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UPPER BOUNDS ON MULTIPLE GENERALIZED MATHIEU SERIES

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Abstract. The main aim of this short note is to obtain new very general upper bounds for multiple generalized Mathieu series considering the related integral representation obtained recently by Pogány and Tomovski [10], by means of the multiple Hardy–Hilbert type integral inequality given in [1].

1. Introduction and preparation

The series

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}\qquad r>0,$$

was introduced by Émile Leonard Mathieu in 1890 in his book [7] devoted to mathematical physics investigations of the elasticity of rigid bodies. Several extensions and unifications such as generalized Mathieu, $\boldsymbol{a}$–, $(\boldsymbol{a},\boldsymbol{\lambda})$–series and their alternating variants have been considered in getting integral representation results are obtained recently by Cerone, Pogány, Srivastava, Tomovski and their coworkers [2, 3, 8, 9, 12, 13, 14, 17], and related bounding inequalities [4, 10, 11, 12, 14, 15, 16]. However, to the best knowledge of the authors the multiple Mathieu series has been considered only in two publications, in ones by Pogány and Tomovski [10] and by Draščić Ban [4], where Draščić Ban focused to the so–called multiple Mathieu's $(\boldsymbol{a},\boldsymbol{\lambda})$–series. In both papers certain sharp bounding inequalities have been obtained in the sense that the well–known sharp bilateral bound $0\leqslant\{u\}<1$ was employed, where $\{u\}$ stands for the fractional part of some real u, see [10] and [4], respectively.

We consider here the *multiple generalized Mathieu series* [10, Eq. (2)] defined by

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu})=\sum_{\mathbf{n}\in\mathbb{N}^m}\frac{2\mathbf{n}^{|\mathbf{s}|}}{(\langle\mathbf{n}^{\boldsymbol{\nu}},\mathbf{n}^{\boldsymbol{\nu}}\rangle+r^2)^{p+1}},$$

where $\mathbf{n}^{\boldsymbol{\nu}}:=(n_1^{\nu_1},\cdots,n_m^{\nu_m})$; $\mathbf{n}^{\boldsymbol{\alpha}|\mathbf{s}|}:=n_1^{\alpha_1 s_1}\cdots n_m^{\alpha_m s_m}$; $\mathbf{s}$, $\boldsymbol{\nu}$ have positive coordinates, i.e. s_ℓ, $\nu_\ell>0$, $\ell=\overline{1,m}$ and $\langle\mathbf{a},\mathbf{b}\rangle$ stands for the inner product in $\mathbb{R}_+^m$.

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Let us introduce few necessary notations and conventions. Denote here, and in what follows, $[x]$ the integer part of some $x \in \mathbb{R}$ and $(a)_m := a(a+1)\cdots(a+m-1)$, $a \in \mathbb{C}$, $m \in \mathbb{N} = \{1,2,\cdots\}$ the Pochhammer symbol (or *shifted factorial*).

The following integral representation formula has been derived by Pogány and Tomovski [10].

LEMMA 1. [10, Theorem 1] *Let $r,\ p+1>0$, let the multiindices $\mathbf{s},\ \boldsymbol{\nu}$ be positive and taken so, that the multiple generalized Mathieu series $\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu})$ converges. Then we have*

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu}) = (p+1)_m \int_{[1,\infty)^m} \frac{\prod\limits_{j=1}^m \left(\dfrac{\left[t_j^{1/(2\nu_j)}\right]^{s_j+1}}{s_j+1} + s_j \displaystyle\int_0^{\left[t_j^{1/(2\nu_j)}\right]} \{u\} u^{s_j-1} du \right)}{(r^2+t_1+\cdots+t_m)^{p+m+1}} \, \mathbf{d}\boldsymbol{t}, \tag{1}$$

where $\mathbf{d}\boldsymbol{t} := \mathrm{d}t_1 \cdots \mathrm{d}t_m$.

Let $p_j > 1,\ j=1,\cdots,m$ be parameters such that $\sum_{j=1}^m p_j^{-1} \geqslant 1$, and let p_j' be the conjugated Hölder parameter related to p_j, that is $p_j^{-1} + (p_j')^{-1} = 1,\ j=1,\cdots,m$. By the standard notation

$$\lambda = \frac{1}{m-1} \sum_{j=1}^m \frac{1}{p_j'} = \sum_{j=1}^m \frac{1}{q_j},$$

where $q_j^{-1} := \lambda - (p_j')^{-1},\ j=1,\cdots,m$. Obviously $0 < \lambda \leqslant 1$ and for $\lambda = 1$ the involved parameters correspond to the conjugated case.

We will also need the following, *here necessarily precised*, Hardy–Hilbert type inequality result by Brnetić *et al.* [1], achieved for the non–conjugated Hölder parameters case.

LEMMA 2. [1, Theorem 1, Eq. (7)] *Let $m \geqslant 2$ be an integer, $\lambda,\ p_j,\ p_j';\ q_j > 0$, $j=1,\cdots,m$ be the real numbers previously defined. Let the domain of integration $\Omega \subseteq \mathbb{R}_+$, and assume that the kernel function $K\colon \Omega^m \mapsto \mathbb{R}_+$ and $\mu_j, j=1,\cdots,m$ are nonnegative, σ–finite measures. If the weight functions $\phi_{ij} \geqslant 0,\ i,j=1,\cdots,m$ satisfy condition $\prod_{i,j=1}^m \phi_{ij}(x_j) = 1$, then the following inequality holds*

$$\int_{\Omega^m} K^\lambda(\boldsymbol{x}) \prod_{j=1}^m f_j(x_j)\mathrm{d}\mu_j(x_j) \leqslant \prod_{j=1}^m \left(\int_\Omega \Big(\phi_{jj}(x_j)F_j(x_j)f_j(x_j)\Big)^{p_j} \mathrm{d}\mu_j(x_j) \right)^{1/p_j} \tag{2}$$

where $f_j\colon \Omega \mapsto \mathbb{R}_+$, $j=1,\cdots,m$; and $\boldsymbol{x} := (x_1,\cdots,x_m)$, while

$$F_i(x_i) = \left(\int_{\Omega^{m-1}} K(\boldsymbol{x}) \prod_{\substack{j=1 \\ j\neq i}}^m \phi_{ij}^{q_i}(x_j)\,\mathrm{d}\mu_j(x_j) \right)^{1/q_i};$$

provided both sides of (2) *there exist.*

REMARK. For the sake of completness we have to mention that Hardy–Hilbert integral inequalities play significant roles in deriving bounding inequalities for alternating Mathieu–type series [16]; they were also important tool in estimating generalized hypergeometric functions ${}_mF_n$ [6].

2. Main results

Our aim is to apply Lemma 2 to the integral representation (1) to derive new upper bounds for $\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu})$. Then, suitably specifying the kernel function K we derive further more elegant corollaries of the main theorem.

THEOREM 1. *Let $m \geqslant 2$ be an integer, $\lambda,\ p_j,\ p_j';\ q_j > 0,\ j = 1,\cdots,m$ be the real numbers defined in the preambula of the* Lemma 2, *and be $A_{ij} + q_i^{-1} > 0,\ i \neq j;$ $i,j = 1,\cdots,m$ such, that $\sum_{i=1}^m A_{ij} = 0$ and*

$$A_{ii} - \alpha_i > \frac{(m-1)\lambda - m - p - 1}{\lambda q_i} \qquad \alpha_i := \sum_{j=1}^m A_{ij}, \quad i = 1,\cdots,m.$$

Then the following inequality holds

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu}) \leqslant \mathsf{C}\cdot\prod_{i=1}^m \left(\int_{\mathbb{R}_+} x_i^{A_{ii}p_i}\left(r^2+m+x_i\right)^{\frac{p_i}{\lambda q_i}\left((m-1)\lambda-m-p-1\right)+p_i(\alpha_i-A_{ii})} f_i^{p_i}(x_i)\mathrm{d}x_i\right)^{1/p_i}, \tag{3}$$

where

$$\mathsf{C} = \frac{2(p+1)_m}{\Gamma^\lambda\left(\frac{p+m+1}{\lambda}\right)}\prod_{i=1}^m \Gamma^{1/q_i}\left(\frac{p+m+1}{\lambda} - m+1-q_i\left(\alpha_i - A_{ii}\right)\right)\cdot\prod_{\substack{i,j=1\\ i\neq j}}^m \Gamma^{1/q_i}\left(q_iA_{ij}+1\right),$$

and

$$f_j(x_j) = \frac{\left[(x_j+1)^{1/(2\nu_j)}\right]^{s_j+1}}{s_j+1} + s_j\int_0^{\left[(x_j+1)^{1/(2\nu_j)}\right]}\{u\}u^{s_j-1}du, \quad j = 1,\cdots,m. \tag{4}$$

Proof. First, by substituting $x_j + 1 \mapsto t_j,\ j = 1,\cdots,m$ the integral formula (1) becomes

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu}) = 2(p+1)_m\int_{\mathbb{R}_+^m}\frac{\prod_{j=1}^m f_j(x_j)}{(r^2+m+x_1+\cdots+x_m)^{p+m+1}}\,\mathbf{d}\boldsymbol{x}, \tag{5}$$

where $f_j(x_j)$ is defined in (4). Choose the kernel function

$$K(\boldsymbol{x}) = \left(r^2+m+x_1+\cdots+x_m\right)^{-(p+m+1)/\lambda},$$

and define $\phi_{ij}(x_j) = x_j^{A_{ij}}$ on $\Omega := \mathbb{R}_+$. Then the normalizing condition of Lemma 2 is satisfied. Indeed

$$\prod_{i,j=1}^{m} \phi_{ij}(x_j) = \prod_{i,j=1}^{m} x_j^{A_{ij}} = \prod_{j=1}^{m} x_j^{\sum\limits_{i=1}^{m} A_{ij}} = 1,$$

by virtue of the assumed $\sum_{i=1}^{m} A_{ij} = 0,\ j = 1,\cdots,m$. Now, we conclude

$$F_i(x_i) = \left(\int_{\mathbb{R}_+^{m-1}} K(\boldsymbol{x}) \prod_{\substack{j=1\\ j\neq i}}^{m} x_j^{q_i A_{ij}} \cdot \frac{\mathbf{d}\boldsymbol{x}}{\mathrm{d}x_i} \right)^{1/q_i}.$$

By substitution $x_k(r^2+m+x_i)^{-1} \mapsto u_k,\ k = 1,\cdots,m,\ k \neq i$, we get

$$F_i(x_i) = \big(r^2+m+x_i\big)^{\sum\limits_{j=1,j\neq i}^{m} A_{ij} + \frac{m-1}{q_i} - \frac{p+m+1}{\lambda q_i}} \left(\int_{\mathbb{R}_+^{m-1}} \frac{\prod\limits_{\substack{j=1\\ j\neq i}}^{m} u_j^{q_i A_{ij}} \cdot \dfrac{\mathbf{d}\boldsymbol{u}}{\mathrm{d}u_i}}{\Big(1+\sum\limits_{\substack{k=1\\ k\neq i}}^{m} u_k\Big)^{(p+m+1)/\lambda}} \right)^{1/q_i}.$$

Employing the familiar formula for the Gamma–function integral [18, Lemma 5.1]

$$\int_{\mathbb{R}_+^{k-1}} \frac{\prod\limits_{j=1}^{k-1} z_j^{a_j-1}}{\Big(1+\sum\limits_{j=1}^{k-1} z_j\Big)^{\sum\limits_{j=1}^{k} a_j}} \mathbf{d}\boldsymbol{z} = \frac{\prod\limits_{j=1}^{k} \Gamma(a_j)}{\Gamma\Big(\sum\limits_{j=1}^{k} a_j\Big)} \qquad \min_{1\leqslant j\leqslant k} \Re\{a_j\} > 0,$$

we deduce

$$F_i^{q_i}(x_i) = \frac{\big(r^2+m+x_i\big)^{\left(m-1-\frac{p+m+1}{\lambda}\right)+q_i(\alpha_i - A_{ii})} \prod\limits_{\substack{j=1\\ j\neq i}}^{m} \Gamma\big(q_i A_{ij}+1\big)}{\Gamma\Big(\dfrac{p+m+1}{\lambda}\Big)} \times \Gamma\Big(\frac{p+m+1}{\lambda} - m + 1 - q_i(\alpha_i - A_{ii})\Big),$$

such that ensures the asserted upper bound (3). $\square$

Choosing $A_{ij} = 0,\ i,j = 1,\cdots,m$ in the Theorem 1 we get the following result.

COROLLARY 1. *Let $m \geqslant 2$ be an integer, $\lambda,\ p_j,\ p_j';\ q_j > 0,\ j = 1,\cdots,m$ be the real numbers defined in* Theorem 1. *Then the following inequality holds*

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu}) \leqslant \mathsf{C}_1 \cdot \prod_{i=1}^{m} \left(\int_{\mathbb{R}_+} \big(r^2+m+x_i\big)^{\frac{p_i}{q_i}\left(m-1-\frac{m-p-1}{\lambda}\right)} f_i^{p_i}(x_i)\mathrm{d}x_i \right)^{1/p_i},$$

where

$$\mathsf{C}_1 = \frac{2(p+1)_m}{\left(\frac{p+m+1}{\lambda} - m + 1\right)_{m-1}^{\lambda}},$$

and functions $f_i(x_i)$ *are given by* (4).

Now, if we put $A_{ii} = A_i$, $A_{i,i+1} = -A_{i+1}$ while $A_{ij} = 0$, $|i-j| > 1$ and the indices are taken modulo m, then we obtain the next result.

COROLLARY 2. *Let* $m \geqslant 2$ *be an integer,* λ, p_j, p'_j; $q_j > 0$, $j = 1, \cdots, m$ *be the real numbers defined in the* Theorem 1. *Then, for all*

$$p + (1-\lambda)m > 2\lambda - 1, \qquad A_i \in \left(m - \frac{p+m+1}{\lambda} - \frac{1}{q_i}, \frac{1}{q_i}\right)$$

and $A_{m,m+1} = -A_1$, *the following inequality holds*

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu}) \leqslant \mathsf{C}_2 \prod_{i=1}^m \left(\int_{\mathbb{R}_+} x_i^{A_i p_i} \left(r^2 + m + x_i\right)^{\frac{p_i}{q_i}\left(m-1-\frac{m+p+1}{\lambda}\right) - p_i A_{i+1}} f_i^{p_i}(x_i)\,\mathrm{d}x_i \right)^{1/p_i},$$

where

$$\mathsf{C}_2 = \frac{2(p+1)_m}{\Gamma^{\lambda}\left(\frac{p+m+1}{\lambda}\right)} \prod_{i=1}^m \left(\Gamma\left(1 - q_i A_{i+1}\right) \Gamma\left(\frac{p+m+1}{\lambda} - m + 1 + q_i A_{i+1}\right) \right)^{1/q_i},$$

and functions $f_i(x_i)$ *remain the same as in* (4).

Finally, we give another variant of upper bounds choosing weight functions ϕ_{ij} to be exponential instead of the power weight function set such that results in Theorem 1.

THEOREM 2. *Let* $m \geqslant 2$, *and* λ, p_j, p'_j; $q_j > 0$, $j = 1, \cdots, m$ *satisfy the non–conjugated Hölder parameters constraints. Assume that* $A_{ij} \in \mathbb{R}$, $i, j = 1, \cdots, m$ *such that* $\sum_{i=1}^m A_{ij} = 0$ *and*

$$0 < q_1 A_{i1} < q_2 A_{i2} < \cdots < q_m A_{im}, \qquad i = 1, \cdots, m.$$

Then the following inequality holds true:

$$\mathscr{S}_p^m(r,\mathbf{s},\boldsymbol{\nu}) \leqslant 2(p+1)_m \cdot \prod_{k=1}^m \left(\int_{\mathbb{R}_+} \mathrm{e}^{-A_{kk} p_k x_k} F_k^{p_k}(x_k) f_k^{p_k}(x_k)\,\mathrm{d}x_k \right)^{1/p_k}, \tag{6}$$

where

$$F_k(x_k) = \left(\sum_{\substack{j=1 \\ j\neq k}}^m \frac{\left(q_j A_{kj}\right)^{\frac{p+m+1-1}{\lambda}} \mathrm{e}^{(r^2+m+x_k) q_j A_{kj}}}{\prod\limits_{\substack{n=1 \\ n\neq j,k}}^m (q_j A_{kj} - q_n A_{kn})} \Gamma\left(1 - \frac{p+m+1}{\lambda}, (r^2+m+x_k) q_j A_{kj}\right) \right)^{1/q_k},$$

and $f_k(x_k)$, $k = 1, \cdots, m$ *remain the same* (4).

Proof. Let us begin the proving procedure in the same manner as for Theorem 1, that is, considering integral expression (5). The kernel function remains also the same.

Now, define $\phi_{kj}(x_j) = \mathrm{e}^{-A_{kj}x_j}$, $k,j = 1,\cdots,m$ on $\Omega := \mathbb{R}_+$. The normalizing constraint of the Lemma 2 is obviously satisfied since $\sum_{i=1}^m A_{kj} = 0$, $j = 1,\cdots,m$. Therefore we have

$$\begin{aligned} F_k^{q_k}(x_k) &= \int_{\mathbb{R}_+^{m-1}} K(\boldsymbol{x}) \exp\Big\{-\sum_{\substack{j=1\\ j\neq k}}^m q_k A_{kj} x_j\Big\} \cdot \frac{\mathbf{d}\boldsymbol{x}}{\mathrm{d}x_k} \\ &= \frac{1}{\Gamma\Big(\frac{p+m+1}{\lambda}\Big)} \int_{\mathbb{R}_+} t^{\frac{p+m+1}{\lambda}-1} \mathrm{e}^{-(r^2+m+x_k)t} \left(\int_{\mathbb{R}_+^{m-1}} \mathrm{e}^{-\sum\limits_{\substack{j=1\\ j\neq k}}^m \big(t+q_kA_{kj}\big)x_j} \frac{\mathbf{d}\boldsymbol{x}}{\mathrm{d}x_k} \right) \mathrm{d}t \\ &= \frac{1}{\Gamma\Big(\frac{p+m+1}{\lambda}\Big)} \int_{\mathbb{R}_+} \frac{t^{\frac{p+m+1}{\lambda}-1}}{\prod\limits_{\substack{j=1\\ j\neq k}}^m (t+q_kA_{kj})} \mathrm{e}^{-(r^2+m+x_k)t}\, \mathrm{d}t\,. \end{aligned}$$

The partial fraction decomposition of the rational part of the integrand gives

$$F_k^{q_k}(x_k) = \frac{1}{\Gamma\Big(\frac{p+m+1}{\lambda}\Big)} \sum_{\substack{j=1\\ j\neq k}}^m \frac{\displaystyle\int_{\mathbb{R}_+} \frac{t^{\frac{p+m+1}{\lambda}-1}}{t+q_jA_{kj}} \mathrm{e}^{-(r^2+m+x_k)t}\, \mathrm{d}t}{\prod\limits_{\substack{n=1\\ n\neq j,k}}^m (q_nA_{kn} - q_jA_{kj})}\,.$$

Employing the formula [5, **3.383** 10.]

$$\begin{aligned} &\int_{\mathbb{R}_+} \frac{x^{\nu-1}\mathrm{e}^{-\mu x}}{x+\beta}\,\mathrm{d}x = \beta^{\nu-1}\mathrm{e}^{\beta\mu}\,\Gamma(\nu)\Gamma(1-\nu,\beta\mu), \\ &|\arg(\beta)| < \pi, \quad \min\big(\Re\{\mu\},\Re\{\nu\}\big) > 0\,, \end{aligned}$$

where $\Gamma(s,z) = \int_z^\infty x^{s-1}\mathrm{e}^{-sx}\,\mathrm{d}x$ is the familiar upper incomplete Gamma function, we deduce

$$F_k^{q_k}(x_k) = \sum_{\substack{j=1\\ j\neq k}}^m \frac{(q_jA_{kj})^{\frac{p+m+1}{\lambda}-1}\,\mathrm{e}^{(r^2+m+x_k)q_jA_{kj}}}{\prod\limits_{\substack{n=1\\ n\neq j,k}}^m (q_nA_{kn} - q_jA_{kj})}\,\Gamma\Big(1-\frac{p+m+1}{\lambda}, (r^2+m+x_k)q_jA_{kj}\Big)\,.$$

By virtue of Lemma 2, taking $\mu_k, k = 1,\cdots,m$ to be ordinary Lebesgue measures, we immediately arrive at the asserted inequality (6). $\square$

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