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FURTHER GROWTH OF ITERATED ENTIRE FUNCTIONS–I

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Abstract. In this paper we study the comparative growth of iterated entire functions and generalise some earlier results.

1. Introduction, Definitions and Notation

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane C. It is well known [1], {[13], p-67, Th-1.46} that $\lim_{r\to\infty} \frac{T(r,f_o g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r,f_o g)}{T(r,g)} = \infty$.

In 1985 Singh [9] proved some comparative growth properties of $\log T(r, f_o g)$ and $T(r, f)$ also he raised the question of investigating the comparative growth of $\log T(r, f_o g)$ and $T(r, g)$. Lahiri [5] proved some result on comparative growth of $\log T(r, f_o g)$ and $T(r, g)$.

Song and Yang [11] established that f and g are any two transcendental entire functions of positive lower order and finite order then

$$\lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(r, f)} = \infty = \lim_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(r, g)}.$$

In 1991 Singh and Baloria [10] asked whether for sufficiently large $R = R(r)$

$$\limsup_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(R, f)} < \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{\log\log M(r, f_o g)}{\log\log M(R, g)} < \infty.$$

Singh and Baloria [10], Lahiri and Sharma [6] worked on this question. Also in a resent paper [2] Dutta study some comparative growth of iterated entire functions. In this paper, we investigate the comparative growth of iterated entire functions. we do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are available in [3], [12] and [13].

The following definitions are well known.

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DEFINITION 1.1. The order ρ_f and lower order λ_f of a meromorphic function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

NOTATION 1.2. [8] $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

DEFINITION 1.3. The p-th order ρ_f^p and lower p-th order λ_f^p of a meromorphic function f is defined as

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}.$$

If f is an entire function then

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Clearly $\rho_f^p \leqslant \rho_f^{p-1}$ and $\lambda_f^p \leqslant \lambda_f^{p-1}$ for all p and when $p = 1$ then p-th order and lower p-th order coincide with classical order and lower order respectively.

DEFINITION 1.4. Let f be an entire function of finite p-th order ρ_f^p then we defined σ_f^p as,

$$\sigma_f^p = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f^p}}$$

According to Lahiri and Banerjee [4] if $f(z)$ and $g(z)$ be entire functions then the iteration of f with respect to g is defined as follows:

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))\\
&\quad\;\; .... \qquad\qquad .... \qquad\qquad ....\\
f_n(z) &= f(g(f........(f(z) \text{ or } g(z))........)),\\
&\quad \text{according as } n \text{ is odd or even,}
\end{aligned}$$

and so

$$\begin{aligned}
g_1(z) &= g(z)\\
g_2(z) &= g(f(z)) = g(f_1(z))\\
g_3(z) &= g(f_2(z)) = g(f(g(z)))\\
&\quad\;\; .... \qquad\qquad ....\\
g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))).
\end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

2. Lemmas

The following lemmas will be needed in the sequel.

LEMMA 2.1. [3] *Let $f(z)$ be an entire function. For $0 \leqslant r < R < \infty$, we have*

$$T(r,f) \leqslant \log^+ M(r,f) \leqslant \frac{R+r}{R-r} T(R,f).$$

LEMMA 2.2. [1] *If f and g are any two entire functions, for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right) - |g(0)|, f\right) \leqslant M(r, f_o g) \leqslant M(M(r,g),f)$$

LEMMA 2.3. [7] *Let $f(z)$ and $g(z)$ be two entire functions. Then we have*

$$T(r, f_o g) \geqslant \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g\right) + O(1), f\right)$$

LEMMA 2.4. *Let $f(z)$ and $g(z)$ be two entire functions of non zero finite p-th order ρ_f^p and ρ_g^p respectively, then for any $\varepsilon > 0$ and $p \geqslant 1$,*

$$\log^{[(n-1)p+1]} M(r,f_n) \leqslant \begin{cases} (\rho_f^p + \varepsilon)\log M(r,g) + O(1) & \textit{when } n \textit{ is even}\\ (\rho_g^p + \varepsilon)\log M(r,f) + O(1) & \textit{when } n \textit{ is odd,}\end{cases}$$

for all sufficiently large values of r.

Proof. First suppose that n is even. Then from second part of Lemma 2.2 and definition of p-th order, it follows that for all sufficiently large values of r,

$$\begin{aligned} M(r,f_n) &\leqslant M(M(r,g_{n-1}),f) \\ \text{i.e., } \log^{[p]} M(r,f_n) &\leqslant \log^{[p]} M(M(r,g_{n-1}),f) \\ &\leqslant [M(r,g_{n-1})]^{\rho_f^p+\varepsilon}. \\ \text{So, } \log^{[p+2]} M(r,f_n) &\leqslant \log^{[2]} M(r,g(f_{n-2}))+O(1). \end{aligned}$$

Taking repeated logarithms $(p-2)$ times, we get

$$\begin{aligned} \log^{[2p]} M(r,f_n) &\leqslant \log^{[p]} M(M(r,f_{n-2}),g)+O(1) \\ &\leqslant [M(r,f_{n-2})]^{\rho_g^p+\varepsilon}+O(1) \\ \text{i.e., } \log^{[2p+2]} M(r,f_n) &\leqslant \log^{[2]} M(r,f_{n-2})+O(1). \end{aligned}$$

Again taking repeated logarithms (p-2) times, we get

$$\log^{[3p]} M(r,f_n) \leqslant [M(r,g_{n-3})]^{\rho_f^p+\varepsilon}+O(1).$$

Finally, after taking repeated logarithms $(n-4)p$ times more, we have for all sufficiently large values of r,

$$\begin{aligned} \log^{[(n-1)p]} M(r,f_n) &\leqslant [M(r,g)]^{\rho_f^p+\varepsilon}+O(1) \\ \text{i.e., } \log^{[(n-1)p+1]} M(r,f_n) &\leqslant (\rho_f^p+\varepsilon)\log M(r,g)+O(1). \end{aligned}$$

Similarly if n is odd then for all sufficiently large values of r

$$\log^{[(n-1)p+1]} M(r,f_n) \leqslant (\rho_g^p+\varepsilon)\log M(r,f)+O(1).$$

This proves the lemma. □

LEMMA 2.5. *Let $f(z)$ and $g(z)$ be two entire functions of non zero finite lower p-th order λ_f^p and λ_g^p respectively, then for any $0<\varepsilon<\min\{\lambda_f^p,\lambda_g^p\}$ and $p\geqslant 1$,*

$$\log^{[(n-1)p+1]} M(r,f_n) \geqslant \begin{cases} (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{2^{n-1}},g\right)+O(1) & \text{when } n \text{ is even} \\ (\lambda_g^p-\varepsilon)\log M\left(\frac{r}{2^{n-1}},f\right)+O(1) & \text{when } n \text{ is odd,} \end{cases}$$

for all sufficiently large values of r.

Proof. First suppose that n is even. Then from first part of Lemma 2.2 we have for all sufficiently large values of r and for any $0 < \varepsilon < \min\{\lambda_f^p,\ \lambda_g^p\}$,

$$\begin{aligned} M(r,f_n) &= M(r,f(g_{n-1})) \\ &\geqslant M\left(\frac{1}{8}M\left(\frac{r}{2},g_{n-1}\right)-|g_{n-1}(0)|,f\right) \\ &\geqslant M\left(\frac{1}{16}M\left(\frac{r}{2},g_{n-1}\right),f\right). \end{aligned}$$

$$\begin{aligned} \therefore\ \log^{[p]} M(r,f_n) &\geqslant \left[\frac{1}{16}M\left(\frac{r}{2},g_{n-1}\right)\right]^{\lambda_f^p-\varepsilon}, \text{ using the Definition 1.3,} \\ \text{i.e. } \log^{[p+1]} M(r,f_n) &\geqslant (\lambda_f^p-\varepsilon)\log\frac{1}{16}M\left(\frac{r}{2},g_{n-1}\right) \\ &\geqslant (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{2},g_{n-1}\right)+O(1) \\ \text{i.e. } \log^{[p+2]} M(r,f_n) &\geqslant \log^{[2]} M\left(\frac{r}{2},g(f_{n-2})\right)+O(1) \\ &\geqslant \log^{[2]} M\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right),g\right)+O(1). \end{aligned}$$

Taking repeated logarithms $(p-2)$ times, we get

$$\begin{aligned} \log^{[2p]} M(r,f_n) &\geqslant \log^{[p]} M\left(\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right),g\right)+O(1) \\ &\geqslant \left[\frac{1}{16}M\left(\frac{r}{2^2},f_{n-2}\right)\right]^{\lambda_g^p-\varepsilon}+O(1) \\ \text{i.e., } \log^{[2p+2]} M(r,f_n) &\geqslant \log^{[2]} M\left(\frac{r}{2^2},f_{n-2}\right)+O(1). \end{aligned}$$

Again taking repeated logarithms $(p-2)$ times, we get

$$\log^{[3p]} M(r,f_n) \geqslant \left[\frac{1}{16}M\left(\frac{r}{2^3},g_{n-3}\right)\right]^{\lambda_f^p-\varepsilon}+O(1).$$

Finally, after taking repeated logarithms $(n-4)p$ times more, we have for all sufficiently large values of r,

$$\begin{aligned} \log^{[(n-1)p]} M(r,f_n) &\geqslant \left[\frac{1}{16}M\left(\frac{r}{2^{n-1}},g\right)\right]^{\lambda_f^p-\varepsilon}+O(1) \\ \text{i.e., } \log^{[(n-1)p+1]} M(r,f_n) &\geqslant (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{2^{n-1}},g\right)+O(1). \end{aligned}$$

Similarly if n is odd then for all sufficiently large values of r

$$\log^{[(n-1)p+1]} M(r,f_n) \geqslant (\lambda_g^p-\varepsilon)\log M\left(\frac{r}{2^{n-1}},f\right)+O(1).$$

This proves the lemma. $\square$

LEMMA 2.6. *Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0<\rho_f^p<\infty$ and $0<\rho_g^p<\infty$. Then for all sufficiently large r and $\varepsilon>0$,*

$$\log^{[(n-1)p]} T(r,f_n) \leqslant \begin{cases} (\rho_f^p+\varepsilon)\log M(r,g)+O(1) & \text{when } n \text{ is even} \\ (\rho_g^p+\varepsilon)\log M(r,f)+O(1) & \text{when } n \text{ is odd} \end{cases}$$

where $p \geqslant 1$.

The lemma follows from Lemma 2.1 and Lemma 2.4.

LEMMA 2.7. *Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0<\lambda_f^p<\infty$ and $0<\lambda_g^p<\infty$. Then for any ε $(0<\varepsilon<\min\{\lambda_f^p,\lambda_g^p\})$ and $p \geqslant 1$,*

$$\log^{[(n-1)p]} T(r,f_n) \geqslant \begin{cases} (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1) & \text{when } n \text{ is even} \\ (\lambda_g^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f\right)+O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of r.

Proof. To prove this lemma we first consider n is even. Then from Lemma 2.1 and Lemma 2.3 we get for ε $(0<\varepsilon<\min\{\lambda_f,\lambda_g\})$ and for all large values of r

$$\begin{aligned} T(r,f_n) &= T(r,f(g_{n-1})) \\ &\geqslant \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right). \\ \therefore\ \log^{[p]} T(r,f_n) &\geqslant \log^{[p+1]} M\left(\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1),f\right)+O(1) \\ &\geqslant \log\left[\frac{1}{8}M\left(\frac{r}{4},g_{n-1}\right)+O(1)\right]^{\lambda_f^p-\varepsilon}+O(1) \\ &\geqslant \log\left[\frac{1}{9}M\left(\frac{r}{4},g_{n-1}\right)\right]^{\lambda_f^p-\varepsilon}+O(1) \\ &\geqslant (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4},g_{n-1}\right)+O(1) \\ &\geqslant (\lambda_f^p-\varepsilon)T\left(\frac{r}{4},g_{n-1}\right)+O(1) \\ &\geqslant (\lambda_f^p-\varepsilon)\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)+O(1), \end{aligned}$$

that is,

$$\begin{aligned}
\log^{[2p]} T(r,f_n) &\geqslant \log^{[p+1]} M\left(\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1),g\right)+O(1)\\
&\geqslant \log\left[\frac{1}{8}M\left(\frac{r}{4^2},f_{n-2}\right)+O(1)\right]^{\lambda_g^p-\varepsilon}+O(1)\\
&\geqslant \log\left[\frac{1}{9}M\left(\frac{r}{4^2},f_{n-2}\right)\right]^{\lambda_g^p-\varepsilon}+O(1).
\end{aligned}$$

$$\text{i.e.,}\quad \log^{[2p]} T(r,f_n) \geqslant (\lambda_g^p-\varepsilon)\log M\left(\frac{r}{4^2},f_{n-2}\right)+O(1)$$

....

....

$$\text{Therefore, } \log^{[(n-2)p]}\ T(r,f_n) \geqslant (\lambda_g^p-\varepsilon)\log M\left(\frac{r}{4^{n-2}},f(g)\right)+O(1). \tag{2.1}$$

$$\text{So,}\quad \log^{[(n-1)p]}\ T(r,f_n) \geqslant (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1) \quad \text{when } n \text{ is even.}$$

Similarly

$$\log^{[(n-1)p]}\ T(r,f_n) \geqslant (\lambda_g^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f\right)+O(1) \quad \text{when } n \text{ is odd.}$$

This proves the lemma. □

3. Theorems

THEOREM 3.1. *Let f and g be two non constant entire functions of non zero finite p-th order and lower p-th order, also $0<\sigma_f^p,\sigma_g^p<\infty$. Then*

$$(i)\quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} \leqslant \frac{4^{\rho_g^p}\rho_f^p}{\lambda_f^p},$$

$$(ii)\quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} \geqslant \frac{\lambda_f^p}{(2^{n-1})^{\rho_g^p}\rho_f^p}$$

when n is even and

$$(iii)\quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,\ g(f))} \leqslant \frac{4^{\rho_f^p}\rho_g^p}{\lambda_g^p},$$

$$(iv)\quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,g(f))} \geqslant \frac{\lambda_g^p}{(2^{n-1})^{\rho_f^p}\rho_g^p}$$

when n is odd.

Proof. First we suppose that n is even, then from Lemma 2.4 we have for all large r and $\varepsilon > 0$,

$$\begin{aligned}\log^{[(n-1)p+1]} M(r, f_n) &\leqslant (\rho_f^p + \varepsilon)\log M(r, g) + O(1)\\ &\leqslant (\rho_f^p + \varepsilon)(\sigma_g^p + \varepsilon) r^{\rho_g^p} + O(1) \text{ by the Definition 1.4.} \quad (3.1)\end{aligned}$$

From Lemma 2.3 we get

$$T(r, f(g)) \geqslant \frac{1}{3}\log M\left(\frac{1}{16}M\left(\frac{r}{4}, g\right), f\right).$$

Since λ_f^p is the lower p-th order of f, so for given ε $(0 < \varepsilon < \lambda_f^p)$ and for all large values of r,

$$\log^{[p]} T(r, f(g)) \geqslant \log^{[p+1]} M\left(\frac{1}{16}M\left(\frac{r}{4}, g\right), f\right) + O(1).$$

$$\therefore \quad \log^{[p]} T(r, f(g)) \geqslant (\lambda_f^p - \varepsilon)\log M\left(\frac{r}{4}, g\right) + O(1). \quad (3.2)$$

Again for a sequence of values of r tending to infinity,

$$\log M\left(\frac{r}{4}, g\right) > (\sigma_g^p - \varepsilon)\left(\frac{r}{4}\right)^{\rho_g^p}. \quad (3.3)$$

Therefore from (3.2) and (3.3) we get for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r, f(g)) \geqslant (\lambda_f^p - \varepsilon)(\sigma_g^p - \varepsilon)\left(\frac{r}{4}\right)^{\rho_g^p} + O(1) \quad (3.4)$$

where $0 < \varepsilon < \min\{\lambda_f^p, \sigma_g^p\}$.

Now from (3.1) and (3.4) we have for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log^{[(n-1)p+1]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\leqslant \frac{(\rho_f^p + \varepsilon)(\sigma_g^p + \varepsilon) r^{\rho_g^p} + O(1)}{(\lambda_f^p - \varepsilon)(\sigma_g^p - \varepsilon)\left(\frac{r}{4}\right)^{\rho_g^p} + O(1)}\\ &= \frac{(\rho_f^p + \varepsilon)(\sigma_g^p + \varepsilon)4^{\rho_g^p} + o(1)}{(\lambda_f^p - \varepsilon)(\sigma_g^p - \varepsilon) + o(1)}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leqslant \frac{4^{\rho_g^p}\rho_f^p}{\lambda_f^p}.$$

Also when n is even then from Lemma 2.5 we get for all sufficiently large values of r

$$\log^{[(n-1)p+1]} M(r, f_n) \geqslant (\lambda_f^p - \varepsilon)\log M\left(\frac{r}{2^{n-1}}, g\right) + O(1).$$

Now for a sequence of values of r tending to infinity we have

$$\log M\left(\frac{r}{2^{n-1}},g\right) > (\sigma_g^p-\varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^p}.$$

Therefore for a sequence of values of r tending to infinity, we get

$$\log^{[(n-1)p+1]} M(r,f_n) \geqslant (\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^p}+O(1) \tag{3.5}$$

where $0<\varepsilon<\min\{\lambda_f^p,\sigma_g^p\}$.

Again by Lemma 2.1 we have for all large values of r,

$$\begin{aligned}
\log^{[p-1]} T(r,f(g)) &\leqslant \log^{[p]} M(r,f(g)) \\
&\leqslant \log^{[p]} M(M(r,g),f) \\
&\leqslant [M(r,g)]^{\rho_f^p+\varepsilon}. \\
\therefore \quad \log^{[p]} T(r,f(g)) &\leqslant (\rho_f^p+\varepsilon)\log M(r,g) \\
&\leqslant (\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)r^{\rho_g^p}. \qquad (3.6)
\end{aligned}$$

Therefore from (3.5) and (3.6) we have for a sequence of values of r tending to infinity,

$$\begin{aligned}
\frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} &\geqslant \frac{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^p}+O(1)}{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)r^{\rho_g^p}} \\
&= \frac{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)+o(1)}{(2^{n-1})^{\rho_g^p}(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)}.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary,

$$\therefore \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} \geqslant \frac{\lambda_f^p}{(2^{n-1})^{\rho_g^p}\rho_f^p}.$$

Similarly for odd n we get the second part of this theorem.

This proves the theorem. □

REMARK 3.2. If f is of regular growth i.e. $\rho_f^p=\lambda_f^p$ and n is even then

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} \leqslant 4^{\rho_g^p},$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} \geqslant \frac{1}{(2^{n-1})^{\rho_g^p}}.$$

Also if g is of regular growth i.e. $\rho_g^p = \lambda_g^p$ and n is odd then

$$(iii)\quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,\ g(f))} \leqslant 4^{\rho_f^p},$$

$$(iv)\quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,g(f))} \geqslant \frac{1}{(2^{n-1})^{\rho_f^p}}.$$

REMARK 3.3. The conditions non zero lower p-th order and finite p-th order are necessary for Theorem 3.1, which are shown by the following examples.

EXAMPLE 3.4. Let $f(z) = \exp^{[p]} z$ and $g(z) = \exp^{[p-1]} z$. Then $\rho_f^p = \lambda_f^p = 1$ and $\rho_g^p = \lambda_g^p = 0$.

Here $f(g) = \exp^{[2p-1]} z$ and

$$\begin{aligned} 3T(2r, f(g)) &\geqslant \log M(r, f(g)) = \exp^{[2p-2]} r \\ \text{i.e.}\quad T(r, f(g)) &\geqslant \frac{1}{3} \exp^{[2p-2]} \frac{r}{2}. \\ \therefore\quad \log^{[p]} T(r, f(g)) &\geqslant \exp^{[p-2]} \frac{r}{2} + O(1). \end{aligned}$$

Now

$$f_n = \begin{cases} \exp^{[np-\frac{n}{2}]} z \ \text{ when n is even} \\ \exp^{[np-\frac{n-1}{2}]} z \ \text{ when n is odd.} \end{cases}$$

So when n is even,

$$\begin{aligned} M(r, f_n) &= \exp^{[np-\frac{n}{2}]} r \\ \text{i.e. } \log^{[(n-1)p+1]} M(r, f_n) &= \log^{[(n-1)p+1]} \exp^{[np-\frac{n}{2}]} r \\ &= \exp^{[p-\frac{n}{2}-1]} r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,f(g))} &\leqslant \frac{\exp^{[p-\frac{n}{2}-1]} r}{\exp^{[p-2]} r} + o(1) \\ &= \frac{1}{\exp^{[\frac{n}{2}-1]} r} + o(1) \to 0 \not> 1 \text{ as } r \to \infty. \end{aligned}$$

Similarly for odd n,

$$\frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,g(f))} \leqslant \frac{\exp^{[p-\frac{n-1}{2}-1]} r}{\exp^{[p-2]} r} + o(1) \to 0 \ \not> \frac{1}{2^{n-1}} \text{as } r \to \infty.$$

EXAMPLE 3.5. Let $f(z) = \exp^{[p]} z$ and $g(z) = \exp^{[p+1]} z$. Then $\rho_f^p = \lambda_f^p = 1$ and $\rho_g^p = \lambda_g^p = \infty$.

Here $g(f) = \exp^{[2p+1]} z$ and

$$T(r,g(f)) \leqslant \log M(r,g(f)) = \exp^{[2p]} r$$
$$\therefore \ \log^{[p]} T(r,g(f)) \leqslant \exp^{[p]} r$$

Now

$$f_n = \begin{cases} \exp^{[np+\frac{n}{2}]} z & \text{when n is even} \\ \exp^{[np+\frac{n-1}{2}]} z & \text{when n is odd.} \end{cases}$$

So when n is odd,

$$\begin{aligned} M(r,f_n) &= \exp^{[np+\frac{n-1}{2}]} r \\ \text{i.e. } \log^{[(n-1)p+1]} M(r,f_n) &= \log^{[(n-1)p+1]} \exp^{[np+\frac{n-1}{2}]} r \\ &= \exp^{[p+\frac{n-1}{2}-1]} r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[p]} T(r,g(f))} &\geqslant \frac{\exp^{[p+\frac{n-1}{2}-1]} r}{\exp^{[p]} r} \\ &= \exp^{[\frac{n-1}{2}-1]} r \to \infty \not< 4 \text{ as } r \to \infty . \end{aligned}$$

THEOREM 3.6. *Let f and g be two non constant entire functions of non zero finite p-th order and lower p-th order, also $0 < \sigma_f^p, \sigma_g^p < \infty$. Then*

$$(i) \ \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f(g))}{\log^{[(n-1)p]} T(r,f_n)} \leqslant \frac{\left(4^{n-1}\right)^{\rho_g^p} \rho_f^p}{\lambda_f^p},$$
$$(ii) \ \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f(g))}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{\lambda_f^p}{2^{\rho_g^p} \rho_f^p}$$

when n is even and

$$(iii) \ \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,g(f))}{\log^{[(n-1)p]} T(r, \ f_n)} \leqslant \frac{\left(4^{n-1}\right)^{\rho_f^p} \rho_g^p}{\lambda_g^p},$$
$$(iv) \ \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,g(f))}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{\lambda_g^p}{2^{\rho_f^p} \rho_g^p}$$

when n is odd.

Proof. When n is even then from Lemma 2.7 we get for all large values of r and any ε $(0 < \varepsilon < \lambda_f^p)$,

$$\log^{[(n-1)p]} T(r,f_n) \geqslant (\lambda_f - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1). \tag{3.7}$$

Again for a sequence of values of r tending to infinity

$$\log M\left(\frac{r}{4^{n-1}},g\right) > (\sigma_g^p-\varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^p}. \tag{3.8}$$

Therefore from (3.7) and (3.8) for a sequence of values of r tending to infinity,

$$\log^{[(n-1)p]} T(r,f_n) \geqslant (\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^p}+O(1) \tag{3.9}$$

where $0<\varepsilon<\min\{\lambda_f^p,\sigma_g^p\}$.

Now from second part of Lemma 2.2 we get for large values of r

$$\begin{aligned}\log^{[p+1]} M(r,f(g)) &\leqslant \log^{[p+1]} M(M(r,g),f)\\ &\leqslant (\rho_f^p+\varepsilon)\log M(r,g)\\ &\leqslant (\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)r^{\rho_g^p}. \end{aligned}\tag{3.10}$$

Now from (3.9) and (3.10) for a sequence of values of r tenging to infinity

$$\begin{aligned}\frac{\log^{[p+1]} M(r,f(g))}{\log^{[(n-1)p]} T(r,f_n)} &\leqslant \frac{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)r^{\rho_g^p}}{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^p}+O(1)}\\ &= \frac{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)\left(4^{n-1}\right)^{\rho_g^p}}{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)+o(1)}.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary,

$$\therefore\quad \liminf_{r\to\infty}\frac{\log^{[p+1]} M(r,f(g))}{\log^{[(n-1)p]} T(r,f_n)} \leqslant \frac{\left(4^{n-1}\right)^{\rho_g^p}\rho_f^p}{\lambda_f^p}.$$

Again for all sufficiently large value of r we get from first part of Lemma 2.2

$$\begin{aligned}\log^{[p+1]} M(r,f(g)) &\geqslant \log^{[p+1]} M\left(\frac{1}{16}M\left(\frac{r}{2},g\right),f\right)\\ &\geqslant \log\left[\frac{1}{16}M\left(\frac{r}{2},g\right)\right]^{\lambda_f^p-\varepsilon}\\ &\geqslant (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{2},g\right)+O(1).\end{aligned}\tag{3.11}$$

Also for a sequence of values of r tending to infinity

$$\log M\left(\frac{r}{2},g\right) > (\sigma_g^p-\varepsilon)\left(\frac{r}{2}\right)^{\rho_g^p}. \tag{3.12}$$

Therefore from (3.11) and (3.12) for a sequence of values of r tending to infinity,

$$\log^{[p+1]} M(r,f(g)) \geqslant (\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{2}\right)^{\rho_g^p}+O(1) \tag{3.13}$$

where $0 < \varepsilon < \min\{\lambda_f^p, \sigma_g^p\}$.

Also when n is even then from Lemma 2.6 we get for r tending to infinity

$$\begin{aligned}\log^{[(n-1)p]} T(r, f_n) &\leqslant (\rho_f^p + \varepsilon)\log M(r,g) + O(1)\\ &\leqslant (\rho_f^p + \varepsilon)(\sigma_g^p + \varepsilon) r^{\rho_g^p} + O(1). \end{aligned} \tag{3.14}$$

Now from (3.13) and (3.14) for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p]} T(r, f_n)} &\geqslant \frac{(\lambda_f^p - \varepsilon)(\sigma_g^p - \varepsilon)\left(\frac{r}{2}\right)^{\rho_g^p} + O(1)}{(\rho_f^p + \varepsilon)(\sigma_g^p + \varepsilon) r^{\rho_g^p} + O(1)}\\ &= \frac{(\lambda_f^p - \varepsilon)(\sigma_g^p - \varepsilon) + o(1)}{2^{\rho_g}(\rho_f^p + \varepsilon)(\sigma_g^p + \varepsilon) + o(1)}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \quad \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p]} T(r, f_n)} \geqslant \frac{\lambda_f^p}{2^{\rho_g^p} \rho_f^p}.$$

Similarly when n is odd we get second part of the theorem.

This proves the theorem. $\square$

REMARK 3.7. If f is of regular growth i.e. $\rho_f^p = \lambda_f^p$ and n is even then

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[p]} M(r, f(g))}{\log^{[(n-1)p]} T(r, f_n)} \leqslant \left(4^{n-1}\right)^{\rho_g^p},$$

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p]} M(r, f(g))}{\log^{[(n-1)p]} T(r, f_n)} \geqslant \frac{1}{2^{\rho_g^p}}.$$

Also if g is of regular growth i.e. $\rho_g^p = \lambda_g^p$ and n is odd then

$$(iii)\ \liminf_{r\to\infty} \frac{\log^{[p]} M(r, g(f))}{\log^{[(n-1)p]} T(r, f_n)} \leqslant \left(4^{n-1}\right)^{\rho_f^p},$$

$$(iv)\ \limsup_{r\to\infty} \frac{\log^{[p]} M(r, g(f))}{\log^{[(n-1)p]} T(r, f_n)} \geqslant \frac{1}{2^{\rho_f^p}}.$$

The next theorem is the generalization of the above theorems.

THEOREM 3.8. *Let f and g be two non constant entire functions of non zero finite p-th order and lower p-th order, also $0 < \sigma_f^p, \sigma_g^p < \infty$. Then*

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r, f_n)}{\log^{[(n-1)p]} T(r, f_n)} \leqslant \frac{\left(4^{n-1}\right)^{\rho_g^p} \rho_f^p}{\lambda_f^p},$$

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r, f_n)}{\log^{[(n-1)p]} T(r, f_n)} \geqslant \frac{\lambda_f^p}{\left(2^{n-1}\right)^{\rho_g} \rho_f^p}$$

when n is even and

$$(iii)\quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,\ f_n)} \leqslant \frac{\left(4^{n-1}\right)^{\rho_f^p} \rho_g^p}{\lambda_g^p},$$
$$(iv)\quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{\lambda_g^p}{(2^{n-1})^{\rho_f^p} \rho_g^p}$$

when n is odd.

Proof. When n is even then from (3.1) and (3.9) we get for a sequence of values of r tending to infinity and for $0<\varepsilon<\min\{\lambda_f^p, \sigma_g^p\}$,

$$\frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \leqslant \frac{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)r^{\rho_g}+O(1)}{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^p}+O(1)}$$
$$= \frac{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)\left(4^{n-1}\right)^{\rho_g^p}+o(1)}{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)+o(1)}.$$

Since $\varepsilon>0$ is arbitrary,

$$\therefore\quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \leqslant \frac{\left(4^{n-1}\right)^{\rho_g^p} \rho_f^p}{\lambda_f^p}.$$

Also from (3.5) and (3.14) we have for a sequence of values of r tending to infinity and for $0<\varepsilon<\min\{\lambda_f^p, \sigma_g^p\}$,

$$\frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)\left(\frac{r}{2^{n-1}}\right)^{\rho_g^p}+O(1)}{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)r^{\rho_g^p}+O(1)}$$
$$= \frac{(\lambda_f^p-\varepsilon)(\sigma_g^p-\varepsilon)+o(1)}{(\rho_f^p+\varepsilon)(\sigma_g^p+\varepsilon)(2^{n-1})^{\rho_g^p}+o(1)}.$$

Since $\varepsilon>0$ is arbitrary,

$$\therefore\quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{\lambda_f^p}{(2^{n-1})^{\rho_g^p} \rho_f^p}.$$

Similarly for odd n we get second part of the theorem.

This proves the theorem. $\square$

REMARK 3.9. If f is regular growth i.e. $\rho_f^p = \lambda_f^p$ and n is even then

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \leqslant \left(4^{n-1}\right)^{\rho_g^p},$$

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{1}{(2^{n-1})^{\rho_g^p}}.$$

Also if g is regular growth i.e. $\rho_g^p = \lambda_g^p$ and n is odd then

$$(iii)\ \liminf_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,\ f_n)} \leqslant \left(4^{n-1}\right)^{\rho_f^p},$$

$$(iv)\ \limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} M(r,f_n)}{\log^{[(n-1)p]} T(r,f_n)} \geqslant \frac{1}{(2^{n-1})^{\rho_f^p}}.$$

THEOREM 3.10. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_{fog}^p \leqslant \rho_{fog}^p < \infty$ and $0 < \lambda_g^p \leqslant \rho_g^p < \infty$. Then for any positive number A,*

$$\frac{\lambda_{fog}^p}{A\rho_g^p} \leqslant \liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_n)}{\log^{[p]} T(r^A,g)} \leqslant \frac{\lambda_{fog}^p}{A\lambda_g^p} \leqslant \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_n)}{\log^{[p]} T(r^A,g)} \leqslant \frac{\rho_{fog}^p}{A\lambda_g^p}$$

when n is even.

Proof. When n is even then from (2.1) we have for all large values of r and arbitrary $\varepsilon(0 < \varepsilon < \min\{\lambda_f^p, \lambda_g^p\})$,

$$\log^{[(n-1)p]} T(r,f_n) \geqslant (\lambda_{fog}^p - \varepsilon)\log\left(\frac{r}{4^{n-2}}\right) + O(1) \tag{3.15}$$

and Definition 1.3,

$$\log^{[p]} T(r^A,g) \leqslant A(\rho_g^p + \varepsilon)\log r. \tag{3.16}$$

So for all large values of r,

$$\frac{\log^{[(n-1)p]} T(r,f_n)}{\log^{[p]} T(r^A,g)} \geqslant \frac{\left(\lambda_{fog}^p - \varepsilon\right)\left(\log r - \log 4^{n-2}\right) + O(1)}{A(\rho_g^p + \varepsilon)\log r}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore\quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_n)}{\log^{[p]} T(r^A,g)} \geqslant \frac{\lambda_{fog}^p}{A\rho_g^p}. \tag{3.17}$$

Again from Lemma 2.6 we have for all large values of r and $\varepsilon > 0$,

$$\log^{[(n-2)p]} T(r,f_n) \leqslant (\rho_g^p + \varepsilon)\log M(r,f(g)) + O(1).$$

Now for a sequence of values of r tending to infinity,

$$\log^{[p+1]} M(r, f(g)) \leqslant (\lambda_{fog}^{p} + \varepsilon) \log r$$

Therefore for a sequence of values of r tending to infinity we get,

$$\log^{[(n-1)p]} T(r, f_n) \leqslant (\lambda_{fog}^{p} + \varepsilon) \log r + O(1)$$

and for all large values of r,

$$\log^{[p]} T(r^A, g) \geqslant A(\lambda_g^p - \varepsilon) \log r \tag{3.18}$$

where $0 < \varepsilon < \lambda_g^p$.

So for a sequence of values of r tending to infinity,

$$\frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\lambda_{fog}^{p} + \varepsilon}{A(\lambda_g^p - \varepsilon)} + o(1).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\lambda_{fog}^{p}}{A\lambda_g^p}. \tag{3.19}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r^A, g) \leqslant A(\lambda_g^p + \varepsilon) \log r. \tag{3.20}$$

Now from (3.15) and (3.20) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \geqslant \frac{\left(\lambda_{fog}^{p} - \varepsilon\right) \left(\log r - \log 4^{n-2}\right) + O(1)}{A(\lambda_g^p + \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{r \to \infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \geqslant \frac{\lambda_{fog}^{p}}{A\lambda_g^p}. \tag{3.21}$$

Again from Lemma 2.6 we have for all large values of r and $\varepsilon > 0$,

$$\begin{aligned} \log^{[(n-1)p]} T(r, f_n) &\leqslant \log^{[p+1]} M(r, f(g)) + O(1) \\ &\leqslant (\rho_{fog}^{p} + \varepsilon) \log r + O(1). \end{aligned} \tag{3.22}$$

So from (3.18) and (3.22) we obtain for all large values of r,

$$\frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\rho_{fog}^{p} + \varepsilon}{A(\lambda_g^p - \varepsilon)} + o(1).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{r \to \infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\rho_{fog}^{p}}{A\lambda_g^p}. \tag{3.23}$$

Therefore the theorem follows from (3.17), (3.19), (3.21) and (3.23).

This proves the theorem. □

THEOREM 3.11. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda^p_{gof} \leqslant \rho^p_{gof} < \infty$ and $0 < \lambda^p_f \leqslant \rho^p_f < \infty$. Then for any positive number A,*

$$\frac{\lambda^p_{gof}}{A\rho^p_f} \leqslant \liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, f)} \leqslant \frac{\lambda^p_{gof}}{A\lambda^p_f} \leqslant \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, f)} \leqslant \frac{\rho^p_{gof}}{A\lambda^p_f}$$

when n is odd.

THEOREM 3.12. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda^p_{fog} \leqslant \rho^p_{fog} < \infty$ and $0 < \rho^p_g < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\rho^p_{fog}}{A\rho^p_g} \leqslant \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)}$$

when n is even.

Proof. From the definition of p-th order, we have for a sequence of values of r tending to infinity and arbitrary $\varepsilon > 0$,

$$\log^{[p]} T(r^A, g) \geqslant A(\rho^p_g - \varepsilon)\log r. \tag{3.24}$$

So from (3.22) and (3.24) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\rho^p_{fog} + \varepsilon}{A(\rho^p_g - \varepsilon)} + o(1).$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \leqslant \frac{\rho^p_{fog}}{A\rho^p_g}. \tag{3.25}$$

Again for (2.1) we get for a sequence of values of r tending to infinity,

$$\log^{[(n-1)p]} T(r, f_n) \geqslant (\rho^p_{fog} - \varepsilon)\log\left(\frac{r}{4^{n-2}}\right) + O(1). \tag{3.26}$$

Now from (3.16) and (3.26) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \geqslant \frac{\left(\rho^p_{fog} - \varepsilon\right)\left(\log r - \log 4^{n-2}\right) + O(1)}{A(\rho^p_g + \varepsilon)\log r}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, g)} \geqslant \frac{\rho^p_{fog}}{A\rho^p_g}. \tag{3.27}$$

So the theorem follows from (3.25) and (3.27).

This proves the theorem. $\square$

THEOREM 3.13. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda^p_{g\circ f} \leqslant \rho^p_{g\circ f} < \infty$ and $0 < \rho^p_f < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, f)} \leqslant \frac{\rho^p_{g\circ f}}{A\rho^p_f} \leqslant \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{\log^{[p]} T(r^A, f)}$$

when n is odd.

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