongs to the space $L^p(d\nu_{n+1})$ and we have*

$$\|f\#g\|_{p,\nu_{n+1}} \leqslant \|f\|_{1,\nu_{n+1}}\|g\|_{p,\nu_{n+1}}. \tag{2.28}$$

ii) For $f \in S_e(\mathbb{R}^{n+1})$ and $g \in L^2(d\nu_{n+1})$, we have

$$\mathscr{F}\big(fg\big) = \mathscr{F}(f)\#\mathscr{F}(g). \tag{2.29}$$

It is not hard to see that all results of this last proposition hold also for the Fourier transform $\widetilde{\mathscr{F}}$ defined by relation (2.17).

DEFINITION 2.9. The Gaussian kernel associated with the spherical mean operator is defined for $t > 0$ by

$$G_t(r,x) = \frac{e^{-\frac{r^2+\|x\|^2}{2t^2}}}{t^{2n+1}}, \quad (r,x) \in \mathbb{R}\times\mathbb{R}^n. \tag{2.30}$$

Thus, one can easily see that the family $\big(G_t\big)_{t>0}$ is an approximation to the identity, in particular for every $f \in L^2(d\nu_{n+1})$ we have

$$\lim_{t\to 0^+} \|G_t\#f - f\|_{2,\nu_{n+1}} = 0. \tag{2.31}$$

3. Uncertainty principle in terms of entropy for the spherical mean operator

The aim of this section is to prove the main result of this work, i.e. the uncertainty principle in terms of entropy for every square integrable functions f with respect to the measure $d\nu_{n+1}$. For this we shall begin by proving an analogue of the Hausdorff-Young theorem for the Fourier transform $\mathscr{F}$, from which we will later deduce the uncertainty principle in terms of entropy, firstly for $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ and then for every $f \in L^2(d\nu_{n+1})$.

THEOREM 3.1. (Hausdorff-Young) *Let p,q be two conjugate exponents such that $1<p\leqslant 2$, then for every function $f\in L^p(d\nu_{n+1})$, $\mathscr{F}(f)$ belongs to $L^q(d\gamma_{n+1})$ and we have*

$$\|\mathscr{F}(f)\|_{q,\gamma_{n+1}}\leqslant A_p\|f\|_{p,\nu_{n+1}}, \tag{3.1}$$

where $A_p=\Big[\dfrac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\Big]^{n+\frac{1}{2}}$.

Proof. Firstly, one can see that according to relation (2.18) and Theorem 2.4, and by using the standard Riesz-Thorin interpolation theorem, that the Fourier transform $\mathscr{F}$ may be extended as a bounded linear operator from $L^p(d\nu_{n+1})$ into $L^q(d\nu_{n+1})$ with $1<p<2$ and $q=p/(p-1)$.

Let $1<p<2$ and $q=p/(p-1)$, and let $f\in L^1(d\nu_{n+1})\cap L^p(d\nu_{n+1})$, then by relations (2.14) and (2.16) we have

$$\big\|\mathscr{F}(f)\big\|^q_{q,\gamma_{n+1}}=\big\|\widetilde{\mathscr{F}}(f)\big\|^q_{q,\nu_{n+1}}. \tag{3.2}$$

Now, according to Theorem 4.1 in [9, p. 880] we have

$$\begin{aligned}\int_0^{+\infty}\Big|\int_0^{+\infty}\Big(\int_{\mathbb{R}^n}f(r,x)e^{-i\langle\lambda|x\rangle}\frac{dx}{(2\pi)^{\frac{n}{2}}}\Big)j_{\frac{n-1}{2}}(r\mu)\frac{r^n dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\Big|^q\frac{\mu^n}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}d\mu\\ \leqslant\Big[\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\Big]^{\frac{q(n+1)}{2}}\Big(\int_0^{+\infty}\Big|\int_{\mathbb{R}^n}f(r,x)e^{-i\langle\lambda|x\rangle}\frac{dx}{(2\pi)^{\frac{n}{2}}}\Big|^p\frac{r^n dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\Big)^{\frac{q}{p}}.\end{aligned}$$

Hence, by using Minkowski's inequality for integrals, see [10, p. 194], and relation (3.2), we get

$$\begin{aligned}\big\|\mathscr{F}(f)\big\|^q_{q,\gamma_{n+1}}&=\int_{\mathbb{R}^n}\Big(\int_0^{+\infty}\Big|\int_0^{+\infty}\big(\int_{\mathbb{R}^n}f(r,x)e^{-i\langle y|x\rangle}\frac{dx}{(2\pi)^{\frac{n}{2}}}\big)j_{\frac{n-1}{2}}(rs)\\ &\quad\times\frac{r^n dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\Big|^q\frac{s^n ds}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\Big)\frac{dy}{(2\pi)^{\frac{n}{2}}}\\ &\leqslant\Big[\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\Big]^{\frac{q(n+1)}{2}}\int_{\mathbb{R}^n}\Big(\int_0^{+\infty}\Big|\int_{\mathbb{R}^n}f(r,x)e^{-i\langle y|x\rangle}\frac{dx}{(2\pi)^{\frac{n}{2}}}\Big|^p\\ &\quad\times\frac{r^n dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\Big)^{\frac{q}{p}}\frac{dy}{(2\pi)^{\frac{n}{2}}}\\ &\leqslant\Big[\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\Big]^{\frac{q(n+1)}{2}}\Big(\int_0^{+\infty}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}f(r,x)e^{-i\langle y|x\rangle}\frac{dx}{(2\pi)^{\frac{n}{2}}}\Big|^q\\ &\quad\times\frac{dy}{(2\pi)^{\frac{n}{2}}}\Big)^{\frac{p}{q}}\frac{r^n dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\Big)^{\frac{q}{p}}.\end{aligned}$$

Using now Theorem 1 in [2, p. 162], we deduce that

$$\|\mathscr{F}(f)\|_{q,\gamma_{n+1}}^{q} \leqslant \left[\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\right]^{q(n+\frac{1}{2})} \int_0^{+\infty} \left(\int_{\mathbb{R}^n} |f(r,x)|^p d\nu_{n+1}(r,x)\right)^{\frac{q}{p}}. \tag{3.3}$$

The result follows then by using the standard density argument. □

In the following we shall prove the uncertainty principle in terms of entropy for a function $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ satisfying $\|f\|_{2,\nu_{n+1}} = 1$. For this we define for every nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^n$, the weighted entropy of f by

$$E_{\nu_{n+1}}(f) = -\int_0^{+\infty}\int_{\mathbb{R}^n} \ln\big(f(r,x)\big) f(r,x) d\nu_{n+1}(r,x), \tag{3.4}$$

whenever the integral on the right hand side is well defined.

Similarly, we define for every nonnegative measurable function h on $\Gamma_{n+1,+}$, the weighted entropy of h by

$$E_{\gamma_{n+1}}(h) = -\int_{\Gamma_{n+1,+}} \ln\big(h(r,x)\big) h(r,x) d\gamma_{n+1}(r,x), \tag{3.5}$$

whenever the integral on the right hand side is well defined.

PROPOSITION 3.2. *Let $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ such that $\|f\|_{2,\nu_{n+1}} = 1$, then we have*

$$E_{\nu_{n+1}}\big(|f|^2\big) + E_{\gamma_{n+1}}\big(|\mathscr{F}(f)|^2\big) \geqslant (2n+1)(1-\ln(2)), \tag{3.6}$$

whenever $E_{\nu_{n+1}}\big(|f|^2\big)$ and $E_{\gamma_{n+1}}\big(|\mathscr{F}(f)|^2\big)$ are finite.

Proof. Let x be a positive real number and let g be the function defined on $[1,2[$ by $g(p) = (x^p - x^2)/(p-2)$, then for every $p \in]1,2[$, we have $g'(p) = (x^p \ln(x)(p-2) - x^p + x^2)/(p-2)^2 \geqslant 0$, and therefore g is increasing on $[1,2[$. This implies in particular that

$$x^2 - x \leqslant \frac{x^p - x^2}{p-2} \leqslant \lim_{p\to 2^-} g(p) = x^2 \ln(x). \tag{3.7}$$

Let $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ such that $\|f\|_{2,\nu_{n+1}} = 1$, and let φ be the function defined on $]1,2]$ by

$$\begin{aligned}\varphi(p) = &\int_{\Gamma_{n+1,+}} |\mathscr{F}(f)(s,y)|^{\frac{p}{p-1}} d\gamma_{n+1}(s,y)\\ &- \left[\frac{p^{\frac{1}{p}}}{\left(\frac{p}{p-1}\right)^{\frac{p-1}{p}}}\right]^{\frac{p(n+\frac{1}{2})}{p-1}} \left(\int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^p d\nu_{n+1}(r,x)\right)^{\frac{1}{p-1}}.\end{aligned}$$

Then by relation (3.1), $\varphi(p) \leqslant 0$ for every $1 < p \leqslant 2$. On the other hand Theorem 2.4 means that $\varphi(2) = 0$. This implies that $\left[\frac{\partial \varphi}{\partial p}\right]_{p=2^-} \geqslant 0$, whenever this derivative exists.

Since $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$, then by a standard interpolation argument, f belongs to $L^p(d\nu_{n+1})$ for every $1 \leqslant p \leqslant 2$. On the other hand, since $E_{\nu_{n+1}}(|f|^2)$ is finite, then according to relation (3.7) and by using the Lebesgue dominated convergence theorem, we deduce that

$$\frac{\partial}{\partial p}\Big[\int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|^p d\nu_{n+1}(x)\Big]_{p=2^-} = \int_0^{+\infty}\int_{\mathbb{R}^n}\lim_{p\to 2^-}\frac{|f(r,x)|^p-|f(r,x)|^2}{p-2}d\nu_{n+1}(r,x),$$

and therefore

$$\frac{\partial}{\partial p}\Big[\int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|^p d\nu_{n+1}(x)\Big]_{p=2^-} = \frac{1}{2}\int_0^{+\infty}\int_{\mathbb{R}^n}\ln\big(|f(r,x)|^2\big)\big|f(r,x)\big|^2 d\nu_{n+1}(r,x). \tag{3.8}$$

Now, since $f \in L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$, by using again the Lebesgue dominated convergence theorem, we get

$$\lim_{p\to 2^-}\int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|^p d\nu_{n+1}(r,x) = \int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|^2 d\nu_{n+1}(r,x) = 1. \tag{3.9}$$

Combining relations (3.8) and (3.9) we get

$$\frac{\partial}{\partial p}\Big[\Big(\int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|^p d\nu_{n+1}(r,x)\Big)^{\frac{1}{p-1}}\Big]_{p=2^-} = -\frac{1}{2}E_{\nu_{n+1}}\big(|f|^2\big). \tag{3.10}$$

In the same way, one can see that

$$\frac{\partial}{\partial p}\Big[\int_{\Gamma_{n+1,+}}\big|\mathscr{F}(f)(s,y)\big|^{\frac{p}{p-1}}d\gamma_{n+1}(s,y)\Big]_{p=2^-} = \frac{1}{2}E_{\gamma_{n+1}}\big(|\mathscr{F}(f)|^2\big). \tag{3.11}$$

Finally, basic calculations show that

$$\frac{\partial}{\partial p}\Big[\Big(\frac{p^{\frac{1}{p}}}{\big(\frac{p}{p-1}\big)^{\frac{p-1}{p}}}\Big)^{\frac{p(n+\frac{1}{2})}{p-1}}\Big]_{p=2} = (n+\frac{1}{2})(1-\ln(2)). \tag{3.12}$$

Thus, according to relations (3.10), (3.11) and (3.12) it follows that

$$\Big[\frac{\partial\varphi}{\partial p}\Big]_{p=2^-} = \frac{1}{2}E_{\nu_{n+1}}\big(|f|^2\big) + \frac{1}{2}E_{\gamma_{n+1}}\big(|\mathscr{F}(f)|^2\big) - (n+\frac{1}{2})(1-\ln(2)), \tag{3.13}$$

which completes the proof. $\square$

LEMMA 3.3. *Let f be a measurable function on $\mathbb{R}^{n+1}$ and let $\omega:\mathbb{R}\longrightarrow\mathbb{R}_+$ be a convex function such that $\omega\circ|f|\in L^1(d\nu_{n+1})$. If $(f_p)_{p\in\mathbb{N}}$ is a sequence of nonnegative measurable functions on $\mathbb{R}^{n+1}$ satisfying $\|f_p\|_{1,\nu_{n+1}}=1$ for every $p\in\mathbb{N}$, and such that the sequence $\big(f_p\#f\big)_{p\in\mathbb{N}}$ converges pointwise to f, then the function $(r,x)\longmapsto \omega\big(|(f_p\#f)(r,x)|\big)$ belongs to $L^1(d\nu_{n+1})$ for every $p\in\mathbb{N}$, and we have*

$$\lim_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\omega\big(\big|(f_p\#f)(r,x)\big|\big)d\nu_{n+1}(r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}\omega\big(|f(r,x)|\big)d\nu_{n+1}(r,x). \tag{3.14}$$

Proof. We have

$$\|\omega\circ|f|\|_{1,\nu_{n+1}}=\int_0^{+\infty}\int_{\mathbb{R}^n}\liminf_{p\to+\infty}\omega\big(\big|(f_p\#f)(r,x)\big|\big)d\nu_{n+1}(r,x), \tag{3.15}$$

and by using Fatou's lemma we can deduce that

$$\|\omega\circ|f|\|_{1,\nu_{n+1}}\leqslant\liminf_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\omega\big(\big|(f_p\#f)(r,x)\big|\big)d\nu_{n+1}(r,x). \tag{3.16}$$

Conversely, according to relation (2.26) we have for every $p\in\mathbb{N}$ and for every $(s,y)\in\mathbb{R}_+\times\mathbb{R}^n$

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\mathscr{T}_{(s,y)}(\check{f}_p)(r,x)d\nu_{n+1}(r,x)=\|f_p\|_{1,\nu_{n+1}}=1, \tag{3.17}$$

which means that for every $(s,y)\in\mathbb{R}_+\times\mathbb{R}^n$, $\mathscr{T}_{(s,y)}(f_p)(r,x)d\nu_{n+1}(r,x)$ is a probability measure on $\mathbb{R}_+\times\mathbb{R}^n$.

Therefore by using Jensen's inequality for convex functions [10], we get for all $(s,y)\in\mathbb{R}_+\times\mathbb{R}^n$

$$\begin{aligned}\omega\Big(\big|(f_p\#f)(s,y)\big|\Big)&=\omega\Big(\Big|\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\mathscr{T}_{(s,-y)}(\check{f}_p)(r,x)d\nu_{n+1}(r,x)\Big|\Big)\\&\leqslant\omega\Big(\int_0^{+\infty}\int_{\mathbb{R}^n}\big|f(r,x)\big|\mathscr{T}_{(s,-y)}(\check{f}_p)(r,x)d\nu_{n+1}(r,x)\Big)\\&\leqslant\int_0^{+\infty}\int_{\mathbb{R}^n}\omega\big(\big|f(r,x)\big|\big)\mathscr{T}_{(s,-y)}(\check{f}_p)(r,x)d\nu_{n+1}(r,x)\\&=f_p\#\big(\omega\circ|f|\big)(s,y).\end{aligned} \tag{3.18}$$

In particular $\omega\circ\big|f_p\#f\big|\in L^1(d\nu_{n+1})$.

Hence, by relations (2.28) and (3.18) we deduce that

$$\begin{aligned}&\limsup_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\omega\big(\big|(f_p\#f)(r,x)\big|\big)d\nu_{n+1}(r,x)\\&\qquad\leqslant\limsup_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}f_p\#\big(\omega\circ|f|\big)(r,x)d\nu_{n+1}(r,x)\\&\qquad=\lim_{p\to+\infty}\|f_p\#\big(\omega\circ|f|\big)\|_{1,\nu_{n+1}}\\&\qquad\leqslant\|\omega\circ|f|\|_{1,\nu_{n+1}}.\end{aligned} \tag{3.19}$$

The proof becomes complete by combining relations (3.16) and (3.19). □

THEOREM 3.4. (Uncertainty principle in terms of entropy) *Let $f \in L^2(d\nu_{n+1})$ such that $\|f\|_{2,\nu_{n+1}} = 1$, then we have*

$$E_{\nu_{n+1}}\left(\left|f\right|^2\right) + E_{\gamma_{n+1}}\left(\left|\mathscr{F}(f)\right|^2\right) \geqslant (2n+1)(1-\ln(2)), \tag{3.20}$$

whenever $E_{\nu_{n+1}}\left(\left|f\right|^2\right)$ and $E_{\gamma_{n+1}}\left(\left|\mathscr{F}(f)\right|^2\right)$ are finite.

Proof. The main idea of this proof is of course to combine Proposition 3.2 with the standard density argument, indeed we will show that for all $f \in L^2(d\nu_{n+1})$ there is a sequence $(f_p)_{p\in\mathbb{N}}$ of non zero functions belonging to $L^1(d\nu_{n+1}) \cap L^2(d\nu_{n+1})$ such that

$$\lim_{p\to+\infty} \|f_p\|_{2,\nu_{n+1}} = \|f\|_{2,\nu_{n+1}}, \tag{3.21}$$

$$\lim_{p\to+\infty} E_{\nu_{n+1}}\left(|f_p|^2\right) = E_{\nu_{n+1}}\left(|f|^2\right), \tag{3.22}$$

and

$$\lim_{p\to+\infty} E_{\gamma_{n+1}}\left(|\mathscr{F}(f_p)|^2\right) = E_{\gamma_{n+1}}\left(|\mathscr{F}(f)|^2\right). \tag{3.23}$$

Let $\left(h_p\right)_{p\in\mathbb{N}}$ be the sequence of functions defined by

$$h_p(r,x) = 2^{n+\frac{1}{2}} p^{2n+1} e^{-p^2(r^2+\|x\|^2)}, \tag{3.24}$$

then by relation (2.31), we have for all $f \in L^2(d\nu_{n+1})$

$$\lim_{p\to+\infty} \|h_p \# f - f\|_{2,\nu_{n+1}} = 0. \tag{3.25}$$

Furthermore, according to Weber's formula [19, 29], we know that for all $p,s>0$ and for all $\mu > -1$

$$\int_0^{+\infty} e^{-p^2r^2} J_\mu(sr) r^{\mu+1} dr = \frac{s^\mu e^{-\frac{s^2}{4p^2}}}{(2p^2)^{\mu+1}}. \tag{3.26}$$

Hence, by relation (3.26) we deduce that for all positive integer p

$$\widetilde{\mathscr{F}}^{-1}\left(h_p\right)(s,y) = e^{-\frac{s^2+\|y\|^2}{(2p)^2}}. \tag{3.27}$$

Let ψ_p be the sequence of functions defined on $\mathbb{R}_+ \times \mathbb{R}^n$ by

$$\psi_p(s,y) = e^{-\frac{s^2+\|y\|^2}{(2p)^2}} = \widetilde{\mathscr{F}}^{-1}\left(h_p\right)(s,y). \tag{3.28}$$

Let $f \in L^2(d\nu_{n+1})$ be such that $\|f\|_{2,\nu_{n+1}} = 1$, then according to relation (3.25), we know that $\lim\limits_{p\to+\infty} \|\widetilde{\mathscr{F}}(\psi_p)\#\widetilde{\mathscr{F}}(f) - \widetilde{\mathscr{F}}(f)\|_{2,\nu_{n+1}} = 0$, in particular there is a subsequence $(\psi_{\sigma(p)})_{p\in\mathbb{N}}$ such that $\widetilde{\mathscr{F}}(\psi_{\sigma(p)})\#\widetilde{\mathscr{F}}(f)$ converges pointwise to $\widetilde{\mathscr{F}}(f)$ almost everywhere.

Let $(f_p)_{p\in\mathbb{N}}$ be the sequence of measurable functions on $\mathbb{R}_+\times\mathbb{R}^n$ defined by

$$f_p=\psi_{\sigma(p)}f. \tag{3.29}$$

Hence, one can see that for all $p\in\mathbb{N}$, f_p is nonzero, and since $\psi_{\sigma(p)}\in L^2(d\nu_{n+1})\cap\mathscr{C}_{e,0}(\mathbb{R}^{n+1})$, f_p belongs to the space $L^1(d\nu_{n+1})\cap L^2(d\nu_{n+1})$, and therefore by Proposition 3.2 we can deduce that

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}^n}&\ln(|f_p(r,x)|^2)|f_p(r,x)|^2d\nu_{n+1}(r,x)\\&+\int_{\Gamma_{n+1,+}}\ln(|\mathscr{F}(f_p)(s,y)|^2)|\mathscr{F}(f_p)(s,y)|^2d\gamma_{n+1}(s,y)\\&\geqslant(2n+1)(1-\ln(2))\|f_p\|^2_{2,\nu_{n+1}}-2\|f_p\|^2_{2,\nu_{n+1}}\ln(\|f_p\|^2_{2,\nu_{n+1}}).\end{aligned} \tag{3.30}$$

Now, by using the Lebesgue dominated convergence theorem we have

$$\lim_{p\to+\infty}\|f_p\|_{2,\nu_{n+1}}=\|f\|_{2,\nu_{n+1}}. \tag{3.31}$$

On the other hand, one can see that for all $p\in\mathbb{N}$ and for almost every $(r,x)\in\mathbb{R}_+\times\mathbb{R}^n$, we have $\left|\ln(|f_p(r,x)|^2)|f_p(r,x)|^2\right|\leqslant C|f(r,x)|^2+\left|\ln(|f(r,x)|^2)|f(r,x)|^2\right|$, for some positive constant C. Hence by again using the Lebesgue dominated convergence theorem, we get that

$$\lim_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\ln(|f_p(r,x)|^2)|f_p(r,x)|^2d\nu_{n+1}(r,x)=E_{\nu_{n+1}}\big(|f|^2\big). \tag{3.32}$$

Let us show now that $\lim\limits_{p\to+\infty}E_{\gamma_{n+1}}\big(|\mathscr{F}(f_p)|^2\big)=E_{\gamma_{n+1}}\big(|\mathscr{F}(f)|^2\big)$.

For this we denote by ω_1,ω_2 the functions defined on $\mathbb{R}$ by

$$\omega_1(t)=\begin{cases}t^2\ln(|t|), & \text{if } |t|>1;\\ 0, & \text{if } |t|\leqslant 1,\end{cases}$$

and

$$\omega_2(t)=\begin{cases}2t^2, & \text{if } |t|\geqslant 1;\\ -t^2\ln(|t|)+2t^2, & \text{if } |t|\leqslant 1,\ t\neq 0;\\ 0, & \text{if } t=0.\end{cases}$$

Then ω_1 and ω_2 are both nonnegative and convex, moreover for every positive real number t we have

$$t^2\ln|t|=\omega_1(t)-\omega_2(t)+2t^2. \tag{3.33}$$

Finally, since $E_{\nu_{n+1}}(|\widetilde{\mathscr{F}}(f)|^2)<+\infty$, the functions $\omega_i\big(|\widetilde{\mathscr{F}}(f)|\big)$ for each $i=1,2$ belong to $L^1(d\nu_{n+1})$. Moreover, according to relation (2.30) we know that for every $p\in\mathbb{N}$, $\widetilde{\mathscr{F}}(\psi_{\sigma(p)})$ is a nonnegative function, and using the inversion formula, we have

$$\widetilde{\mathscr{F}}\big(\widetilde{\mathscr{F}}(\psi_{\sigma(p)})\big)(0,0)=\psi_{\sigma(p)}(0,0)=1. \tag{3.34}$$

Hence by applying Lemma 3.3 and relations (2.29) and (3.29), we deduce that for $i=1,2$

$$\lim_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\omega_i\big(\widetilde{\mathscr{F}}(f_p)\big)(r,x)d\nu_{n+1}(r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}\omega_i\big(\widetilde{\mathscr{F}}(f)\big)(r,x)d\nu_{n+1}(r,x), \tag{3.35}$$

and therefore by relations (3.31) and (3.33), we get

$$\lim_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n}\ln(|\widetilde{\mathscr{F}}(f_p)(r,x)|^2)|\widetilde{\mathscr{F}}(f_p)(r,x)|^2d\nu_{n+1}(r,x)=E_{\nu_{n+1}}\big(|\widetilde{\mathscr{F}}(f)|^2\big), \tag{3.36}$$

which means by relations (2.14) and (2.16) that

$$\lim_{p\to+\infty}\int_{\Gamma_{n+1}}\ln(|\mathscr{F}(f_p)(s,y)|^2)|\mathscr{F}(f_p)(s,y)|^2d\gamma_{n+1}(r,x)=E_{\gamma_{n+1}}\big(|\mathscr{F}(f)|^2\big). \tag{3.37}$$

The proof is complete by combining relations (3.30), (3.31) and (3.32) with (3.37). □

4. Conclusion

As has been stated in the introduction, the spherical mean operator plays an important role in the study of many interesting physical problems, indeed according to Lavrentiev, Romanov and Vasiliev [18, 21] and aslo to Schuster [27], the technical study of sound navigation and radiation strongly involves the spherical mean operator. Similarly, Hellsten and Andersson [14] show, how the measured data in Synthetic-aperture radar can be interpreted as spherical means of the ground reflectivity, the role of the Fourier transform $\mathscr{F}$ is then essential in solving inverse problems related to signal recovering.

The uncertainty principle is one of the central ideas in signal theory giving in general a lower bound for the simultaneous localization of waves in phase and frequency spaces. There are many advantageous ways to study the spreading out of a distribution of the waves [12], however the entropy has been shown to be a particularly sensitive measure.

Inequality (3.20) gives a bound for the maximum localization of the entropy in phase and frequency spaces related to the spherical mean operator $\mathscr{R}$. As application of relation (3.20), one may deduce the well known Heisenberg-Pauli-Weyl uncertainty principle [13] for the Fourier transform $\mathscr{F}$. Indeed, let α be a positive real number and let $d\delta^{\alpha}_{n+1}$ be the probability measure defined on $\mathbb{R}_+\times\mathbb{R}^n$ by

$$d\delta^{\alpha}_{n+1}(r,x)=G_\alpha(r,x)d\nu_{n+1}(r,x), \tag{4.1}$$

where $G_\alpha(r,x)$ is the Gaussian kernel defined by relation (2.30).

Let $f\in L^2(d\nu_{n+1})$ such that $\|f\|_{2,\nu_{n+1}}=1$, since the map $t\longmapsto t\ln t$ is convex on $]0,+\infty[$, then according to Jensen's inequality for convex functions [10], we deduce that

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)|^2}{G_\alpha(r,x)}\ln\left(\frac{|f(r,x)|^2}{G_\alpha(r,x)}\right)d\delta^{\alpha}_{n+1}(r,x)\geqslant 0, \tag{4.2}$$

hence,

$$E_{\nu_{n+1}}(|f|^2) \leqslant \ln\left(\alpha^{2n+1}\right) + \frac{1}{2\alpha^2}\int_0^{+\infty}\int_{\mathbb{R}^n} |f(r,x)|^2(r^2+\|x\|^2)d\nu_{n+1}(r,x), \tag{4.3}$$

and by Plancherel's theorem,

$$E_{\gamma_{n+1}}(|\mathscr{F}(f)|^2) \leqslant \ln\left(\alpha^{2n+1}\right) + \frac{1}{2\alpha^2}\int_{\Gamma_{n+1,+}} |\mathscr{F}(f)(s,y)|^2(s^2+2\|y\|^2)d\gamma_{n+1}(s,y). \tag{4.4}$$

Combining now relation (3.20) with relations (4.3) and (4.4), we deduce that

$$\begin{aligned}\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|^2_{2,\nu_{n+1}} &+ \left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|^2_{2,\gamma_{n+1}}\\ &\geqslant 2\alpha^2\left((2n+1)-\ln(2\alpha^2)^{2n+1}\right).\end{aligned} \tag{4.5}$$

Replacing f by $\dfrac{f}{\|f\|_{2,\nu_{n+1}}}$ in relation (4.5), we deduce that for every non zero function $f \in L^2(d\nu_{n+1})$,

$$\begin{aligned}\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|^2_{2,\nu_{n+1}} &+ \left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|^2_{2,\gamma_{n+1}}\\ &\geqslant 2\alpha^2\left((2n+1)-\ln(2\alpha^2)^{2n+1}\right)\|f\|^2_{2,\nu_{n+1}}.\end{aligned} \tag{4.6}$$

Inequality (4.6) being true for every $\alpha > 0$, holds in particular for the upper bound of the quantity $2\alpha^2\left((2n+1)-\ln(2\alpha^2)^{2n+1}\right)$, attained for $\alpha = \dfrac{1}{\sqrt{2}}$, hence

$$\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|^2_{2,\nu_{n+1}} + \left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|^2_{2,\gamma_{n+1}} \geqslant (2n+1)\|f\|^2_{2,\nu_{n+1}}. \tag{4.7}$$

On the other hand let $f \in L^2(d\nu_{n+1})$, and for every $t > 0$ we denote by f_t the dilation of f defined on $\mathbb{R}_+ \times \mathbb{R}^n$ by $f_t(r,x) = f(tr,tx)$. Then, f_t belongs to $L^2(d\nu_{n+1})$ and we have

$$\|f_t\|^2_{2,\nu_{n+1}} = \frac{1}{t^{n+2}}\|f\|^2_{2,\nu_{n+1}}. \tag{4.8}$$

Moreover,

$$\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f_t\right\|^2_{2,\nu_{n+1}} = \frac{1}{t^{n+4}}\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|^2_{2,\nu_{n+1}}, \tag{4.9}$$

and

$$\left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f_t)\right\|^2_{2,\gamma_{n+1}} = \frac{1}{t^n}\left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|^2_{2,\gamma_{n+1}}. \tag{4.10}$$

Now, without loss of generality, one may assume that $\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|_{2,\nu_{n+1}}$ and $\left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|_{2,\gamma_{n+1}}$ are both non zero and finite, and hence the same holds for f_t for every $t > 0$ and we have

$$\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f_t\right\|^2_{2,\nu_{n+1}} + \left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f_t)\right\|^2_{2,\gamma_{n+1}} \geqslant (2n+1)\|f_t\|^2_{2,\nu_{n+1}}. \tag{4.11}$$

Then, by relations (4.8), (4.9) and (4.10) we get, for every $t > 0$

$$\frac{1}{t^2}\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|^2_{2,\nu_{n+1}}+t^2\left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|^2_{2,\gamma_{n+1}}\geqslant(2n+1)\|f\|^2_{2,\nu_{n+1}}. \tag{4.12}$$

In particular for $t = t_0 = \sqrt{\dfrac{\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|_{2,\nu_{n+1}}}{\left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|_{2,\gamma_{n+1}}}}$, we obtain the following relation known as Heisenberg-Pauli-Weyl inequality for the spherical mean operator, see [23],

$$\left\|(r^2+\|x\|^2)^{\frac{1}{2}}f\right\|_{2,\nu_{n+1}}\left\|(s^2+2\|y\|^2)^{\frac{1}{2}}\mathscr{F}(f)\right\|_{2,\gamma_{n+1}}\geqslant\frac{2n+1}{2}\|f\|^2_{2,\nu_{n+1}}. \tag{4.13}$$

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