Journal of
Mathematical
Inequalities
Volume 5, Number 4 (2011), 507–521

ESTIMATING THE NORMALIZED JENSEN FUNCTIONAL

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(*Communicated by S. Abramovich*)

Abstract. We establish upper and lower bounds for the normalized Jensen functional in the context of $\left(M_{[\varphi]},A\right)$-convexity. In connection with these results, a refinement of the triangle inequality is proved.

1. Introduction

The normalized Jensen functional for convex functions f is defined by

$$\mathscr{J}_n(f,\mathbf{p},\mathbf{x})=\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right).$$

The aim of this paper is to estimate the normalized Jensen functional in the framework of $\left(M_{[\varphi]},A\right)$-convex functions and to state some results related to the lower and upper bounds for it (for details see [4]). We give the analogue bounds for the integral case also. In the last section some resources helpful when dealing with norm inequalities can be found.

For the convenience of the reader, before stating the results, we establish the notation and we briefly recall some basic notions related to our goal:

Throughout the paper $I\subseteq\mathbb{R}$ and $J\subseteq\mathbb{R}$ are intervals.

DEFINITION 1. Let $\varphi:I\rightarrow J$ a continuous, increasing and bijective function. The *weighted quasi-arithmetic mean* $M_{[\varphi]}$ of a nonempty set of data $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in I^n$ with weights $\mathbf{p}=(p_1,p_2,\ldots,p_n)$, where $p_i\geqslant 0$, $\sum_{i=1}^{n}p_i=1$, is defined by the formula

$$M_{[\varphi]}(\mathbf{x};\mathbf{p})=\varphi^{-1}\left(\sum_{i=1}^{n}p_i\varphi(x_i)\right).$$

Particularly the weighted arithmetic mean $A(\mathbf{x};\mathbf{p})=\sum_{i=1}^{n}p_ix_i$ corresponds to $\varphi(x)=x$, and the weighted geometric mean $G(\mathbf{x};\mathbf{p})=\prod_{i=1}^{n}x_i^{p_i}$ corresponds to $\varphi(x)=\log x$.

Mathematics subject classification (2010): Primary 26B25, Secondary 26E60, 26D15.
Keywords and phrases: Normalized Jensen functional, quasi-arithmetic means.

DEFINITION 2. Suppose that we have two continuous, bijective and increasing functions $\varphi : I \to I, \psi : J \to J$. A function $f : I \to J$ is called $(M_{[\varphi]}, M_{[\psi]})$ - *convex* if for every two points $a, b \in I$ and all $\lambda \in [0,1]$

$$f\left(\varphi^{-1}\left((1-\lambda)\varphi(a)+\lambda\varphi(b)\right)\right) \leqslant \psi^{-1}\left((1-\lambda)\psi(f(a))+\lambda\psi(f(b))\right).$$

The function is called *strictly* $(M_{[\varphi]}, M_{[\psi]})$ - *convex* if the inequality is strict for all $a \neq b$ and $\lambda \in (0,1)$.

According to the definition, we observe some particular cases, depending on which type of mean, arithmetic (A) or geometric (G), it is given on its domain and codomain (see also C. P. Niculescu [8]):

- (A,A)-convex functions (the usual convex functions)
- (A,G)-convex functions (the log-convex functions)
- (G,A)-convex functions
- (G,G)-convex functions (the multiplicatively convex functions).

A basic result concerning the convex functions is *Jensen's inequality*. Its formal statement is as follows:

PROPOSITION 1. (Jensen's inequality) *A real valued function f defined on I is convex if and only if for all $x_1, x_2, ..., x_n \in I$ and $p_1, p_2, ..., p_n \in (0,1)$ with $\sum_{i=1}^{n} p_i = 1$ we have*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i).$$

If f is strictly convex then the equality holds if and only if $x_1 = ... = x_n$.

If we restate it as

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geqslant 0$$

we can see that the lower bound zero depends only on f and the interval I.

We introduce now a generalization of the Jensen functional defined above, namely:

DEFINITION 3. *The normalized Jensen functional* is defined by

$$\mathscr{T}(f,\mathbf{p},\mathbf{x}) = \sum_{i=1}^{n} p_i f(x_i) - \left(f \circ \varphi^{-1}\right)\left(\sum_{i=1}^{n} p_i \varphi(x_i)\right),$$

where f is a $(M_{[\varphi]}, A)$-convex function.

The following results will be extended in later sections.

THEOREM 1. *For* $i = 1,...,n$, *suppose that* $x_i \in I$, $p_i > 0$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $q_i > 0$ *with* $\sum_{i=1}^{n} q_i = 1$. *Then*

$$\min_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\mathscr{T}(f,\mathbf{q},\mathbf{x}) \leqslant \mathscr{T}(f,\mathbf{p},\mathbf{x}) \leqslant \max_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\mathscr{T}(f,\mathbf{q},\mathbf{x})$$

for all f *which are* $\left(M_{[\varphi]},A\right)-$*convex functions.*

If f *is strictly* $\left(M_{[\varphi]},A\right)-$*convex then both sides of the inequality are equalities if and only if* $x_1 = ... = x_n$ *or* $p_i = q_i$ *for all* $i = 1,...,n$.

See [7] for the details of the proof. An interesting proof is also developed for the particular case $f(x) = x$ and $\varphi(x) = \log x$ by J. M. Aldaz [2].

Theorem 1 corresponds in the context of $\left(M_{[\varphi]},A\right)$-convex functions to a result proved by S. S. Dragomir for convex functions (see [4, Theorem 1]). By the substitution $\varphi(x_i) = z_i$ Theorem 1 is [4, Theorem 1] for the convex function $f \circ \varphi^{-1}$.

We generalize this result also for the continuous case as follows.

THEOREM 2. (Integral analogue of Theorem 1) *Let* $p(x)\,dx$ *and* $q(x)\,dx$ *be two absolutely continuous measures, where* $p,q : [a,b] \to (0,\infty)$ *are increasing such that* $\int_a^b p(x)\,dx = 1$ *and* $\int_a^b q(x)\,dx = 1$. *Define*

$$\mathscr{T}(f,p) := \int_a^b f(x)\,p(x)\,dx - \left(f\circ\varphi^{-1}\right)\left(\int_a^b \varphi(x)\,p(x)\,dx\right).$$

Then the following inequalities hold

$$\inf_{t,s\in[a,b];s\neq t}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx}\right\}\mathscr{T}(f,q) \leqslant \mathscr{T}(f,p) \leqslant \sup_{t,s\in[a,b];s\neq t}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx}\right\}\mathscr{T}(f,q),$$

for every $f : [a,b] \to \mathbb{R}$ *a* $\left(M_{[\varphi]},A\right)$*-convex function.*

Theorem 2 (see its proof in Section 2.2) corresponds in the framework of $\left(M_{[\varphi]},A\right)$-convex functions to a result due to J. Barić, M. Matić and J. Pecarić [3].

2. Main results

Motivated by the above results, we introduce in the present paper a more general functional and we establish its bounds. The techniques of the proofs are similar to the technique used in [1, Theorem 6, Theorem 7].

2.1. The discrete case

DEFINITION 4. Assume that we have a real valued function f defined on an interval I, the real numbers p_{ij}, $i = 1,...,k$ and $j = 1,...,n_i$ such that $p_{ij} > 0$, $\sum_{j=1}^{n_i} p_{ij} = 1$ for all $i = 1,...,k$ (we denote $\mathbf{p}_i = (p_{i1}, p_{i2},..., p_{in_i})$), $\mathbf{x}_i = (x_{i1}, x_{i2},..., x_{in_i}) \in I^{n_i}$ for all $i=1,...,k$ and $\mathbf{q}=(q_1,q_2,...,q_k)$, $q_i>0$ such that $\sum_{i=1}^{k} q_i=1$. We define the *generalized Jensen functional* by

$$\mathscr{J}_k(f,\mathbf{p}_1,...,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,...,\mathbf{x}_k) := \sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k} f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) - f\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}\right).$$

First of all observe that, by Jensen's inequality, the functional $\mathscr{J}_k$ is nonnegative. We will need the following lemma.

LEMMA 1. *Let* $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ *be as in Definition* 4. *Then*

$$\sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k} \sum_{i=1}^{k} q_i x_{ij_i} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}. \tag{2.1}$$

Proof. The proof is straightforward, by a simple computation we get the claimed result:

$$\begin{aligned} &\sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k} \sum_{i=1}^{k} q_i x_{ij_i} \\ &= q_1 \sum_{j_1=1}^{n_1} p_{1j_1} x_{1j_1} \sum_{j_2,...,j_k=1}^{n_2,...,n_k} p_{2j_2}...p_{kj_k} + ... \\ &\quad + q_k \sum_{j_k=1}^{n_k} p_{kj_k} x_{kj_k} \sum_{j_1,...,j_{k-1}=1}^{n_1,...,n_{k-1}} p_{1j_1}...p_{(k-1)j_{k-1}} \\ &= q_1 \sum_{j_1=1}^{n_1} p_{1j_1} x_{1j_1} + ... + q_k \sum_{j_k=1}^{n_k} p_{kj_k} x_{kj_k} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij} x_{ij}. \qquad \square \end{aligned}$$

The following theorem is a generalization of Theorem 1, for $\varphi(x)=x$.

THEOREM 3. *Let* f, $\mathbf{p}_i$, $\mathbf{x}_i$ *and* $\mathbf{q}$ *be as in Definition* 4 *and the positive real numbers* r_{ij}, $i=1,...,k$ *and* $j=1,...,n_i$ *such that* $\sum_{j=1}^{n_i} r_{ij}=1$ *for all* $i=1,...,k$. *We denote*

$$\begin{aligned} \mathbf{r}_i &= (r_{i1},r_{i2},...,r_{in_i}) \ \text{for all } i=1,...,k, \\ m &= \min_{\substack{1\leqslant j_1\leqslant n_1 \\ \cdots \\ 1\leqslant j_k\leqslant n_k}} \left\{ \frac{p_{1j_1}...p_{kj_k}}{r_{1j_1}...r_{kj_k}} \right\}, \\ M &= \max_{\substack{1\leqslant j_1\leqslant n_1 \\ \cdots \\ 1\leqslant j_k\leqslant n_k}} \left\{ \frac{p_{1j_1}...p_{kj_k}}{r_{1j_1}...r_{kj_k}} \right\}. \end{aligned}$$

If f is a convex function then we have

$$\begin{aligned} & m\mathscr{J}_k\left(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k\right) \\ \leqslant\ & \mathscr{J}_k\left(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k\right) \leqslant M\mathscr{J}_k\left(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k\right). \end{aligned}$$

Proof. If $m \geqslant 1$ then

$$p_{1j_1}\ldots p_{kj_k} - r_{1j_1}\ldots r_{kj_k} \geqslant p_{1j_1}\ldots p_{kj_k} - mr_{1j_1}\ldots r_{kj_k} \geqslant 0,$$

$1 \leqslant j_i \leqslant n_i,\ 1 \leqslant i \leqslant k$. On the other hand

$$\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k} - r_{1j_1}\ldots r_{kj_k}\right) = 0.$$

Therefore $p_{1j_1}\ldots p_{kj_k} = r_{1j_1}\ldots r_{kj_k},\ 1 \leqslant j_i \leqslant n_i,\ 1 \leqslant i \leqslant k$ that is $\mathbf{p}_i = \mathbf{r}_i,$ for all $i = 1,\ldots,k$ and

$$\mathscr{J}_k\left(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k\right) = \mathscr{J}_k\left(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k\right).$$

It remains to consider the case $m < 1$.

We prove the first inequality. Indeed, applying twice the equality (2.1) we get the desired result:

$$\begin{aligned} & \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k} - mr_{1j_1}\ldots r_{kj_k}\right) f\left(\sum_{i=1}^{k} q_i x_{ij_i}\right) + mf\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} r_{ij}x_{ij}\right) \\ \geqslant\ & f\left(\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k} - mr_{1j_1}\ldots r_{kj_k}\right)\sum_{i=1}^{k} q_i x_{ij_i} + m\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} r_{ij}x_{ij}\right) \\ =\ & f\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}x_{ij}\right). \end{aligned}$$

The proof of the other inequality goes likewise and we omit the details. □

The following particular case is of interest: $\mathbf{p}_1 = \ldots = \mathbf{p}_k = \mathbf{p}$ and $\mathbf{x}_1 = \ldots = \mathbf{x}_k = \mathbf{x}$. Then we have $\mathscr{J}_k\left(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k\right) = \mathscr{J}_k\left(f,\mathbf{p},\mathbf{q},\mathbf{x}\right)$, where

$$\mathscr{J}_k\left(f,\mathbf{p},\mathbf{q},\mathbf{x}\right) := \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

For this particular case we get the following result:

COROLLARY 1. *Assume that we have* $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in I^n,\ \mathbf{p} = (p_1,p_2,\ldots,p_n)$ *such that* $p_i > 0,\ i = 1,\ldots,n,\ \sum_{i=1}^{n} p_i = 1,\ \mathbf{q} = (q_1,q_2,\ldots,q_k)$ *such that* $q_i > 0,\ i =$

$1,\dots,n,\ \sum_{i=1}^{k} q_i = 1\ (1 \leqslant k \leqslant n)$ *and* $\mathbf{r} = (r_1, r_2, \dots, r_n)$ *such that* $r_i > 0,\ i = 1,\dots,n,$ $\sum_{i=1}^{n} r_i = 1$. *If* f *is a convex function then we have*

$$\min_{1 \leqslant i_1,\dots,i_k \leqslant n} \left\{ \frac{p_{i_1} \cdots p_{i_k}}{r_{i_1} \dots r_{i_k}} \right\} \mathscr{J}_k(f, \mathbf{r}, \mathbf{q}, \mathbf{x}) \leqslant \mathscr{J}_k(f, \mathbf{p}, \mathbf{q}, \mathbf{x}) \leqslant \max_{1 \leqslant i_1,\dots,i_k \leqslant n} \left\{ \frac{p_{i_1} \cdots p_{i_k}}{r_{i_1} \dots r_{i_k}} \right\} \mathscr{J}_k(f, \mathbf{r}, \mathbf{q}, \mathbf{x}).$$

Proof. It is a special case of Theorem 3 for $\mathbf{p}_1 = \dots = \mathbf{p}_k = \mathbf{p}$ and $\mathbf{x}_1 = \dots = \mathbf{x}_k = \mathbf{x}$. □

Corollary 1 is a result due to S. Abramovich and S. S. Dragomir (see [1]).

We denote in the sequel

$$\mathscr{J}_f(a,b) = \max_{p,q} \{ pf(a) + qf(b) - f(pa + qb);\ p,q > 0,\ p + q = 1 \}.$$

The right side of Proposition 2 appears in [9] and earlier as an immediate consequence of theorems proved in the paper [6] by A. Matković and J. Pečarić:

PROPOSITION 2. *Assume that* $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$, $\mathbf{p} = (p_1, p_2, \dots, p_n)$ *are such that* $p_i > 0,\ \sum_{i=1}^{n} p_i = 1$. *Let* f *be a convex function on* $[a,b]$. *Then*

$$\mathscr{J}(f, \mathbf{p}, \mathbf{x}) \leqslant \mathscr{J}_f(a,b) \leqslant f(a) + f(b) - 2f\left(\frac{a+b}{2}\right).$$

Our next result reads as follows.

THEOREM 4. *Let* $\mathbf{p}_i$, $\mathbf{x}_i$ *and* $\mathbf{q}$ *be as in Definition* 4. *If* f *is a real valued convex function on* $I = [a,b]$, *then*

$$\mathscr{J}_k(f, \mathbf{p}_1, \dots, \mathbf{p}_k, \mathbf{q}, \mathbf{x}_1, \dots, \mathbf{x}_k) \leqslant \mathscr{J}_f(a,b).$$

Proof. Since $x_{ij} \in [a,b]$, $i = 1,\dots,k$, $j = 1,\dots,n_i$, we may consider $\lambda_{ij} \in [0,1]$ such that

$$x_{ij} = (1 - \lambda_{ij})a + \lambda_{ij} b$$

for all $i = 1,\dots,k$ and $j = 1,\dots,n_i$.

By Jensen's inequality we have

$$\begin{aligned}
&\mathscr{J}_k(f,\mathbf{p}_1,\dots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\dots,\mathbf{x}_k)\\
&=\sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k} f\left(\sum_{i=1}^{k} q_i\left[(1-\lambda_{ij_i})a+\lambda_{ij_i}b\right]\right)\\
&\quad -f\left(\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}\left[(1-\lambda_{ij})a+\lambda_{ij}b\right]\right)\\
&\leqslant \sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k} \sum_{i=1}^{k} q_i\left[(1-\lambda_{ij_i})f(a)+\lambda_{ij_i}f(b)\right]\\
&\quad -f\left(a\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}(1-\lambda_{ij})+b\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}\lambda_{ij}\right)\\
&= f(a)\left(1-\sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k}\sum_{i=1}^{k} q_i\lambda_{ij_i}\right)+f(b)\sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k}\sum_{i=1}^{k} q_i\lambda_{ij_i}\\
&\quad -f\left(a\left(1-\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}\lambda_{ij}\right)+b\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}\lambda_{ij}\right).
\end{aligned}$$

Since

$$\sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k}\sum_{i=1}^{k} q_i\lambda_{ij_i}=\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}\lambda_{ij}$$

we finally get

$$\mathscr{J}_k(f,\mathbf{p}_1,\dots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\dots,\mathbf{x}_k)\leqslant f(a)p+f(b)q-f(pa+qb),$$

where $p=1-\sum_{i=1}^{k}\left(q_i\sum_{j=1}^{n_i}p_{ij}\lambda_{ij}\right)$, $q=\sum_{i=1}^{k}\left(q_i\sum_{j=1}^{n_i}p_{ij}\lambda_{ij}\right)$.

Obviously we have

$$0\leqslant \min_{i,j}\{\lambda_{ij}\}\leqslant \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} p_{ij}\lambda_{ij}\leqslant \max_{i,j}\{\lambda_{ij}\}\leqslant 1.$$

Then $0\leqslant p,q\leqslant 1$, $p+q=1$ and we conclude that the first inequality is valid. □

The theorem above can be used to infer the following corollary:

COROLLARY 2. *Assume that* $\mathbf{x}=(x_1,x_2,\dots,x_n)\in[a,b]^n$, $\mathbf{p}=(p_1,p_2,\dots,p_n)$ *are such that* $p_i>0$, $\sum_{i=1}^{n}p_i=1$ *and* $\mathbf{q}=(q_1,q_2,\dots,q_k)$ *such that* $q_i>0$, $\sum_{i=1}^{k}q_i=1$. *If* f *is a convex real valued function defined on* $[a,b]$ *then*

$$\mathscr{J}_k(f,\mathbf{p},\mathbf{q},\mathbf{x})\leqslant \mathscr{J}_f(a,b)\leqslant f(a)+f(b)-2f\left(\frac{a+b}{2}\right).$$

Proof. The first inequality is a special case of Theorem 4 for $\mathbf{p}_1 = ... = \mathbf{p}_k = \mathbf{p}$ and $\mathbf{x}_1 = ... = \mathbf{x}_k = \mathbf{x}$. $\square$

For $k = 1$ we state as an immediate consequence that Proposition 2 holds.

REMARK 1. Let $\mathbf{p}_i$, $\mathbf{x}_i$ and $\mathbf{q}$ be as in Definition 4. Let f be $\left(M_{[\varphi]}, A\right)$ – convex and define the functional

$$\mathscr{T}_k\left(f,\mathbf{p}_1,...,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,...,\mathbf{x}_k\right) := \sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k} q_i\varphi\left(x_{ij_i}\right)\right) - \left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k} q_i\sum_{j=1}^{n_i} p_{ij}\varphi\left(x_{ij}\right)\right).$$

We can easily obtain analogues results for $\mathscr{T}_k$, simply by substituting $\mathscr{J}_k$ by $\mathscr{T}_k$, the function f by $f\circ\varphi^{-1}$ (obviously it is convex) and x by $\varphi(x)$.

The following proposition was stated for the case of convex functions by L. Horváth at the Conference on Inequalities and Applications '10, Hajdúszoboszló, Hungary (September 19-25, 2010). We give the statement for $\left(M_{[\varphi]}, A\right)$ - convex functions and we prove it:

PROPOSITION 3. *Let $\mathbf{p}_i$, $\mathbf{x}_i$ and $\mathbf{q}$ be as in Definition 4. If f is $\left(M_{[\varphi]}, A\right)$ - convex function then*

$$\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k} q_i\sum_{j=1}^{n_i} p_{ij}\varphi\left(x_{ij}\right)\right) \leqslant \sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k} q_i\varphi\left(x_{ij_i}\right)\right) \leqslant \sum_{i=1}^{k}\left(q_i\sum_{j=1}^{n_i} p_{ij}f\left(x_{ij}\right)\right).$$

Proof. The first inequality can be proved taking into account that

$$\prod_{i=1}^{k}\left(p_{i1}+...+p_{in_i}\right) = \sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k} = 1.$$

Since f is $\left(M_{[\varphi]}, A\right)$ - convex, we have

$$\left(f\circ\varphi^{-1}\right)\left(\sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k}\sum_{i=1}^{k} q_i\varphi\left(x_{ij_i}\right)\right) \leqslant \sum_{j_1,...,j_k=1}^{n_1,...,n_k} p_{1j_1}...p_{kj_k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k} q_i\varphi\left(x_{ij_i}\right)\right).$$

Combining this inequality with Lemma 1, we may conclude that the first inequality holds.

The second inequality also follows from the convexity of $f\circ\varphi^{-1}$ and from Lemma 1:

$$\begin{aligned}&\sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k} q_i\varphi\left(x_{ij_i}\right)\right)\\ &\leqslant \sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} p_{1j_1}\dots p_{kj_k}\sum_{j=1}^{k} q_j f\left(x_{ij_i}\right)=\sum_{i=1}^{k}\left(q_i\sum_{j=1}^{n_i} p_{ij} f\left(x_{ij}\right)\right).\end{aligned}$$

This concludes the proof. □

COROLLARY 3. *Assume that* f *is a real valued function defined on an interval* I, $\mathbf{x}=(x_1,x_2,\dots,x_n)\in I^n$, $\mathbf{p}=(p_1,p_2,\dots,p_n)$ *such that* $p_i>0$, $\sum_{i=1}^n p_i=1$ *and* $\mathbf{q}=(q_1,q_2,\dots,q_k)$ *such that* $q_i>0$, $\sum_{i=1}^k q_i=1$ $(1\leqslant k\leqslant n)$. *If* f *is* $\left(M_{[\varphi]},A\right)$ *- convex then we have*

$$\begin{aligned}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{n} p_i\varphi\left(x_i\right)\right) &\leqslant \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k}\left(f\circ\varphi^{-1}\right)\left(\sum_{j=1}^{k} q_j\varphi\left(x_{i_j}\right)\right)\\ &\leqslant \sum_{i=1}^{n} p_i f\left(x_i\right).\end{aligned}$$

Proof. It follows immediately from Proposition 3 considering $\mathbf{p}_1=\dots=\mathbf{p}_k=\mathbf{p}$ and $\mathbf{x}_1=\dots=\mathbf{x}_k=\mathbf{x}$. □

If $k=1$ and f is a convex function then Corollary 3 reduces to Jensen's inequality.

If $f:I\subset(0,\infty)\to(0,\infty)$ is a $\left(M_{[\varphi]},M_{[\psi]}\right)-$convex function, then $g:=\psi\circ f\circ\varphi^{-1}$ is convex. This means that from results obtained for $\mathscr{J}_k$ for convex functions one can derive results related to $\left(M_{[\varphi]},M_{[\psi]}\right)-$convex functions.

2.2. The integral case

In what follows we shall concentrate on the integral analogue of the results from the previous sections.

Proof. [Proof of Theorem 2] We will prove the first inequality. We denote

$$m=\inf_{t,s\in[a,b],s\neq t}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx}\right\}>0.$$

If $m\geqslant 1$ then

$$\int_s^t (p(x)-q(x))\,dx\geqslant\int_s^t (p(x)-mq(x))\,dx\geqslant 0,$$

$s\neq t$. As $\int_a^b (p(x)-q(x))\,dx=0$, this implies

$$\int_s^t p(x)\,dx=\int_s^t q(x)\,dx,$$

for $s \neq t$, that is $p = q$.

It remains to consider the case $m < 1$. Since the function $f \circ \varphi^{-1}$ is convex, we deduce that the required inequality holds:

$$
\begin{aligned}
&\int_a^b f(x)(p(x) - mq(x))\,dx + m\left(f \circ \varphi^{-1}\right)\left(\int_a^b \varphi(x) q(x)\,dx\right) \\
\geqslant &\int_a^b (p(x) - mq(x))\,dx \\
&\cdot \left(f \circ \varphi^{-1}\right)\left(\frac{1}{\int_a^b (p(x) - mq(x))\,dx}\int_a^b \varphi(x)(p(x) - mq(x))\,dx\right) \\
&+ m\left(f \circ \varphi^{-1}\right)\left(\int_a^b \varphi(x) q(x)\,dx\right) \\
= &(1-m)\left(f \circ \varphi^{-1}\right)\left(\frac{1}{1-m}\int_a^b \varphi(x)(p(x) - mq(x))\,dx\right) \\
&+ m\left(f \circ \varphi^{-1}\right)\left(\int_a^b \varphi(x) q(x)\,dx\right) \\
\geqslant &\left(f \circ \varphi^{-1}\right)\left(\int_a^b \varphi(x) p(x)\,dx\right).
\end{aligned}
$$

The proof of the second inequality goes likewise and has been omitted. $\square$

We consider $p_i(x)\,dx$ and $r_i(x)\,dx$, $i = 1, ..., k$, to be absolutely continuous measures, where $p_i,\ r_i : [a,b] \to (0,\infty)$ are increasing functions such that $\int_a^b p_i(x)\,dx = 1$, $\int_a^b r_i(x)\,dx = 1$. We also consider $\mathbf{q} = (q_1, q_2, ..., q_k)$, $q_i > 0$ with $\sum_{i=1}^k q_i = 1$. We denote

$$
\begin{aligned}
\mathscr{I}_k(f, p_1, ..., p_k, \mathbf{q}) := &\int_{[a,b]^k} f\left(\sum_{i=1}^k q_i x_i\right)\prod_{i=1}^k (p_i(x_i)\,dx_i) \\
&- f\left(\sum_{i=1}^k q_i \int_a^b x p_i(x)\,dx\right)(\geqslant 0).
\end{aligned}
$$

THEOREM 5. (Integral analogue of Theorem 3) *If f is convex then we have*

$$
\begin{aligned}
&\inf_{t,s\in[a,b],s\neq t}\left\{\frac{\int_{[t,s]^k}\prod_{i=1}^k (p_i(x_i)\,dx_i)}{\int_{[t,s]^k}\prod_{i=1}^k (r_i(x_i)\,dx_i)}\right\}\mathscr{I}_k(f, r_1, ..., r_k, \mathbf{q}) \\
\leqslant &\ \mathscr{I}_k(f, p_1, ..., p_k, \mathbf{q}) \\
\leqslant &\sup_{t,s\in[a,b],s\neq t}\left\{\frac{\int_{[t,s]^k}\prod_{i=1}^k (p_i(x_i)\,dx_i)}{\int_{[t,s]^k}\prod_{i=1}^k (r_i(x_i)\,dx_i)}\right\}\mathscr{I}_k(f, r_1, ..., r_k, \mathbf{q}).
\end{aligned}
$$

Proof. We will prove *the first inequality*. We denote

$$
m = \inf_{t,s\in[a,b],s\neq t}\left\{\frac{\int_{[t,s]^k}\prod_{i=1}^k (p_i(x_i)\,dx_i)}{\int_{[t,s]^k}\prod_{i=1}^k (r_i(x_i)\,dx_i)}\right\} > 0.
$$

If $m \geqslant 1$ then

$$\int_{[t,s]^k}\left(\prod_{i=1}^k p_i(x_i)-\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i$$
$$\geqslant \int_{[t,s]^k}\left(\prod_{i=1}^k p_i(x_i)-m\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i \geqslant 0,$$

$s \neq t$. As $\int_{[a,b]^k}\left(\prod_{i=1}^k p_i(x_i)-\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i = 0$, this implies

$$\int_{[t,s]^k}\prod_{i=1}^k (p_i(x_i)\,dx_i) = \int_{[t,s]^k}\prod_{i=1}^k (r_i(x_i)\,dx_i),$$

$s \neq t$, that is $p_i = r_i$, for all $i = 1,\dots,k$ and

$$\mathscr{J}_k(f,p_1,\dots,p_k,\mathbf{q}) = \mathscr{J}_k(f,r_1,\dots,r_k,\mathbf{q}).$$

It remains to consider the case $m < 1$. Since we have

$$\int_{[a,b]^k}\sum_{i=1}^k q_i x_i \prod_{i=1}^k p_i(x_i)\,dx_i = \sum_{i=1}^k q_i \int_a^b x p_i(x)\,dx$$

and the function f is convex we deduce that the required inequality holds:

$$\int_{[a,b]^k} f\left(\sum_{i=1}^k q_i x_i\right)\left(\prod_{i=1}^k p_i(x_i)-m\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i + mf\left(\sum_{i=1}^k q_i\int_a^b x r_i(x)\,dx\right)$$
$$\geqslant \int_{[a,b]^k}\left(\prod_{i=1}^k p_i(x_i)-m\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i$$
$$\cdot f\left(\frac{\int_{[a,b]^k}\sum_{i=1}^k q_i x_i\left(\prod_{i=1}^k p_i(x_i)-m\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i}{\int_{[a,b]^k}\left(\prod_{i=1}^k p_i(x_i)-m\prod_{i=1}^k r_i(x_i)\right)\prod_{i=1}^k dx_i}\right)$$
$$+mf\left(\sum_{i=1}^k q_i\int_a^b x r_i(x)\,dx\right)$$
$$= (1-m)f\left(\frac{\sum_{i=1}^k q_i\left(\int_a^b x\left(p_i(x)-mr_i(x)\right)dx\right)}{1-m}\right)+mf\left(\sum_{i=1}^k q_i\int_a^b x r_i(x)\,dx\right)$$
$$\geqslant f\left(\sum_{i=1}^k q_i\int_a^b x p_i(x)\,dx\right).$$

The proof of the second inequality goes likewise and has been omitted. □

If $p_1 = \dots = p_k = p$ then we have $\mathscr{J}_k(f,p_1,\dots,p_k,\mathbf{q}) = \mathscr{J}_k(f,p,\mathbf{q})$, where

$$\mathscr{J}_k(f,p,\mathbf{q}) := \int_{[a,b]^k} f\left(\sum_{i=1}^k q_i x_i\right)\prod_{i=1}^k (p(x_i)\,dx_i) - f\left(\int_a^b x p(x)\,dx\right).$$

COROLLARY 4. (Integral analogue of Corollary 1) *Let* $p(x)\,dx$, *and* $r(x)\,dx$ *be absolutely continuous measures, where* $p,\ r:[a,b]\to(0,\infty)$ *are increasing functions such that* $\int_a^b p(x)\,dx=1$, $\int_a^b r(x)\,dx=1$. *Then, under the assumptions of Corollary* 5, *we have*

$$\inf_{t,s\in[a,b];s\neq t}\left\{\frac{\int_{[t,s]^k}\prod_{i=1}^k (p(x_i)\,dx_i)}{\int_{[t,s]^k}\prod_{i=1}^k (r(x_i)\,dx_i)}\right\}\mathscr{J}_k(f,r,\mathbf{q})$$
$$\leqslant \mathscr{J}_k(f,p,\mathbf{q})\leqslant \sup_{t,s\in[a,b];s\neq t}\left\{\frac{\int_{[t,s]^k}\prod_{i=1}^k (p(x_i)\,dx_i)}{\int_{[t,s]^k}\prod_{i=1}^k (r(x_i)\,dx_i)}\right\}\mathscr{J}_k(f,r,\mathbf{q}).$$

Proof. It follows immediately from Theorem 5, with $p_1=\dots=p_k=p$. □

If $k=1$, since we have $\mathscr{J}_k(f,p,\mathbf{q})=\mathscr{J}(f,p)$, this corollary reduces to a result due to J. Barić, M. Matić and J. Pecarić (see [3]).

REMARK 2. Consider that f is $\left(M_{[\varphi]},A\right)-$convex and define the functional

$$\mathscr{T}_k(f,p_1,\dots,p_k,\mathbf{q}):=\int_{[a,b]^k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^k q_i\varphi(x_i)\right)\prod_{i=1}^k (p_i(x_i)\,dx_i)$$
$$-\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^k q_i\int_a^b \varphi(x)\,p_i(x)\,dx\right)(\geqslant 0).$$

Then we can easily obtain similar results for $\mathscr{T}_k$, simply by substituting $\mathscr{J}_k$ by $\mathscr{T}_k$, the function f by $f\circ\varphi^{-1}$ (obviously it is convex) and x by $\varphi(x)$.

PROPOSITION 4. (Integral analogue of Proposition 3) *Let* $p_i(x)\,dx$, $i=1,\dots,k$, *be absolutely continuous measures,* $p_i:[a,b]\to(0,\infty)$ *increasing functions such that* $\int_a^b p_i(x)\,dx=1$. *Suppose that we have* $\mathbf{q}=(q_1,q_2,\dots,q_k)$ *such that* $q_i>0$, $\sum_{i=1}^k q_i=1$ $(1\leqslant k)$.

If f *is* $\left(M_{[\varphi]},A\right)$ *- convex then we have*

$$\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^k q_i\int_a^b \varphi(x)\,p_i(x)\,dx\right)$$
$$\leqslant \int_{[a,b]^k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^k q_i\varphi(x_i)\right)\prod_{i=1}^k (p_i(x_i)\,dx_i)\leqslant \sum_{i=1}^k q_i\int_a^b f(x)\,p_i(x)\,dx$$

for all positive integers k.

Proof. The first inequality is proved taking into account that

$$\int_a^b p_i(x)\,dx=1,\ \text{for all } i=1,\dots,k.$$

Since f is $\left(M_{[\varphi]},A\right)$ - convex we have

$$\begin{aligned}&\left(f\circ\varphi^{-1}\right)\left(\int_{[a,b]^k}\sum_{i=1}^{k}q_i\varphi\left(x_i\right)\prod_{i=1}^{k}\left(p_i\left(x_i\right)dx_i\right)\right)\\ &\leqslant\int_{[a,b]^k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k}q_i\varphi\left(x_i\right)\right)\prod_{i=1}^{k}\left(p_i\left(x_i\right)dx_i\right).\end{aligned}$$

Combining this inequality with the following result

$$\int_{[a,b]^k}\sum_{i=1}^{k}q_i\varphi\left(x_i\right)\prod_{i=1}^{k}\left(p_i\left(x_i\right)dx_i\right)=\sum_{i=1}^{k}q_i\int_a^b\varphi\left(x\right)p_i\left(x\right)dx$$

we obtain the conclusion.

The second inequality follows by applying the same technique:

$$\begin{aligned}&\int_{[a,b]^k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k}q_i\varphi\left(x_i\right)\right)\prod_{i=1}^{k}\left(p_i\left(x_i\right)dx_i\right)\\ &\leqslant\int_{[a,b]^k}\sum_{i=1}^{k}q_if\left(x_i\right)\prod_{i=1}^{k}\left(p_i\left(x_i\right)dx_i\right)=\sum_{i=1}^{k}q_i\int_a^b f\left(x\right)p_i\left(x\right)dx.\qquad\square\end{aligned}$$

COROLLARY 5. (Integral analogue of Corollary 3) *Let $p\left(x\right)dx$ be an absolutely continuous measure, $p:[a,b]\to(0,\infty)$ increasing such that $\int_a^b p\left(x\right)dx=1$. Suppose that we have $\mathbf{q}=\left(q_1,q_2,...,q_k\right)$ such that $q_i>0$, $\sum_{i=1}^{k}q_i=1$ ($k\geqslant 1$).*

If f is $\left(M_{[\varphi]},A\right)$-convex then we have

$$\begin{aligned}\left(f\circ\varphi^{-1}\right)\left(\int_a^b\varphi\left(x\right)p\left(x\right)dx\right)&\leqslant\int_{[a,b]^k}\left(f\circ\varphi^{-1}\right)\left(\sum_{i=1}^{k}q_i\varphi\left(x_i\right)\right)\prod_{i=1}^{k}\left(p\left(x_i\right)dx_i\right)\\ &\leqslant\int_a^b f\left(x\right)p\left(x\right)dx\end{aligned}$$

for all positive integers k.

Proof. It follows immediately from Proposition 4, with $p_1=...=p_k=p$. $\square$

3. Applications for norm inequalities

Let $(X;\|.\|)$ be a normed linear space, $x=\left(x_1,...,x_n\right)$ an n-tuple in X^n; $\mathbf{p}=\left(p_1,p_2,...,p_n\right)$ such that $p_i>0$, $\sum_{i=1}^{n}p_i=1$, $\mathbf{q}=\left(q_1,q_2,...,q_k\right)$ such that $q_i>0$, $\sum_{i=1}^{k}q_i=1$ $(1\leqslant k\leqslant n)$ and $\mathbf{r}=\left(r_1,r_2,...,r_n\right)$ such that $r_i>0$, $\sum_{i=1}^{n}r_i=1$ are probability distributions.

Then we have

$$\begin{aligned}
&\min_{1\leqslant i_1,\dots,i_k\leqslant n}\left\{\frac{p_{i_1}\dots p_{i_k}}{r_{i_1}\dots r_{i_k}}\right\}\mathscr{T}_k(\|.\|,\mathbf{r},\mathbf{q},\mathbf{x})\\
&\leqslant \mathscr{T}_k(\|.\|,\mathbf{p},\mathbf{q},\mathbf{x})\leqslant \max_{1\leqslant i_1,\dots,i_k\leqslant n}\left\{\frac{p_{i_1}\dots p_{i_k}}{r_{i_1}\dots r_{i_k}}\right\}\mathscr{T}_k(\|.\|,\mathbf{r},\mathbf{q},\mathbf{x}),
\end{aligned}$$

where

$$\mathscr{T}_k(\|.\|,\mathbf{p},\mathbf{q},\mathbf{x}):=\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}\left\|\sum_{j=1}^{k}q_jx_{i_j}\right\|-\left\|\sum_{i=1}^{n}p_ix_i\right\|.$$

In particular, considering $\mathbf{r}$ to be the uniform distribution,

$$\begin{aligned}
&\min_{1\leqslant i_1,\dots,i_k\leqslant n}\{p_{i_1}\dots p_{i_k}\}\left(\sum_{i_1,\dots,i_k=1}^{n}\left\|\sum_{j=1}^{k}q_jx_{i_j}\right\|-n^{k-1}\left\|\sum_{i=1}^{n}x_i\right\|\right)\\
&\leqslant \mathscr{T}_k(\|.\|,\mathbf{p},\mathbf{q},\mathbf{x})\\
&\leqslant \max_{1\leqslant i_1,\dots,i_k\leqslant n}\{p_{i_1}\dots p_{i_k}\}\left(\sum_{i_1,\dots,i_k=1}^{n}\left\|\sum_{j=1}^{k}q_jx_{i_j}\right\|-n^{k-1}\left\|\sum_{i=1}^{n}x_i\right\|\right).
\end{aligned}$$

If we take $x_i\neq 0,\ i=1,\dots,n$ and

$$p_i=\frac{\frac{1}{\|x_i\|}}{\sum_{j=1}^{n}\frac{1}{\|x_j\|}}$$

by rearranging the inequality, we get

$$\begin{aligned}
&\min_{1\leqslant i_1,\dots,i_k\leqslant n}\left\{\|x_{i_1}\|\cdot\ldots\cdot\|x_{i_k}\|\right\}\\
&\cdot\left[\sum_{i_1,\dots,i_k=1}^{n}\frac{1}{\|x_{i_1}\|\cdot\ldots\cdot\|x_{i_k}\|}\left\|\sum_{j=1}^{k}q_jx_{i_j}\right\|-\left(\sum_{j=1}^{n}\frac{1}{\|x_j\|}\right)^{k-1}\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right]\\
&\leqslant \sum_{i_1,\dots,i_k=1}^{n}\left\|\sum_{j=1}^{k}q_jx_{i_j}\right\|-n^{k-1}\left\|\sum_{i=1}^{n}x_i\right\|\\
&\leqslant \max_{1\leqslant i_1,\dots,i_k\leqslant n}\left\{\|x_{i_1}\|\cdot\ldots\cdot\|x_{i_k}\|\right\}\\
&\cdot\left[\sum_{i_1,\dots,i_k=1}^{n}\frac{1}{\|x_{i_1}\|\cdot\ldots\cdot\|x_{i_k}\|}\left\|\sum_{j=1}^{k}q_jx_{i_j}\right\|-\left(\sum_{j=1}^{n}\frac{1}{\|x_j\|}\right)^{k-1}\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right],
\end{aligned}$$

(a refinement of the triangle inequality).

If $\|x_1\|=\dots=\|x_n\|$ then it becomes an equality.

We conclude this discussion by stressing that for $k=1$ we easily infer a particular result that appeared in a paper of M. Kato et al. [5]:

$$\min_{1\leqslant i\leqslant n}\{\|x_i\|\}\cdot\left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\leqslant\sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|$$
$$\leqslant\max_{1\leqslant i\leqslant n}\{\|x_i\|\}\cdot\left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right).$$

Acknowledgement. We thank the anonymous referees for their constructive suggestions for improving the paper. Also the author acknowledge the support of CNCSIS Grant $420/2008$.

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(Received October 10, 2010)

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