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NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF THE RIESZ POTENTIAL IN MODIFIED MORREY SPACES

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Abstract. We prove that the fractional maximal operator M_α and the Riesz potential operator I_α, $0<\alpha<n$ are bounded from the modified Morrey space $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to the weak modified Morrey space $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if, $\alpha/n \leqslant 1-1/q \leqslant \alpha/(n-\lambda)$ and from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if, $\alpha/n \leqslant 1/p-1/q \leqslant \alpha/(n-\lambda)$.

As applications, we establish the boundedness of some Schödinger type operators on modified Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class. As an another application, we prove the boundedness of various operators on modified Morrey spaces which are estimated by Riesz potentials.

Introduction

For $x\in\mathbb{R}^n$ and $t>0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t)=\mathbb{R}^n\setminus B(x,t)$.

One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x)=\sup_{t>0}|B(x,t)|^{-1+\alpha/n}\int_{B(x,t)}|f(y)|dy,\quad 0\leqslant\alpha<n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

The fractional maximal function $M_\alpha f$ coincides for $\alpha=0$ with the Hardy-Littlewood maximal function $Mf\equiv M_0 f$ and is intimately related to the Riesz potential operator

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-\alpha}},\qquad 0<\alpha<n$$

(see, for example, [1] and [23]).

The operators M_α and I_α play important role in real and harmonic analysis (see, for example [26, 29, 34, 35]).

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In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [24]). Later, Morrey spaces found important applications to Navier-Stokes ([21], [37]) and Schrödinger ([25], [27], [28], [31], [32]) equations, elliptic problems with discontinuous coefficients ([7], [13]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the book [16].

DEFINITION 1. Let $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n$, $[t]_1 = \min\{1,t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} \left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p}$$

respectively.

Note that
$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$
$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset\!\!\!\succ L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \quad \max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \leqslant \|f\|_{\widetilde{L}_{p,\lambda}}$$
and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

DEFINITION 2. [5, 9, 10, 11] Let $1 \leqslant p < \infty, 0 \leqslant \lambda \leqslant n$. We denote by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space and by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified weak Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norms

$$\|f\|_{WL_{p,\lambda}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \left(t^{-\lambda}\ |\{y \in B(x,t) :\ |f(y)| > r\}|\right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda}\ |\{y \in B(x,t) :\ |f(y)| > r\}|\right)^{1/p}$$

respectively.

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0}(\mathbb{R}^n) = W\widetilde{L}_{p,0}(\mathbb{R}^n),$$
$$L_{p,\lambda}(\mathbb{R}^n) \subset WL_{p,\lambda}(\mathbb{R}^n) \text{ and } \|f\|_{WL_{p,\lambda}} \leqslant \|f\|_{L_{p,\lambda}},$$
$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset W\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \text{ and } \|f\|_{W\widetilde{L}_{p,\lambda}} \leqslant \|f\|_{\widetilde{L}_{p,\lambda}}.$$

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{q}\right)$. D. R. Adams [1] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement

THEOREM A. *Let* $0 < \alpha < n$ *and* $0 \leqslant \lambda < n - \alpha$, $1 \leqslant p < \frac{n-\lambda}{\alpha}$.

1) If $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $L_{1,\lambda}(\mathbb{R}^n)$ *to* $WL_{q,\lambda}(\mathbb{R}^n)$.

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of Theorem A reduces to the aforementioned result by Hardy-Littlewood-Sobolev.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \leqslant v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x), \tag{1}$$

hence Theorem A also implies the boundedness of the fractional maximal operator M_α, where v_n is the volume of the unit ball in $\mathbb{R}^n$. F. Chiarenza and M. Frasca [8] proved that the maximal operator M is also bounded from $L_{p,\lambda}$ to $L_{p,\lambda}$ for all $1 < p < \infty$ and $0 < \lambda < n$.

In this paper we study the fractional maximal integral and the Riesz potential in the modified Morrey space. In the case $p = 1$ we prove that the operator I_α is bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if, $\alpha/n \leqslant 1 - 1/q \leqslant \alpha/(n-\lambda)$. In the case $1 < p < (n-\lambda)/\alpha$ we prove that the operator I_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ if and only if, $\alpha/n \leqslant 1/p - 1/q \leqslant \alpha/(n-\lambda)$.

The structure of the paper is as follows. In section 1 the boundedness of the maximal operator in modified Morrey space $\widetilde{L}_{p,\lambda}$ is proved. The main result of the paper is the Hardy-Littlewood-Sobolev inequality in modified Morrey space for the Riesz potential, established in section 2. In section 3 by using the $\left(\widetilde{L}_{p,\lambda}, \widetilde{L}_{q,\lambda}\right)$ boundedness of the fractional maximal operators we establish the boundedness of some Schödinger type operators on modified Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class. In section 4 we give some applications of the results obtained in section 2 to certain operators which are majorized by the Riesz potential.

1. $\widetilde{L}_{p,\lambda}$-boundedness of the maximal operator

In this section we study the $\widetilde{L}_{p,\lambda}$-boundedness of the maximal operator M.

THEOREM 1. *1. If* $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$, $0 \leqslant \lambda < n$, *then* $Mf \in W\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{W\widetilde{L}_{1,\lambda}} \leqslant C_{1,\lambda} \|f\|_{\widetilde{L}_{1,\lambda}},$$

where $C_{1,\lambda}$ depends only on λ and n.

2. If $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, $1 < p < \infty, 0 \leqslant \lambda < n$, then $Mf \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ and

$$\|Mf\|_{\widetilde{L}_{p,\lambda}} \leqslant C_{p,\lambda} \|f\|_{\widetilde{L}_{p,\lambda}},$$

where $C_{p,\lambda}$ depends only on p, λ and n.

Proof. By the Fefferman-Stein inequality

$$\int_{\mathbb{R}^n} (Mf(y))^p g(y)dy \leqslant C_1 \int_{\mathbb{R}^n} |f(y)|^p Mg(y)dy$$

valid for all non-negative functions $g \in L_1^{loc}(\mathbb{R}^n)$ (see [15]), we get

$$\begin{aligned}\int_{B(x,t)} (Mf(y))^p dy &= \int_{\mathbb{R}^n} (Mf(y))^p \chi_{B(x,t)}(y)dy \\ &\leqslant C_1 \int_{\mathbb{R}^n} |f(y)|^p M\chi_{B(x,t)}(y)dy.\end{aligned}$$

As is known (see, [5], Lemma 2, p. 160), for all $t > 0$ and $x, y \in \mathbb{R}^n$

$$\left(\frac{t}{|x-y|+t}\right)^n \leqslant M\chi_{B(x,t)}(y) \leqslant \left(\frac{4t}{|x-y|+t}\right)^n.$$

Therefore, we have the following inequalities

$$\int_{B(x,t)} (Mf(y))^p dy$$

$$\begin{aligned}&\leqslant C_1 \left(\int_{B(x,t)} |f(y)|^p dy + \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}t)\setminus B(x,2^j t)} \frac{t^n |f(y)|^p dy}{(|x-y|+t)^n} \right) \\ &\leqslant C_1 \left([t]_1^\lambda \; \|f\|_{\widetilde{L}_{p,\lambda}}^p + \; \|f\|_{\widetilde{L}_{p,\lambda}}^p \sum_{j=0}^{\infty} \frac{[2^{j+1}t]_1^\lambda}{(2^j+1)^n} \right) \\ &\leqslant C_1 \|f\|_{\widetilde{L}_{p,\lambda}}^p \left([t]_1^\lambda + \begin{cases} \left(2^\lambda t^\lambda \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-n)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-nj}\right)^{1/p}, & 0 < t < \frac{1}{2}, \\ \left(\sum\limits_{j=0}^{\infty} 2^{-nj}\right)^{1/p}, & t \geqslant \frac{1}{2} \end{cases} \right) \\ &\leqslant C_1 \|f\|_{\widetilde{L}_{p,\lambda}}^p \left([t]_1^\lambda + \begin{cases} \left(C_2 t^\lambda + C_3 t^n\right)^{1/p}, & 0 < t < \frac{1}{2}, \\ C_3^{1/p}, & t \geqslant \frac{1}{2} \end{cases} \right) \\ &\leqslant C_4 \; [t]_1^\lambda \; \|f\|_{\widetilde{L}_{p,\lambda}}^p. \qquad \square\end{aligned}$$

2. Hardy-Littlewood-Sobolev inequality in modified Morrey spaces

The following Hardy-Littlewood-Sobolev inequality in modified Morrey spaces is valid.

THEOREM 2. *Let* $0<\alpha<n$, $0\leqslant\lambda<n-\alpha$ *and* $1\leqslant p<\frac{n-\lambda}{\alpha}$.

1) If $1<p<\frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n}\leqslant\frac{1}{p}-\frac{1}{q}\leqslant\frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

2) If $p=1<\frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n}\leqslant 1-\frac{1}{q}\leqslant\frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Proof. 1) *Sufficiency.* Let $0<\alpha<n$, $0<\lambda<n-\alpha$, $f\in\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ and $1<p<\frac{n-\lambda}{\alpha}$. Then

$$I_\alpha f(x)=\left(\int_{B(x,t)}+\int_{\complement_{B(x,t)}}\right)f(y)|x-y|^{\alpha-n}dy\equiv A(x,t)+C(x,t).$$

For $A(x,t)$ we have

$$\begin{aligned}|A(x,t)|&\leqslant\int_{B(x,t)}|f(y)||x-y|^{\alpha-n}dy\\&\leqslant\sum_{j=1}^{\infty}\left(2^{-j}t\right)^{\alpha-n}\int_{B(x,2^{-j+1}t)\setminus B(x,2^{-j}t)}|f(y)|dy.\end{aligned}$$

Hence

$$|A(x,t)|\leqslant C_5t^\alpha Mf(x)\quad\text{with}\quad C_5=\frac{v_n2^n}{2_\alpha-1}. \tag{2}$$

In the second integral by the Hölder's inequality we have

$$\begin{aligned}|C(x,t)|\leqslant&\left(\int_{\complement_{B(x,t)}}|x-y|^{-\beta}|f(y)|^pdy\right)^{1/p}\\&\times\left(\int_{\complement_{B(x,t)}}|x-y|^{\left(\frac{\beta}{p}+\alpha-n\right)p'}dy\right)^{1/p'}=J_1\cdot J_2.\end{aligned}$$

Let $\lambda<\beta<n-\alpha p$. For J_1 we get

$$\begin{aligned}J_1&=\left(\sum_{j=0}^{\infty}\int_{B(x,2^{j+1}t)\setminus B(x,2^jt)}|f(y)|^p|x-y|^{-\beta}dy\right)^{1/p}\\&\leqslant t^{-\frac{\beta}{p}}\|f\|_{\widetilde{L}_{p,\lambda}}\left(\sum_{j=0}^{\infty}2^{-\beta j}[2^{j+1}t]_1^\lambda\right)^{1/p}\\&=t^{-\frac{\beta}{p}}\|f\|_{\widetilde{L}_{p,\lambda}}\begin{cases}\left(2^\lambda t^\lambda\sum\limits_{j=0}^{[\log_2\frac{1}{2t}]}2^{(\lambda-\beta)j}+\sum\limits_{j=[\log_2\frac{1}{2t}]+1}^{\infty}2^{-\beta j}\right)^{1/p}, & 0<t<\frac{1}{2},\\ \left(\sum\limits_{j=0}^{\infty}2^{-\beta j}\right)^{1/p}, & t\geqslant\frac{1}{2}\end{cases}\end{aligned}$$

$$= t^{-\frac{\beta}{p}}\,\|f\|_{\widetilde{L}_{p,\lambda}} \begin{cases} \left(C_6 t^{\lambda} + C_7 t^{\beta}\right)^{1/p}, & 0 < t < \frac{1}{2}, \\ C_7^{1/p}, & t \geqslant \frac{1}{2} \end{cases}$$

$$= \|f\|_{\widetilde{L}_{p,\lambda}} \begin{cases} (C_6 + C_7)^{\frac{1}{p}} t^{\frac{\lambda-\beta}{p}}, & 0 < t < \frac{1}{2}, \\ C_7^{\frac{1}{p}} t^{-\frac{\beta}{p}}, & t \geqslant \frac{1}{2} \end{cases}$$

$$= C_8 [2t]_1^{\frac{\lambda}{p}}\; t^{-\frac{\beta}{p}}\; \|f\|_{\widetilde{L}_{p,\lambda}}, \tag{3}$$

where $C_6 = \frac{2^{\beta}}{2^{\beta-\lambda}-1}$, $C_7 = \frac{2^{2\beta}}{2^{\beta}-1}$ and

$$C_8 = \begin{cases} 2^{-\frac{\lambda}{p}} (C_2 + C_3)^{\frac{1}{p}}, & 0 < t < \frac{1}{2}, \\ C_6^{\frac{1}{p}}, & t \geqslant \frac{1}{2}. \end{cases}$$

For J_2 we obtain

$$J_2 = \left(\int_{\mathbb{S}^{n-1}} d\xi \int_t^{\infty} r^{n-1+\left(\frac{\beta}{p}+\alpha-n\right)p'} dr \right)^{\frac{1}{p'}} = C_9 t^{\frac{\beta}{p}+\alpha-\frac{n}{p}}. \tag{4}$$

From (3) and (4) we have

$$|C(x,t)| \leqslant C_{10} [t]_1^{\frac{\lambda}{p}}\; t^{\alpha-\frac{n}{p}}\; \|f\|_{\widetilde{L}_{p,\lambda}}. \tag{5}$$

Thus

$$|I_{\alpha} f(x)| \leqslant C_{11} \left(t^{\alpha} M f(x) + [t]_1^{\frac{\lambda}{p}}\; t^{\alpha-\frac{n}{p}}\; \|f\|_{\widetilde{L}_{p,\lambda}} \right)$$

$$\leqslant C_{11} \min\left\{ t^{\alpha} M f(x) + t^{\alpha-\frac{n}{p}}\; \|f\|_{\widetilde{L}_{p,\lambda}},\; t^{\alpha} M f(x) + t^{\alpha-\frac{n-\lambda}{p}}\; \|f\|_{\widetilde{L}_{p,\lambda}} \right\}, \qquad t > 0.$$

Minimizing with respect to t, at

$$t = \left[(M f(x))^{-1} \|f\|_{\widetilde{L}_{p,\lambda}} \right]^{p/(n-\lambda)}$$

and

$$t = \left[(M f(x))^{-1} \|f\|_{\widetilde{L}_{p,\lambda}} \right]^{p/n}$$

we have

$$|I_{\alpha} f(x)| \leqslant C_{11} \min\left\{ \left(\frac{M f(x)}{\|f\|_{\widetilde{L}_{p,\lambda}}} \right)^{1-\frac{p\alpha}{n-\lambda}}, \left(\frac{M f(x)}{\|f\|_{\widetilde{L}_{p,\lambda}}} \right)^{1-\frac{p\alpha}{n}} \right\} \|f\|_{\widetilde{L}_{p,\lambda}}.$$

Then

$$|I_{\alpha} f(x)| \leqslant C_{11} \left(M f(x) \right)^{p/q} \|f\|_{\widetilde{L}_{p,\lambda}}^{1-p/q}.$$

Hence, by Theorem 1, we have

$$\begin{aligned}\int_{B(x,t)} |I_\alpha f(y)|^q \, dy &\leqslant C_{12} \, \|f\|_{\widetilde{L}_{p,\lambda}}^{q-p} \int_{B(x,t)} (Mf(y))^p \, dy \\ &\leqslant C_{13} [t]_1^\lambda \, \|f\|_{\widetilde{L}_{p,\lambda}}^q ,\end{aligned}$$

which implies that I_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Necessity. Let $1 < p < \frac{n-\lambda}{\alpha}$, $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ and I_α bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1, t\}$. Then

$$\begin{aligned}\|f_t\|_{\widetilde{L}_{p,\lambda}} &= \sup_{r>0, x\in\mathbb{R}^n} \left([r]_1^{-\lambda} \int_{B(x,r)} |f_t(y)|^p \, dy \right)^{1/p} \\ &= t^{-\frac{n}{p}} \sup_{x\in\mathbb{R}^n, r>0} \left([r]_1^{-\lambda} \int_{B(x,tr)} |f(y)|^p dy \right)^{1/p} \\ &= t^{-\frac{n}{p}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1} \right)^{\lambda/p} \sup_{r>0, x\in\mathbb{R}^n} \left([tr]_1^{-\lambda} \int_{B(x,tr)} |f(y)|^p \, dy \right)^{1/p} \\ &= t^{-\frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda}} ,\end{aligned}$$

and

$$I_\alpha f_t(x) = t^{-\alpha} I_\alpha f(tx),$$

$$\begin{aligned}\|I_\alpha f_t\|_{\widetilde{L}_{q,\lambda}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n, r>0} \left([r]_1^{-\lambda} \int_{B(x,r)} |I_\alpha f(ty)|^q \, dy \right)^{1/q} \\ &= t^{-\alpha-\frac{n}{q}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1} \right)^{\lambda/q} \sup_{r>0, x\in\mathbb{R}^n} \left([tr]_1^{-\lambda} \int_{B(tx,tr)} |I_\alpha f(y)|^q \, dy \right)^{1/q} \\ &= t^{-\alpha-\frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} .\end{aligned}$$

By the boundedness of I_α from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$

$$\begin{aligned}\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} &= t^{\alpha+\frac{n}{q}} \, [t]_{1,+}^{-\frac{\lambda}{q}} \, \|I_\alpha f_t\|_{\widetilde{L}_{q,\lambda}} \\ &\leqslant t^{\alpha+\frac{n}{q}} \, [t]_{1,+}^{-\frac{\lambda}{q}} \, \|f_t\|_{\widetilde{L}_{p,\lambda}} \\ &\leqslant C_{p,q,\lambda} t^{\alpha+\frac{n}{q}-\frac{n}{p}} [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{p,\lambda}} ,\end{aligned}$$

where $C_{p,q,\lambda}$ depends only on p, q, λ and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n}$, then in the case $t \to 0$ we have $\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$.

2) *Sufficiency.* Let $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$. We have

$$|\{y \in B(x,t) \,:\, |I_\alpha f(y)| > 2\beta\}| \leqslant |\{y \in B(x,t) \,:\, |A(y,t)| > \beta\}| \\ + |\{y \in B(x,t) \,:\, |C(y,t)| > \beta\}|.$$

Then

$$\begin{aligned} C(y,t) &= \sum_{j=0}^{\infty} \int_{B(y,2^{j+1}t)\setminus B(y,2^j t)} |f(z)||y-z|^{\alpha-n} dz \\ &\leqslant t^{\alpha-n} \|f\|_{\widetilde{L}_{1,\lambda}} \sum_{j=0}^{\infty} 2^{-(n-\alpha)j} [2^{j+1}t]_1^{\lambda} \\ &= t^{\alpha-n} \|f\|_{\widetilde{L}_{1,\lambda}} \begin{cases} 2^{\lambda} t^{\lambda} \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-n+\alpha)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-(n-\alpha)j}, & 0 < t < \frac{1}{2}, \\ \sum\limits_{j=0}^{\infty} 2^{-(n-\alpha)j}, & t \geqslant \frac{1}{2} \end{cases} \\ &= t^{\alpha-n} \|f\|_{\widetilde{L}_{1,\lambda}} \begin{cases} C_{14} t^{\lambda} + C_{15} t^{n-\alpha}, & 0 < t < \frac{1}{2}, \\ C_{15}, & t \geqslant \frac{1}{2} \end{cases} \\ &= \|f\|_{\widetilde{L}_{1,\lambda}} \begin{cases} (C_{14} + C_{15}) t^{\lambda+\alpha-n}, & 0 < t < \frac{1}{2}, \\ C_{15} t^{\alpha-n}, & t \geqslant \frac{1}{2} \end{cases} \\ &= C_{16} [2t]_1^{\lambda}\, t^{\alpha-n}\, \|f\|_{\widetilde{L}_{1,\lambda}}, \end{aligned}$$

where $C_{14} = \frac{2^{n-\alpha}}{2^{n-\alpha-\lambda}-1}$, $C_{15} = \frac{2^{2(n-\alpha)}}{2^{n-\alpha}-1}$ and

$$C_{16} = \begin{cases} 2^{-\lambda}(C_{14} + C_{15}), & 0 < t < \frac{1}{2}, \\ C_{15}, & t \geqslant \frac{1}{2}. \end{cases}$$

Taking into account inequality (2) and Theorem 2, we have

$$|\{y \in B(x,t) \,:\, |A(y,t)| > \beta\}| \leqslant \left| \left\{ y \in B(x,t) \,:\, Mf(y) > \frac{\beta}{C_5 t^{\alpha}} \right\} \right| \\ \leqslant \frac{C_{17} t^{\alpha}}{\beta} \cdot [t]_1^{\lambda} \, \|f\|_{\widetilde{L}_{1,\lambda}},$$

where $C_{17} = C_5 \cdot C_{1,\lambda}$ and thus if $C_{16}[2t]_1^{\lambda}\, t^{\alpha-n}\, \|f\|_{\widetilde{L}_{1,\lambda}} = \beta$, then $|C(y,t)| \leqslant \beta$ and consequently, $|\{y \in B(x,t) \,:\, |C(y,t)| > \beta\}| = 0$.

Then

$$|\{y\in B(x,t)\ :\ |I_\alpha f(y)|>2\beta\}| \leqslant \frac{C_{17}}{\beta}[t]_1^\lambda t^\alpha \|f\|_{\widetilde{L}_{1,\lambda}} \leqslant C_{18}[t]_1^\lambda \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{n-\lambda}{n-\lambda-\alpha}}, \text{ if } 2t<1$$

and

$$|\{y\in B(x,t)\ :\ |I_\alpha f(y)|>2\beta\}| \leqslant \frac{C_{17}}{\beta}[t]_1^\lambda t^\alpha \|f\|_{\widetilde{L}_{1,\lambda}} \leqslant C_{19}[t]_1^\lambda \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{n}{n-\alpha}}, \text{ if } 2t\geqslant 1,$$

where $C_{18}=C_{17}\cdot C_{16}^{\frac{\alpha}{n-\lambda-\alpha}}$ and $C_{19}=C_{17}\cdot C_{16}^{\frac{\alpha}{n-\alpha}}$.

Finally we have

$$|\{y\in B(x,t)\ :\ |I_\alpha f(y)|>2\beta\}| \leqslant C_{20}[t]_1^\lambda \min\left\{\left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{n-\lambda}{n-\lambda-\alpha}}, \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{n}{n-\alpha}}\right\} \leqslant C_{20}[t]_1^\lambda \left(\frac{1}{\beta}\|f\|_{\widetilde{L}_{1,\lambda}}\right)^q,$$

where $C_{20}=\max\{C_{18},C_{19}\}$.

Necessity. Let I_α is bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$. We have

$$\begin{aligned}
\|I_\alpha f_t\|_{W\widetilde{L}_{q,\lambda}} &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\,\tau>0}\left([\tau]_1^{-\lambda}\int_{\{y\in B(x,\tau)\ :\ |I_\alpha f_t(y)|>r\}} dy\right)^{1/q} \\
&= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\,\tau>0}\left([\tau]_1^{-\lambda}\int_{\{y\in B(tx,\tau)\ :\ |I_\alpha f(ty)|>rt^\alpha\}} dy\right)^{1/q} \\
&= t^{-\alpha-\frac{n}{q}} \sup_{\tau>0}\left(\frac{[t\tau]_1}{[\tau]_1}\right)^{\lambda/q} \sup_{r>0} rt^\alpha \\
&\quad\times \sup_{x\in\mathbb{R}^n,\,\tau>0}\left([t\tau]_1^{-\lambda}\int_{\{y\in B(x,t\tau)\ :\ |I_\alpha f(y)|>rt^\alpha\}} dy\right)^{1/q} \\
&= t^{-\alpha-\frac{n}{q}}[t]_{1,+}^{\frac{\lambda}{q}}\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}}.
\end{aligned}$$

By the boundedness of I_α from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$

$$\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}} \leqslant C_{1,q,\lambda} t^{\alpha+\frac{n}{q}-n}[t]_{1,+}^{\lambda-\frac{\lambda}{q}}\|f\|_{\widetilde{L}_{1,\lambda}},$$

where $C_{1,q,\lambda}$ depends only on q,λ and n.

If $1 < \frac{1}{q} + \frac{\alpha}{n}$, then in the case $t \to 0$ we have $\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then for $t \to \infty$ we obtain $\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n} \leqslant 1 - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$. $\square$

COROLLARY 1. *Let $0 < \alpha < n$, $0 \leqslant \lambda < n - \alpha$ and $1 \leqslant p \leqslant \frac{n-\lambda}{\alpha}$.*

1) If $1 < p < \frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n} \leqslant 1 - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Proof. Sufficiency of Corollary 1 follows from Theorem 2 and inequality (1).

Necessity. (1) Let M_α be bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ for $1 < p < \frac{n-\lambda}{\alpha}$. Then we have

$$M_\alpha f_t(x) = t^{-\alpha} M_\alpha f(tx),$$

and

$$\|M_\alpha f_t\|_{\widetilde{L}_{q,\lambda}} = t^{-\alpha - \frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \|M_\alpha f\|_{\widetilde{L}_{q,\lambda}}.$$

By the same argument in Theorem 2 we obtain $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$.

(2) Let M_α be bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$. Then we have

$$M_\alpha f_t(x) = t^{-\alpha} M_\alpha f(tx),$$

and

$$\|M_\alpha f_t\|_{W\widetilde{L}_{q,\lambda}} = t^{-\alpha - \frac{n}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \|M_\alpha f\|_{W\widetilde{L}_{q,\lambda}}.$$

Hence we obtain $\frac{\alpha}{n} \leqslant 1 - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$. $\square$

3. The modified Morrey estimates for the operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$, where the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for some $q_1 \geqslant n$. The modified Morrey $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ estimates for the operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$ are obtained.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [14, 30, 38]). Shen [30] studied the Schrödinger operator

$-\Delta+V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \geqslant n/2$ and he proved the L_p boundedness of the operators $(-\Delta+V)^{i\gamma}$, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$. Kurata and Sugano generalized Shen's results to uniformly elliptic operators in [17]. Sugano [36] also extended some results of Shen to the operator $V^\gamma(-\Delta+V)^{-\beta}$, $0 \leqslant \gamma \leqslant \beta \leqslant 1$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$, $0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1$ and $\beta-\gamma \geqslant \frac{1}{2}$. Later, Lu [20] and Li [18] investigated the Schrödinger operators in a more general setting.

We investigate the modified Morrey $\widetilde{L}_{p,\lambda} - \widetilde{L}_{q,\lambda}$ boundedness of the operators

$$T_1 = V^\gamma(-\Delta+V)^{-\beta}, \ 0 \leqslant \gamma \leqslant \beta \leqslant 1,$$

$$T_2 = V^\gamma\nabla(-\Delta+V)^{-\beta}, \ 0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1, \ \beta-\gamma \geqslant \frac{1}{2}.$$

Note that the operators $V(-\Delta+V)^{-1}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ in [18] are the special case of T_1 and T_2, respectively.

It is worth pointing out that we need to establish pointwise estimates for T_1, T_2 and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [18]. And we prove the modified Morrey estimates by using $\left(\widetilde{L}_{p,\lambda}, \widetilde{L}_{q,\lambda}\right)$ boundedness of the fractional maximal operators.

DEFINITION 3. 1) A nonnegative locally L_q integrable function V on $\mathbb{R}^n$ is said to belong to the reverse Hölder class B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{\frac{1}{q}} \leqslant \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

2) Let $V \geqslant 0$. We say $V \in B_\infty$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_\infty(B)} \leqslant \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

Clearly, $B_\infty \subset B_q$ for $1 < q < \infty$. But it is important that the B_q class has a property of "self-improvement"; that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$ (see [18]).

The following two pointwise estimates for T_1 and T_2 which proven in [38], Lemma 3.2 with the potential $V \in B_\infty$.

THEOREM B. *Suppose $V \in B_\infty$ and $0 \leqslant \gamma \leqslant \beta \leqslant 1$. Then there exists a constant $C > 0$ such that*

$$|T_1 f(x)| \leqslant CM_\alpha f(x), \quad f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta-\gamma)$.

THEOREM C. *Suppose $V \in B_\infty$, $0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1$ and $\beta - \gamma \geqslant \frac{1}{2}$. Then there exists a constant $C > 0$ such that*

$$|T_2 f(x)| \leqslant C M_\alpha f(x), \quad f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - \gamma) - 1$.

Note that the similar estimates for the adjoint operators T_1^* and T_2^* with the potential $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$ also valid (see [19]).

THEOREM D. *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$, $0 \leqslant \gamma \leqslant \beta \leqslant 1$ and let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$. Then there exists a constant $C > 0$ such that*

$$|T_1^* f(x)| \leqslant C \left(M_{\alpha q_2} \left(|f|^{q_2} \right)(x) \right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - \gamma)$.

THEOREM E. *Suppose $V \in B_{q_1}$ for some $q_1 > \frac{n}{2}$, $0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1$ and $\beta - \gamma \geqslant \frac{1}{2}$. And let*

$$\frac{1}{q_1} = \begin{cases} 1 - \frac{\gamma}{q_1}, & \text{if } q_1 > n, \\ 1 - \frac{\alpha+1}{q_1} + \frac{1}{n}, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$|T_2^* f(x)| \leqslant C \left(M_{\alpha q_2} \left(|f|^{q_2} \right)(x) \right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^n),$$

where $\alpha = 2(\beta - \gamma) - 1$.

The above theorems will yield the modified Morrey estimates for T_1 and T_2.

COROLLARY 2. *Assume that $V \in B_\infty$, and $0 \leqslant \gamma \leqslant \beta \leqslant 1$. Let $1 \leqslant p \leqslant \frac{n}{\gamma}$, $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$ and $0 \leqslant \lambda < n$, where $\alpha = 2(\beta - \gamma) < n$.*

1) Let $p = 1 < \frac{n-\lambda}{\alpha}$. Then there exists a positive constant C such that for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_1 f\|_{W\widetilde{L}_{q,\lambda}} \leqslant C \|f\|_{\widetilde{L}_{1,\lambda}}.$$

2) Let $1 < p < \frac{n-\lambda}{\alpha}$. Then there exists a positive constant C such that for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_1 f\|_{\widetilde{L}_{q,\lambda}} \leqslant C \|f\|_{\widetilde{L}_{p,\lambda}}.$$

COROLLARY 3. *Assume that $V \in B_\infty$, $0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1$ and $\beta - \gamma \geqslant \frac{1}{2}$. Let $1 \leqslant p \leqslant \frac{n}{\alpha}$, $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$ and $0 \leqslant \lambda < n$, where $\alpha = 2(\beta - \gamma) - 1 < n$.*

1) Let $p = 1 < \frac{n-\lambda}{\alpha}$. Then there exists a positive constant C such that for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_2 f\|_{W\widetilde{L}_{q,\lambda}} \leqslant C \|f\|_{\widetilde{L}_{1,\lambda}}.$$

2) Let $1 < p < \frac{n-\lambda}{\alpha}$. *Then there exists a positive constant* C *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_2 f\|_{\widetilde{L}_{q,\lambda}} \leqslant C\|f\|_{\widetilde{L}_{p,\lambda}}.$$

COROLLARY 4. *Assume that* $V \in B_{q_1}$ *for* $q_1 > \frac{n}{2}$, *and* $0 \leqslant \gamma \leqslant \beta \leqslant 1$.

Let $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, $1 \leqslant p < \frac{1}{\frac{\alpha}{q_1}+\frac{\alpha}{n}}$, $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{\frac{n}{q_2}-\lambda}$ *and* $0 \leqslant \lambda < nq_2$, *where* $\alpha = 2(\beta - \gamma) < n$.

1) Let $p = 1 < \frac{\frac{n}{q_2}-\lambda}{\alpha}$. *Then there exists a positive constant* C *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_1 f\|_{W\widetilde{L}_{q,\lambda}} \leqslant C\|f\|_{\widetilde{L}_{1,\lambda}}.$$

2) Let $1 < p < \frac{\frac{n}{q_2}-\lambda}{\alpha}$. *Then there exists a positive constant* C *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_1 f\|_{\widetilde{L}_{q,\lambda}} \leqslant C\|f\|_{\widetilde{L}_{p,\lambda}}.$$

COROLLARY 5. *Assume that* $V \in B_{q_1}$ *for* $q_1 > \frac{n}{2}$, *and*

$$\begin{cases} 0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1, & \text{if } q_1 > n, \\ 0 \leqslant \gamma \leqslant \frac{1}{2} < \beta \leqslant 1, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

Let $\alpha = 2(\beta - \gamma) - 1 < n$ *and* $\beta - \gamma \geqslant \frac{1}{2}$, *and let* $1 \leqslant p < \frac{1}{\frac{\alpha}{q_1}+\frac{\alpha}{n}}$, $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{\frac{n}{q_2}-\lambda}$, $\frac{1}{q_2} = 1 - \frac{\alpha}{q_1}$, *and* $0 \leqslant \lambda < nq_2$, *where*

$$\frac{1}{p_1} = \begin{cases} \frac{\alpha}{q_1}, & \text{if } q_1 > n, \\ \frac{\alpha+1}{q_1} + \frac{1}{n}, & \text{if } \frac{n}{2} < q_1 < n. \end{cases}$$

1) Let $p = 1 < \frac{n-\lambda}{\alpha}$. *Then there exists a positive constant* C *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_2 f\|_{W\widetilde{L}_{q,\lambda}} \leqslant C\|f\|_{\widetilde{L}_{1,\lambda}}.$$

2) Let $1 < p < \frac{\frac{n}{q_2}-\lambda}{\alpha}$. *Then there exists a positive constant* C *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\|T_2 f\|_{\widetilde{L}_{q,\lambda}} \leqslant C\|f\|_{\widetilde{L}_{p,\lambda}}.$$

4. Some applications

The theorems of the section 2 can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \leqslant \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{6}$$

for $x,y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, Chapter 5 in [34].

THEOREM 3. *Let* $0 < \alpha < n$, $0 \leqslant \lambda < n$ *and condition* (6) *be satisfied.*

1) If $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$ *is sufficient for the boundedness of* $L^{-\alpha/2}$ *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \leqslant 1 - \frac{1}{q} \leqslant \frac{\alpha}{n-\lambda}$ *is sufficient for the boundedness of* $L^{-\alpha/2}$ *from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (6), it follows that

$$|L^{-\alpha/2} f(x)| \leqslant C I_\alpha |f|(x)$$

for all $x \in \mathbb{R}^n$, where $C > 0$ is independent of x (see [12]). Hence by Theorem 2 we have

$$\|L^{-\alpha/2} f\|_{\widetilde{L}_{q,\lambda}} \leqslant C \|I_\alpha |f|\|_{\widetilde{L}_{q,\lambda}} \leqslant C \|f\|_{\widetilde{L}_{p,\lambda}},$$

where the constant $C > 0$ is independent of f. $\square$

Property (6) is satisfied for large classes of differential operators (see, for example [6]). In [6] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorem 2 is also applicable for proving boundedness of those operators from $\widetilde{L}_{p,\lambda}$ to $\widetilde{L}_{q,\lambda}$.

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