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SOME NONLINEAR INTEGRAL INEQUALITIES AND THEIR DISCRETE ANALOGUES

WEIHONG SHENG AND WEI NIAN LI

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Abstract. The purpose of this paper is to investigate some nonlinear integral inequalities and their discrete analogues. The inequalities given here can be used as handy tools in the qualitative theory of certain differential equations, integral equations and difference equations.

1. Introduction

It is well known that the integral inequalities and the finite difference inequalities play a fundamental role in the development of the theory of differential equations, integral equations and difference equations. In the past few years, many such inequalities have been discovered, which are motivated by certain applications. For example, see the monographes [1–5], papers [6–11] and the references therein. The main purpose of this paper is to investigate some nonlinear integral inequalities and their discrete analogues. Our paper gives, in some sense, an extension of the results of Pachpatte [6].

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$ is the given subset of $\mathbb{R}$, $\mathbb{N}_0$ denotes the set of nonnegative integers, $C(M,S)$ denotes the class of all continuous functions defined on set M with range in the set S, and we use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the sums involved exist on the respective domains of their definitions, and we always assume that $p \geqslant q > 0$, p and q are real constants.

The following lemma is useful in our main results.

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LEMMA 1. ([12]) *Let* $a \in \mathbb{R}_+$. *Then*

$$a^{\frac{q}{p}} \leqslant \left(\frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}\right) \text{ for any } K > 0. \tag{2.1}$$

Next, we establish our main results.

THEOREM 2. *Assume that* $u(t)$, $a(t)$, $b(t)$, $g(t)$, $h_i(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, \cdots, n$, *and there exists a series of positive real numbers* $q_1, q_2, \cdots, q_n$ *such that* $p \geqslant q_i > 0$, $i = 1, 2, \cdots, n$. *If* $w(t,s)$ *and its partial derivative* $\frac{\partial}{\partial t}w(t,s)$ *are real–valued nonnegative continuous functions for* $t, s \in \mathbb{R}_+$ *with* $s \leqslant t$, *then the inequality*

$$u^p(t) \leqslant a(t) + b(t)\int_0^t w(t,\tau)[g(\tau)u^p(\tau) + \sum_{i=1}^n h_i(\tau)u^{q_i}(\tau)]\mathrm{d}\tau, \ \ t \in \mathbb{R}_+, \tag{2.2}$$

implies

$$u(t) \leqslant \left\{a(t) + b(t)\int_0^t B(\tau)\exp\left(\int_\tau^t A(s)\mathrm{d}s\right)\mathrm{d}\tau\right\}^{\frac{1}{p}} \text{ for any } K > 0, \ \ t \in \mathbb{R}_+, \tag{2.3}$$

where

$$A(t) = w(t,t)b(t)\left(g(t) + \sum_{i=1}^n \frac{q_i h_i(t)}{pK^{\frac{p-q_i}{p}}}\right) + \int_0^t \frac{\partial w(t,\tau)}{\partial t}b(\tau)\left(g(\tau) + \sum_{i=1}^n \frac{q_i h_i(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\mathrm{d}\tau, \tag{2.4}$$

and

$$\begin{aligned} B(t) = {} & w(t,t)\left[a(t)g(t) + \sum_{i=1}^n h_i(t)\left(\frac{K(p-q_i) + q_i a(t)}{pK^{\frac{p-q_i}{p}}}\right)\right] \\ & + \int_0^t \frac{\partial w(t,\tau)}{\partial t}\left[a(\tau)g(\tau) + \sum_{i=1}^n h_i(\tau)\left(\frac{K(p-q_i) + q_i a(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\right]\mathrm{d}\tau, \ \ t \in \mathbb{R}_+. \end{aligned} \tag{2.5}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_0^t w(t,\tau)[g(\tau)u^p(\tau) + \sum_{i=1}^n h_i(\tau)u^{q_i}(\tau)]\mathrm{d}\tau, \ \ t \in \mathbb{R}_+. \tag{2.6}$$

Then (2.2) can be restated as

$$u^p(t) \leqslant a(t) + b(t)z(t), \ \ t \in \mathbb{R}_+. \tag{2.7}$$

Using Lemma 1, from (2.7), for any $K > 0$, we easily obtain

$$\begin{aligned} u^{q_i}(t) & \leqslant (a(t) + b(t)z(t))^{\frac{q_i}{p}} \\ & \leqslant \frac{K(p-q_i) + q_i a(t)}{pK^{\frac{p-q_i}{p}}} + \frac{q_i b(t)z(t)}{pK^{\frac{p-q_i}{p}}}, \ \ t \in \mathbb{R}_+, \ i = 1, 2, \cdots, n. \end{aligned} \tag{2.8}$$

Combining (2.6)–(2.8), we have

$$\begin{aligned}
z'(t) &= w(t,t)\left[g(t)u^p(t)+\sum_{i=1}^n h_i(t)u^{q_i}(t)\right]+\int_0^t w_t'(t,\tau)\left[g(\tau)u^p(\tau)\right.\\
&\quad\left.+\sum_{i=1}^n h_i(\tau)u^{q_i}(\tau)\right]\mathrm{d}\tau\\
&\leqslant w(t,t)\left[a(t)g(t)+\sum_{i=1}^n h_i(t)\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}+b(t)\left(g(t)+\sum_{i=1}^n\frac{q_ih_i(t)}{pK^{\frac{p-q_i}{p}}}\right)z(t)\right]\\
&\quad+\int_0^t w_t'(t,\tau)\left[a(\tau)g(\tau)+\sum_{i=1}^n h_i(\tau)\frac{K(p-q_i)+q_ia(\tau)}{pK^{\frac{p-q_i}{p}}}\right.\\
&\quad\left.+b(\tau)\left(g(\tau)+\sum_{i=1}^n\frac{q_ih_i(\tau)}{pK^{\frac{p-q_i}{p}}}\right)z(\tau)\right]\mathrm{d}\tau\\
&\leqslant \left[w(t,t)b(t)\left(g(t)+\sum_{i=1}^n\frac{q_ih_i(t)}{pK^{\frac{p-q_i}{p}}}\right)+\int_0^t w_t'(t,\tau)b(\tau)\left(g(\tau)+\sum_{i=1}^n\frac{q_ih_i(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\mathrm{d}\tau\right]z(t)\\
&\quad+w(t,t)\left[a(t)g(t)+\sum_{i=1}^n h_i(t)\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right]\\
&\quad+\int_0^t w_t'(t,\tau)\left[a(\tau)g(\tau)+\sum_{i=1}^n h_i(\tau)\frac{K(p-q_i)+q_ia(\tau)}{pK^{\frac{p-q_i}{p}}}\right]\mathrm{d}\tau\\
&= A(t)z(t)+B(t),\ \ t\in\mathbb{R}_+,
\end{aligned}$$

where $A(t)$ and $B(t)$ are defined as in (2.4) and (2.5) respectively. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t)\exp\left(-\int_0^t A(s)\mathrm{d}s\right)\right]\leqslant B(t)\exp\left(-\int_0^t A(s)\mathrm{d}s\right),\ \ t\in\mathbb{R}_+. \tag{2.9}$$

It follows from (2.9) that

$$z(t)\exp\left(-\int_0^t A(s)\mathrm{d}s\right)\leqslant\int_0^t B(\tau)\exp\left(-\int_0^\tau A(s)\mathrm{d}s\right)\mathrm{d}\tau,$$

i.e.,

$$\begin{aligned}
z(t)&\leqslant\exp\left(\int_0^t A(s)\mathrm{d}s\right)\int_0^t B(\tau)\exp\left(-\int_0^\tau A(s)\mathrm{d}s\right)\mathrm{d}\tau\\
&=\int_0^t B(\tau)\exp\left(\int_\tau^t A(s)\mathrm{d}s\right)\mathrm{d}\tau,\ \ t\in\mathbb{R}_+.
\end{aligned}\tag{2.10}$$

It is easy to see that the desired inequality (2.3) follows from (2.7) and (2.10). The proof of Theorem 2 is complete. □

Letting $w(t,s)=1$ in Theorem 2, we obtain the following corollary.

COROLLARY 3. *Assume that* $u(t),\ a(t),\ b(t),\ g(t),\ h_i(t)\in C(\mathbb{R}_+,\mathbb{R}_+),\ i=1,2,\cdots,n$. *If there exists a series of positive real numbers* $q_1,q_2,\cdots,q_n$ *such that* $p\geqslant q_i>0,\ i=1,2,\cdots,n$, *then*

$$u^p(t)\leqslant a(t)+b(t)\int_0^t\left[g(s)u^p(s)+\sum_{i=1}^n h_i(s)u^{q_i}(s)\right]\mathrm{d}s,\ \ t\in\mathbb{R}_+, \tag{2.11}$$

implies

$$u(t) \leqslant \left\{ a(t) + b(t) \int_0^t \left[a(\tau)g(\tau) + \sum_{i=1}^n h_i(\tau) \left(\frac{K(p-q_i)+q_i a(\tau)}{pK^{\frac{p-q_i}{p}}} \right) \right] \times \exp\left(\int_\tau^t F(s)\mathrm{d}s \right) \mathrm{d}\tau \right\}^{\frac{1}{p}} \text{ for any } K>0,\ t \in \mathbb{R}_+, \tag{2.12}$$

where

$$F(t) = b(t) \left(g(t) + \sum_{i=1}^n \frac{q_i h_i(t)}{pK^{\frac{p-q_i}{p}}} \right).$$

THEOREM 4. *Assume that* $u,\ a,\ b \in C(\mathbb{R}_+, \mathbb{R}_+)$, *and* $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ *is a continuous function such that*

$$0 \leqslant f_i(t,x) - f_i(t,y) \leqslant \phi_i(t,y)(x-y), \tag{2.13}$$

for $t \in \mathbb{R}_+$ *and* $x \geqslant y \geqslant 0$, *where* $\phi_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ *is a continuous function,* $i = 1,2,\cdots,n$. *If there exists a series of positive real numbers* $q_1, q_2, \cdots, q_n$ *such that* $p \geqslant q_i > 0,\ i = 1,2,\cdots,n$, *then*

$$u^p(t) \leqslant a(t) + b(t) \sum_{i=1}^n \int_0^t f_i(\tau, u^{q_i}(\tau))\mathrm{d}\tau,\ \ t \in \mathbb{R}_+, \tag{2.14}$$

implies

$$u(t) \leqslant \left\{ a(t) + b(t) \sum_{i=1}^n \int_0^t \exp\left(\int_\tau^t M_i(s)\mathrm{d}s \right) f_i\left(\tau, \frac{K(p-q_i)+q_i a(\tau)}{pK^{\frac{p-q_i}{p}}} \right) \mathrm{d}\tau \right\}^{\frac{1}{p}} \text{for any } K>0,\ t \in \mathbb{R}_+, \tag{2.15}$$

where

$$M_i(t) = \phi_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}} \right) \frac{q_i b_i(t)}{pK^{\frac{p-q_i}{p}}},\ \ i = 1,2,\cdots,n. \tag{2.16}$$

Proof. Define $z(t)$ by

$$z(t) = \sum_{i=1}^n \int_0^t f_i(\tau, u^{q_i}(\tau))\mathrm{d}\tau, \quad t \in \mathbb{R}_+. \tag{2.17}$$

Then (2.14) can be written as (2.7). As in the proof of Theorem 2, from (2.7), we easily obtain (2.8). Obviously, it follows from (2.17), (2.8) and (2.13) that

$$
\begin{aligned}
z'(t) &= \sum_{i=1}^{n} f_i(t, u^{q_i}(t)) \\
&\leqslant \sum_{i=1}^{n} \Bigg[f_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}} + \frac{q_i b(t) z(t)}{pK^{\frac{p-q_i}{p}}}\right) - f_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}}\right) \\
&\qquad + f_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}}\right)\Bigg] \\
&\leqslant \sum_{i=1}^{n} \left[\phi_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}}\right) \frac{q_i b(t)}{pK^{\frac{p-q_i}{p}}} z(t) + f_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}}\right)\right] \\
&= z(t) \sum_{i=1}^{n} M_i(t) + \sum_{i=1}^{n} f_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}}\right), \ t \in \mathbb{R}_+,
\end{aligned} \tag{2.18}
$$

where $M_i(t)$ is defined as in (2.16). So we get

$$
z(t) \leqslant \sum_{i=1}^{n} \int_0^t \exp\left(\int_\tau^t M_i(s)\mathrm{d}s\right) f_i\left(\tau, \frac{K(p-q_i)+q_i a(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\mathrm{d}\tau, \ t \in \mathbb{R}_+. \tag{2.19}
$$

It is easy to see that the desired inequality (2.15) follows from (2.7) and (2.19). The proof of Theorem 4 is complete. $\square$

THEOREM 5. *Assume that $u(t)$, $a(t)$, $b(t)$, $g(t)$ and $h_i(t)$, are nonnegative functions defined for $t \in \mathbb{N}_0$, $i = 1,2,\cdots,n$, $w(t,s)$ and $\mathbf{\Delta}_1 w(t,s)$ are real–valued nonnegative functions for $t,s \in \mathbb{N}_0$ with $s \leqslant t$. If there exists a series of positive real numbers $q_1, q_2, \cdots, q_n$ such that $p \geqslant q_i > 0$, $i = 1,2,\cdots,n$, then the inequality*

$$
u^p(t) \leqslant a(t) + b(t) \sum_{\tau=0}^{t-1} w(t,\tau)[g(\tau)u^p(\tau) + \sum_{i=1}^{n} h_i(\tau) u^{q_i}(\tau)], \ \ t \in \mathbb{N}_0, \tag{2.20}
$$

implies

$$
u(t) \leqslant \left\{ a(t) + b(t) \sum_{\tau=0}^{t-1} \widetilde{B}(\tau) \prod_{s=\tau+1}^{t-1} (1 + \widetilde{A}(s)) \right\}^{\frac{1}{p}} \ \textit{for any } K > 0, \ t \in \mathbb{N}_0, \tag{2.21}
$$

where $\mathbf{\Delta}_1 w(t,s) = w(t+1,s) - w(t,s)$ for $t,s \in \mathbb{N}_0$ with $s \leqslant t$,

$$
\widetilde{A}(t) = w(t+1,t) b(t) \left(g(t) + \sum_{i=1}^{n} \frac{q_i h_i(t)}{pK^{\frac{p-q_i}{p}}} \right) + \sum_{\tau=0}^{t-1} \mathbf{\Delta}_1 w(t,\tau) b(\tau) \left(g(\tau) + \sum_{i=1}^{n} \frac{q_i h_i(\tau)}{pK^{\frac{p-q_i}{p}}} \right), \tag{2.22}
$$

and

$$
\begin{aligned}
\widetilde{B}(t) &= w(t+1,t) \left[a(t) g(t) + \sum_{i=1}^{n} h_i(t) \left(\frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}} \right) \right] \\
&\quad + \sum_{\tau=0}^{t-1} \mathbf{\Delta}_1 w(t,\tau) \left[a(\tau) g(\tau) + \sum_{i=1}^{n} h_i(t) \left(\frac{K(p-q_i)+q_i a(t)}{pK^{\frac{p-q_i}{p}}} \right) \right], \ \ t \in \mathbb{N}_0.
\end{aligned} \tag{2.23}
$$

Proof. Define a function $z(t)$ by

$$z(t)=\sum_{s=0}^{t-1}w(t,\tau)\left[g(\tau)u^{p}(\tau)+\sum_{i=1}^{n}h_i(\tau)u^{q_i}(\tau)\right],\ \ t\in\mathbb{N}_0. \tag{2.24}$$

As in the proof of Theorem 2, we easily obtain (2.7) and (2.8). Combining (2.24), (2.7) and (2.8), we have

$$\begin{aligned}
z(t+1)-z(t) &= w(t+1,t)\left[g(t)u^{p}(t)+\sum_{i=1}^{n}h_i(t)u^{q_i}(t)\right]+\sum_{\tau=0}^{t-1}\boldsymbol{\Delta}_1 w(t,\tau)\left[g(\tau)u^{p}(\tau)\right.\\
&\quad\left.+\sum_{i=1}^{n}h_i(\tau)u^{q_i}(\tau)\right]\\
&\leqslant \left[w(t+1,t)b(t)\left(g(t)+\sum_{i=1}^{n}\frac{q_ih_i(t)}{pK^{\frac{p-q_i}{p}}}\right)\right.\\
&\quad\left.+\sum_{\tau=0}^{t-1}\boldsymbol{\Delta}_1 w(t,\tau)b(\tau)\left(g(\tau)+\sum_{i=1}^{n}\frac{q_ih_i(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\right]z(t)\\
&\quad+w(t+1,t)\left[a(t)g(t)+\sum_{i=1}^{n}h_i(t)\left(\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right)\right]\\
&\quad+\sum_{\tau=0}^{t-1}\boldsymbol{\Delta}_1 w(t,\tau)\left[a(\tau)g(\tau)+\sum_{i=1}^{n}h_i(t)\left(\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right)\right]\\
&=\widetilde{A}(t)z(t)+\widetilde{B}(t),\ \ t\in\mathbb{N}_0.
\end{aligned}$$

Therefore, we have

$$z(t+1)-(1+\widetilde{A}(t))z(t)\leqslant\widetilde{B}(t),\ \ t\in\mathbb{N}_0. \tag{2.25}$$

Multiplying both sides of (2.25) by $\prod_{s=0}^{t}[1+\widetilde{A}(s)]^{-1}$, taking $t=\tau$, and summing up both sides of the resulting inequality from 0 to $t-1$, we get

$$z(t)\prod_{s=0}^{t-1}[1+\widetilde{A}(s)]^{-1}\leqslant\sum_{\tau=0}^{t-1}\left\{\widetilde{B}(\tau)\prod_{s=0}^{\tau}[1+\widetilde{A}(s)]^{-1}\right\},\ \ t\in\mathbb{N}_0,$$

which implies

$$z(t)\leqslant\sum_{\tau=0}^{t-1}\left\{\widetilde{B}(\tau)\prod_{s=\tau+1}^{t-1}[1+\widetilde{A}(s)]\right\},\ \ t\in\mathbb{N}_0. \tag{2.26}$$

Using (2.26) in (2.7) we get the required inequality (2.21). This completes the proof of Theorem 5. $\square$

Letting $w(t,s)=1$ in Theorem 5, we can easily obtain the following corollary.

COROLLARY 6. *Assume that $u(t)$, $a(t)$, $b(t)$, $g(t)$ and $h_i(t)$ are nonnegative functions defined for $t\in\mathbb{N}_0$, $i=1,2,\cdots,n$. If there exists a series of positive real numbers $q_1,q_2,\cdots,q_n$ such that $p\geqslant q_i>0$, $i=1,2,\cdots,n$, Then the inequality*

$$u^{p}(t)\leqslant a(t)+b(t)\sum_{s=0}^{t-1}\left[g(s)u^{p}(s)+\sum_{i=1}^{n}h_i(s)u^{q_i}(s)\right],\ \ t\in\mathbb{N}_0, \tag{2.27}$$

implies

$$u(t) \leqslant \left\{a(t)+b(t)\sum_{\tau=0}^{t-1}\left[a(\tau)g(\tau)+\sum_{i=1}^{n}h_i(\tau)\left(\frac{K(p-q_i)+q_ia(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\right]\right.$$
$$\left.\times\prod_{s=\tau+1}^{t-1}(1+F(s))\right\}^{\frac{1}{p}} \text{ for any } K>0,\ t\in\mathbb{N}_0, \tag{2.28}$$

where $F(t)$ is defined as in Corollary 3.

THEOREM 7. *Assume that $u(t)$, $a(t)$, $b(t)$ are nonnegative functions defined for $t\in\mathbb{N}_0$, and f_i is a real–valued nonnegative function for $t,s\in\mathbb{N}_0$ such that*

$$0\leqslant f_i(t,x)-f_i(t,y)\leqslant \phi_i(t,y)(x-y), \tag{2.29}$$

for $x\geqslant y$, where ϕ_i is a nonnegative function, $i=1,2,\cdots,n$. If there exists a series of positive real numbers $q_1,q_2,\cdots,q_n$ such that $p\geqslant q_i>0$, $i=1,2,\cdots,n$, then

$$u^p(t)\leqslant a(t)+b(t)\sum_{i=1}^{n}\sum_{\tau=0}^{t-1}f_i(\tau,u^{q_i}(\tau)),\ \ t\in\mathbb{N}_0, \tag{2.30}$$

implies

$$u(t)\leqslant\left\{a(t)+b(t)\sum_{i=1}^{n}\left[\sum_{\tau=0}^{t-1}\left(\prod_{s=\tau+1}^{t-1}(1+M_i(t))\right)f_i\left(\tau,\frac{K(p-q_i)+q_ia(\tau)}{pK^{\frac{p-q_i}{p}}}\right)\right]\right\}^{\frac{1}{p}}$$
$$\text{for any } K>0,\ t\in\mathbb{N}_0, \tag{2.31}$$

where $M_i(t)$ is defined as in Theorem 4.

Proof. Define $z(t)$ by

$$z(t)=\sum_{i=1}^{n}\sum_{\tau=0}^{t-1}f_i(\tau,u^{q_i}(\tau)),\quad t\in\mathbb{N}_0. \tag{2.32}$$

As in the proof of Theorem 2, we easily obtain (2.7) and (2.8). It follows from (2.32), (2.8) and (2.29) that

$$\begin{aligned}
&z(t+1)-z(t)=\sum_{i=1}^{n}f_i(t,u^{q_i}(t))\\
&\leqslant\sum_{i=1}^{n}\left[f_i\left(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}+\frac{q_ib(t)z(t)}{pK^{\frac{p-q_i}{p}}}\right)-f_i\left(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right)\right.\\
&\quad\left.+f_i\left(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right)\right]\\
&\leqslant\sum_{i=1}^{n}\left[\phi_i\left(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right)\frac{q_ib(t)}{pK^{\frac{p-q_i}{p}}}z(t)+f_i\left(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right)\right]\\
&=z(t)\sum_{i=1}^{n}M_i(t)+\sum_{i=1}^{n}f_i\left(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\right),\ t\in\mathbb{N}_0.
\end{aligned}$$

Therefore,

$$z(t+1)-\Big(1+\sum_{i=1}^{n}M_i(t)\Big)z(t)\leqslant\sum_{i=1}^{n}f_i\Big(t,\frac{K(p-q_i)+q_ia(t)}{pK^{\frac{p-q_i}{p}}}\Big),\ t\in\mathbb{N}_0. \tag{2.33}$$

From (2.33), we easily obtain

$$z(t)\leqslant\sum_{i=1}^{n}\Big[\sum_{\tau=0}^{t-1}\Big(\prod_{s=\tau+1}^{t-1}(1+M_i(t))\Big)f_i\Big(\tau,\frac{K(p-q_i)+q_ia(\tau)}{pK^{\frac{p-q_i}{p}}}\Big)\Big],\ t\in\mathbb{N}_0. \tag{2.34}$$

Therefore, the desired inequality (2.31) follows from (2.7) and (2.34). The proof of Theorem 7 is complete. □

REMARK 8. Letting $p>1, n=1, K=q_1=1$ in Corollary 3, Theorems 2 and 4, Corollary 6, Theorems 5 and 7, we easily obtain Theorem 1(a$_1$), (a$_3$), Theorem 2(b$_1$), Theorem 3(c$_1$), (c$_3$) and Theorem 4(d$_1$) established by Pachpatte[6], respectively.

3. An application

In this section, we present an application of Corollary 3 to obtain the explicit bound on the solution of a certain differential equation.

Consider the following differential equation

$$u^{p-1}(t)u'(t)+H(t,u^p(t),u^{q_1}(t),u^{q_2}(t))=r(t),\quad u(0)=u_0, \tag{3.1}$$

where p, q_i, u_0 are real constants and $p\geqslant q_i>0$, $i=1,2$, $u(t), r(t)\in C(\mathbb{R}_+,\mathbb{R})$, $H\in C(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

Assume that $|H(t,u^p,u^{q_1},u^{q_2})|\leqslant g(t)|u^p|+h_1(t)|u^{q_1}|+h_2(t)|u^{q_2}|$, where $g(t)$, $h_i(t)\in C(\mathbb{R}_+,\mathbb{R}_+)$, $i=1,2$. It is easy to see that the problem (3.1) is equivalent to the integral equation

$$\frac{u^p(t)-u_0^p}{p}+\int_0^t H(s,u^p(s),u^{q_1}(s),u^{q_2}(s))\mathrm{d}s=\int_0^t r(s)\mathrm{d}s. \tag{3.2}$$

Then we have

$$|u(t)|^p\leqslant\bar{a}(t)+p\int_0^t\Big(g(s)|u^p(s)|+h_1(s)|u^{q_1}(s)|+h_2(t)|u^{q_2}(s)|\Big)\mathrm{d}s, \tag{3.3}$$

where $\bar{a}(t)=|u_0|^p+p\int_0^t|r(s)|\mathrm{d}s$. Now a suitable application of Corollary 3 with $b(t)=1$ yields

$$\begin{aligned}|u(t)|\leqslant\Big\{\bar{a}(t)+\int_0^t\Big[\bar{a}(\tau)g(\tau)+\sum_{i=1}^{2}h_i(s)\Big(\frac{K(p-q_i)+q_i\bar{a}(\tau)}{pK^{\frac{p-q_i}{p}}}\Big)\\ \times\exp\Big(\int_\tau^t\Big(g(s)+\sum_{i=1}^{2}\frac{q_ih_i(s)}{pK^{\frac{p-q_i}{p}}}\Big)\mathrm{d}s\Big)\Big]\Big)\mathrm{d}\tau\Big\}^{\frac{1}{p}}\ \text{for any } K>0,\end{aligned} \tag{3.4}$$

which gives the bound on the solution of (3.1) in the terms of the known quantities.

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REFERENCES

[1] D. BAINOV, P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.
[2] R. P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker Inc., New York, 1997.
[3] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.
[4] B. G. PACHPATTE, *Inequalities for Finite Difference Equations*, Marcel Dekker Inc., New York, 2002.
[5] B. G. PACHPATTE, *Integral and Finite Difference Inequalities and Applications*, Elsevier Science B. V., Amsterdam, 2006.
[6] B. G. PACHPATTE, *On some new inequalities related to a certain inequality arising in the theory of differential equation*, J. Math. Anal. Appl. **251** (2000), 736–751.
[7] T. LÜ, Y. HUANG, *A generalization of discrete Gronwall inequality and its application to weakly singular Volterra integral equation of the second kind*, J. Math. Anal. Appl. **282** (2003), 56–62.
[8] W. SHENG, W. N. LI, *Bounds on certain nonlinear discrete inequalities*, J. Math. Inequal. **2** (2008), 279–286.
[9] K. ZHENG, Y. WU, S. ZHONG, *Discrete nonlinear integral inequalities in two variables and their applications*, Appl. Math. Comput. **207** (2009), 140–147.
[10] Q. H. MA, J. PEČARIĆ, *Explicit bounds on some new nonlinear retarded integral inequalities and their applicationsm*, Taiwanese J. Math. **13** (2009), 287–306.
[11] W. N. LI, *Some nonlinear integral inequalities in two independent variables*, Advance in Difference Equations **2010** (2010), Artical ID 984141.
[12] W. N. LI, M. HAN, *Bounds for certain nonlinear dynamic inequalities on time scales*, Discrete Dynamics in Nature and Society **2009** (2009), Artical ID 897087.

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Weihong Sheng
Institute of Differential Equations & Dynamical Systems
Binzhou University
Shandong 256603
China
e-mail: wh-sheng@163.com

Wei Nian Li
Department of Mathematics
Binzhou University
Shandong 256603
China
e-mail: wnli@263.net