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REFINED YOUNG INEQUALITY WITH KANTOROVICH CONSTANT

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Abstract. The Specht ratio $S(h)$ is the optimal constant in the reverse of the arithmetic-geometric mean inequality, i.e., if $0 < m \leqslant a,b \leqslant M$ and $h = \frac{M}{m}$, then $(1-\mu)a + \mu b \leqslant S(h)a^{1-\mu}b^{\mu}$ for all $\mu \in [0,1]$. Recently S. Furuichi proved that $(1-\mu)a+\mu b \geqslant S(h^r)a^{1-\mu}b^{\mu}$ for $a,b>0$, $\mu \in [0,1]$, where $h = \frac{b}{a}$ and $r = \min\{\mu, 1-\mu\}$. In this paper, we improve it by virtue of the Kantorovich constant, utilizing the refined scalar Young inequality we establish a weighted arithmetic-geometric-harmonic mean inequality for two positive operators. In the remainder of this work we focus on extending the refined weighted arithmetic-harmonic mean inequality to an operator version for another type of improvement.

1. Introduction

Throughout this paper, A,B are positive operators on a Hilbert space, we use the following notations: $A\nabla_{\mu}B = (1-\mu)A + \mu B$, $A\sharp_{\mu}B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\mu}A^{1/2}$, and $A!_{\mu}B = ((1-\mu)A^{-1} + \mu B^{-1})^{-1}$, see F. Kubo and T. Ando [6]. When $\mu = 1/2$ we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively. The Kontorovich constant is defined as $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$, while the Specht ratio [9] is denoted by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}} \quad \text{for} \quad t>0, t \neq 1; \quad \text{and} \quad S(1) = \lim_{t\to 1} S(t) = 1.$$

We start from the famous Young inequality:

$$a\nabla_{\mu}b \geqslant a^{1-\mu}b^{\mu} \tag{1}$$

for positive numbers a,b and $\mu \in [0,1]$. The inequality (1) is also called a weighted arithmetic-geometric mean inequality and its reverse inequality was given in [10] with the Specht ratio as follows:

$$a\nabla_{\mu}b \leqslant S(h)a^{1-\mu}b^{\mu} \tag{2}$$

for all $\mu \in [0,1]$, where $0 < m \leqslant a,b \leqslant M$ and $h = \frac{M}{m}$.

Recently, an improvement of the inequality (1) was given in [2] as follows:

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THEOREM F. *For $a,b>0$, if $\mu\in[0,1]$, $r=\min\{\mu,1-\mu\}$ and $h=\frac{b}{a}$, then*

$$a\nabla_\mu b \geqslant S(h^r)a^{1-\mu}b^\mu. \tag{3}$$

Based on this, the refined weighted arithmetic-geometric operator mean inequality is given by

$$A\nabla_\mu B \geqslant S(h^r)A\sharp_\mu B. \tag{4}$$

See [3, 4] for recent developments of the improved Young inequality. See also [5] for another type of improvement for the classical Young inequality.

In this short paper, we improve the inequality (3) via the Kantorovich constant as follows:

$$a\nabla_\mu b \geqslant K(h,2)^r a^{1-\mu}b^\mu$$

for all $\mu\in[0,1]$, where $r=\min\{\mu,1-\mu\}$ and $h=\frac{b}{a}$. It admits an operator extension

$$A\nabla_\mu B \geqslant K(h,2)^r A\sharp_\mu B$$

for positive operators A, B on a Hilbert space. While we provide a new viewpoint and method which is different from that of the refinement given in [2].

2. Refinement of Young Inequalities

First of all, we cite a refinement of the weighted arithmetic-geometric mean inequality for n positive numbers, which was shown by Pečarić et.al., see [7; Theorem 1, P.717] and also [1, 8].

LEMMA 1. *Let $x_1,\cdots,x_n$ belong to a fixed closed interval $I=[a,b]$ with $a<b$, $p_1,\cdots,p_n\geqslant 0$ with $\sum\limits_{i=1}^{n} p_i=1$ and $\lambda=\min\{p_1,\cdots,p_n\}$. If f is a convex function on I, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big) \geqslant n\lambda\left[\sum_{i=1}^{n}\frac{1}{n}f(x_i) - f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big)\right]. \tag{5}$$

We will use lemma 1 as the following form by applying $f(x)=-\log x$:

COROLLARY 2. *If $x_i\in[a,b]$, $0<a<b$, $p_1,\cdots,p_n\geqslant 0$ with $\sum_{i=1}^{n} p_i=1$ and $\lambda=\min\{p_1,\cdots,p_n\}$, then*

$$\frac{\sum_{i=1}^{n} p_i x_i}{\prod_{i=1}^{n} x_i^{p_i}} \geqslant \left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i^{\frac{1}{n}}}\right)^{n\lambda}. \tag{6}$$

The case $n=2$ in (6) is simplified to the following one, which is a loose extension of [2].

COROLLARY 3. *If* $a,b>0$, $\mu\in[0,1]$, *then*

$$a\nabla_\mu b \geqslant K(h,2)^r a^{1-\mu}b^\mu, \tag{7}$$

where $r=\min\{\mu,1-\mu\}$ *and* $h=\frac{b}{a}$.

Replacing a, b by a^{-1}, b^{-1}, respectively, we have the counterpart of (7) itself.

COROLLARY 4. *If* $a,b>0$ *and* $\mu\in[0,1]$, *then*

$$a^{1-\mu}b^\mu \geqslant K(h,2)^r a!_\mu b. \tag{8}$$

Furthermore Corollary 3 implies Theorem F because of the following fact.

LEMMA 5. *If* $t>0$ *and* $0\leqslant r\leqslant\frac{1}{2}$, *then*

$$K(t,2)^r \geqslant S(t^r). \tag{9}$$

To prove Lemma 5, we need the following lemma.

LEMMA 6. ([2] Lemma 2.3) *If* $t>0$ *and* $t\neq 1$, *then*

$$\frac{t^{\frac{t}{t-1}}}{e} \leqslant \frac{t^2+1}{t+1}. \tag{10}$$

Proof. We give it a proof for convenience. By taking logarithms in (10), it is enough to prove that $f(t)=\log(t^2+1)-\log(t+1)-\frac{t}{t-1}\log t+1\geqslant 0$ for $t>0$ and $t\neq 1$.

Since $f'(t)=\frac{2t}{t^2+1}-\frac{1}{t+1}-\frac{1}{t-1}++\frac{\log t}{(t-1)^2}=\frac{4t}{t^4-1}+\frac{\log t}{(t-1)^2}$, it follows that $f'(t)\leqslant 0$ for $0<t<1$ and $f'(t)\geqslant 0$ for $t>1$. Thus we have $f(t)\geqslant\lim_{t\to 1}f(t)=0$ for all $t>0$ with $t\neq 1$. $\square$

Proof of Lemma 5. If $t=1$, then it is easy to get $S(1)=1=K(1,2)$.

If $t>0$ and $t\neq 1$, then, logarithmic-arithmetic mean inequality implies

$$\frac{t^r-1}{\log t^r}\leqslant\frac{t^r+1}{2} \quad \text{for} \quad 0\leqslant r\leqslant\frac{1}{2}.$$

Combining with (10) we have

$$S(t^r)=\frac{t^{r\frac{1}{t^r-1}}}{e}\frac{t^r-1}{\log t^r}=\frac{1}{t^r}\frac{t^{r\frac{t^r}{t^r-1}}}{e}\frac{t^r-1}{\log t^r}\leqslant\frac{1}{t^r}\frac{t^{2r}+1}{t^r+1}\frac{t^r+1}{2}=\frac{t^{2r}+1}{2t^r}.$$

Since $f(x)=x^{2r}(x\geqslant 0)$ is concave for $0\leqslant r\leqslant\frac{1}{2}$, it follows that

$$\frac{t^{2r}+1}{2}\leqslant\left(\frac{t+1}{2}\right)^{2r}=\left[\frac{(t+1)^2}{4}\right]^r.$$

Hence we have

$$S(t^r)\leqslant\frac{t^{2r}+1}{2}\frac{1}{t^r}\leqslant\left[\frac{(t+1)^2}{4t}\right]^r=K(t,2)^r. \quad \square$$

3. Applications to Operator Young Inequality

THEOREM 7. *Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either of the following conditions:*

(i) $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$

(ii) $0 < m'I \leqslant B \leqslant mI < MI \leqslant A \leqslant M'I$.

Then

$$A\nabla_\mu B \geqslant K(h,2)^r A\sharp_\mu B \tag{11}$$

for all $\mu \in [0,1]$, *where* $r = \min\{\mu, 1-\mu\}$, $h \equiv \frac{M}{m}$ *and* $h' \equiv \frac{M'}{m'}$.

Proof. From Corollary 3, we have

$$(1-\mu) + \mu x \geqslant K(x,2)^r x^\mu$$

for any $x > 0$. And hence

$$(1-\mu)I + \mu X \geqslant \min_{h \leqslant x \leqslant h'} K(x,2)^r X^\mu$$

for the positive operator X such that $0 < hI \leqslant X \leqslant h'I$.

Substituting $A^{-1/2}BA^{-1/2}$ for X in the above inequality we have:

In the case of (i), $1 < h = \frac{M}{m} \leqslant A^{-1/2}BA^{-1/2} \leqslant \frac{M'}{m'} = h'$, we have

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geqslant \min_{h \leqslant x \leqslant h'} K(x,2)^r (A^{-1/2}BA^{-1/2})^\mu.$$

It is easy to check that $K(x,2)$ is an increasing function for $x > 1$, then

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geqslant K(h,2)^r (A^{-1/2}BA^{-1/2})^\mu. \tag{12}$$

In the case of (ii), we have $0 < 1/h' \leqslant A^{-1/2}BA^{-1/2} \leqslant 1/h < 1$, then

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geqslant \min_{1/h' \leqslant x \leqslant 1/h} K(x,2)^r (A^{-1/2}BA^{-1/2})^\mu.$$

Since $K(x,2)$ is a decreasing function for $0 < x < 1$, we have

$$(1-\mu)I + \mu A^{-1/2}BA^{-1/2} \geqslant K(1/h,2)^r (A^{-1/2}BA^{-1/2})^\mu. \tag{13}$$

Multiplying both sides by $A^{1/2}$ to inequality (12) and (13) and using $K(1/h,2) = K(h,2)$ for $h > 0$, we obtain the refined arithmetic-geometric operator mean inequality. $\square$

By replacing A, B by A^{-1}, B^{-1}, respectively, then the noncommutative geometric-harmonic mean inequality can be obtained as follows:

THEOREM 8. *Assume the conditions as in Theorem 7. Then*

$$A\sharp_\mu B \geqslant K(h,2)^r A!_\mu B. \tag{14}$$

From Lemma 5, it's easy to get the following

COROLLARY 9. [2] *Assume the conditions as in Theorem 7. Then*

$$A\nabla_\mu B \geqslant S(h^r)A\sharp_\mu B. \tag{15}$$

In the remainder, we focus on extending the refined weighted arithmetic-harmonic mean inequality to an operator version for another type of improvement.

LEMMA 10. *If* $x_1,\cdots,x_n>0$ *and* $p_1,\cdots,p_n \geqslant 0$ *with* $\sum\limits_{i=1}^{n} p_i = 1$, *then*

$$\sum_{i=1}^{n} p_i x_i^{-1} - \left(\sum_{i=1}^{n} p_i x_i\right)^{-1} \geqslant n\lambda\left[\sum_{i=1}^{n}\frac{1}{n}x_i^{-1} - \left(\sum_{i=1}^{n}\frac{1}{n}x_i\right)^{-1}\right], \tag{16}$$

where $\lambda = \min\{p_1,p_2,\cdots p_n\}$.

Proof. Let $f(x)=x^{-1}$ in lemma 1, then the desired inequality is obtained. $\square$

THEOREM 11. *If* $\mu\in[0,1]$, *A and B are positive operators, then*

$$A\nabla_\mu B \geqslant A!_\mu B + 2r(A\nabla B - A!B), \tag{17}$$

where $r=\min\{\mu,1-\mu\}$.

Proof. From the case $n=2$ in Lemma 10, we have, for $x>0$ and $\mu\in[0,1]$,

$$(1-\mu)+\mu x^{-1} - ((1-\mu)+\mu x)^{-1} \geqslant 2r\left[\frac{1+x^{-1}}{2} - \left(\frac{1+x}{2}\right)^{-1}\right].$$

Thus it follows that

$$(1-\mu)I+\mu T^{-1} \geqslant ((1-\mu)I+\mu T)^{-1} + 2r\left[\frac{I+T^{-1}}{2} - \left(\frac{I+T}{2}\right)^{-1}\right] \tag{18}$$

for a strictly positive operator T and $\mu\in[0,1]$.

We may assume that A, B are invertible. Put $T=A^{\frac12}B^{-1}A^{\frac12}$ in (18), then

$$\begin{aligned}(1-\mu)I+\mu(A^{\frac12}B^{-1}A^{\frac12})^{-1} &\geqslant ((1-\mu)I+\mu A^{\frac12}B^{-1}A^{\frac12})^{-1}\\ &\quad +2r\left[\frac{I+(A^{\frac12}B^{-1}A^{\frac12})^{-1}}{2} - \left(\frac{I+A^{\frac12}B^{-1}A^{\frac12}}{2}\right)^{-1}\right].\end{aligned}$$

Multiplying both sides by $A^{\frac12}$ we have

$$(1-\mu)A+\mu B \geqslant ((1-\mu)A^{-1}+\mu B^{-1})^{-1} + 2r\left[\frac{A+B}{2} - \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right],$$

so that

$$A\nabla_\mu B \geqslant A!_\mu B + 2r(A\nabla B - A!B). \qquad \square$$

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