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FREQUENCY VARIANT OF EULER TYPE IDENTITIES AND THE PROBLEM OF SIGN–CONSTANCY OF THE KERNEL IN ASSOCIATED QUADRATURE FORMULAS

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Abstract. In the recent years many authors used extended Euler identities to obtain generalizations of some classical quadrature formulas with the best possible error estimates. The main step in obtaining the best possible error estimates was to control zeros of the kernel in the error term which consists of the affine combinations of the translates of periodic Bernoulli polynomials. This was done for some low degrees of Bernoulli polynomials. The main goal of this paper is to consider a general case. The frequency variant of extended Euler identities is found to be more tractable for this problem.

1. Introduction

Extended Euler identities obtained in [1] generalize the well known formula for the expansion of a function in Bernoulli polynomials (see for example [8]). Namely, for $n \in \mathbb{N}$ and $f : [0,1] \to \mathbb{R}$ such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$, the following identities hold for every $x \in [0,1]$:

$$\int_0^1 f(t)dt = f(x) - T_n(x) + \frac{1}{n!}\int_0^1 B_n^*(x-t)\,df^{(n-1)}(t), \tag{1}$$

$$\int_0^1 f(t)dt = f(x) - T_{n-1}(x) + \frac{1}{n!}\int_0^1 \left[B_n^*(x-t) - B_n^*(x)\right]df^{(n-1)}(t), \tag{2}$$

where $T_0(x) = 0$ and for $1 \leqslant m \leqslant n$

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right],$$

where $B_k(x)$ are Bernoulli polynomials and $B_k^*(x) = B_k(x - [x])$.

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In the recent years many authors (see for example [1], [3], [4], [5], [9] and references therein) used (1) and (2) to obtain generalizations of some classical quadrature formulas with the best possible error estimates.

The procedure of deducing the quadrature formulas can be summarized as follows. Using symmetric (with respect to $1/2$) nodes $0 \leqslant x_1 < x_2 < \cdots x_k \leqslant 1/2 \leqslant x_{k+1} < \cdots < x_{2k} \leqslant 1$ and affine combinations of (1) it follows

$$\int_0^1 f(t)dt = \sum_{i=1}^{2k} \lambda_i f(x_i) - \widetilde{T}_n + \frac{1}{n!} \int_0^1 \left(\sum_{i=1}^{2k} \lambda_i B_n^*(x_i - t) \right) df^{(n-1)}(t), \tag{3}$$

where $\widetilde{T}_n = \sum_{i=1}^n \frac{\widetilde{B}_i}{i!} \left[f^{(i-1)}(1) - f^{(2i-1)}(0) \right]$, $\widetilde{B}_i = \sum_{j=1}^{2k} \lambda_j B_i(x_j)$, $\sum_{i=1}^{2k} \lambda_i = 1$ and $\lambda_j = \lambda_{2k+1-j}$, $j = 1, \ldots, k$. Notice that chosen symmetry implies $\widetilde{B}_{2i-1} = 0$, $i \in \mathbb{N}$, consequently (3) is usually written as

$$\int_0^1 f(t)dt = \sum_{i=1}^{2k} \lambda_i f(x_i) - \widetilde{T}_{2n} + \frac{1}{(2n)!} \int_0^1 G_{2n+1}(t) df^{(2n)}(t), \tag{4}$$

$$\int_0^1 f(t)dt = \sum_{i=1}^{2k} \lambda_i f(x_i) - \widetilde{T}_{2n} + \frac{1}{(2n+2)!} \int_0^1 F_{2n+2}(t) df^{(2n+1)}(t), \tag{5}$$

where $\widetilde{T}_{2n} = \sum_{i=1}^n \frac{\widetilde{B}_{2i}}{(2i)!} \left[f^{(2i-1)}(1) - f^{(2i-1)}(0) \right]$, $G_{2n+1}(t) = \sum_i^{2k} \lambda_i B_{2n+1}^*(x_i - t)$, $F_{2n+2}(t) = \sum_{i=1}^{2k} \lambda_i \left[B_{2n+2}^*(x_i - t) - B_{2n+2}^*(x_i) \right]$.

To produce quadrature formulas for preassigned nodes the following conditions are usually imposed:

$$\widetilde{B}_{2n} = \widetilde{B}_{2n-2} = \cdots = \widetilde{B}_{2(n-k+2)} = 0, \; n \geqslant k-1. \tag{6}$$

Unperturbed (uncorrected) quadrature formulas are obtained for $n = k-1$, i.e. formulas which do not involve derivatives at boundary points. Notice that (6) is equivalent to

$$G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2k-3)}(0) = 0. \tag{7}$$

The main step in obtaining the best possible error estimates is to prove that

$$G_{2n+1}(t) = \sum_{i=1}^{2k} \lambda_i B_{2n+1}^*(x_i - t)$$

has some "nice" zeros in $(0, 1/2)$ (usually G_{2n+1} has no zeros at all in $(0, 1/2)$). We formulate the following problem which seems to be interesting independently of the present context.

PROBLEM 1. *Find the distribution of nodes* $0 \leqslant x_1 < x_2 < \cdots < x_k \leqslant 1/2$, *such that* $G_{2n+1}(t) = \sum_{i=1}^{2k} \lambda_i B_{2n+1}^*(x_i - t)$ *has no zeros in* $(0, 1/2)$, *if* $\sum_{i=1}^{2k} \lambda_i = 1$, $x_{2k-j+1} = 1 - x_k$, $j = 1, \ldots, k$, $G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2k-3)}(0) = 0$, *where* $n \geqslant k-1$.

Some partial results can be found in [1], [3], [4], [5], [9], where nodes and weights are explicitly calculated or a priori given, thus allowing explicit expression of G_{2n+1} for some small n. An exception is [5], where some elementary motivations for the present chapter can be found.

To prove some special cases of Problem 1 (but of a general nature as stated above), we found the "frequency" variant of identities (1) and (2) more tractable. An easy consequence of Multiplication Theorem for periodic Bernoulli functions B_n^* in the form

$$B_n^*(x-mt) = m^{n-1}\sum_{k=0}^{m-1} B_n^*\left(\frac{x+k}{m}-t\right),\ n \geqslant 0,\ m \geqslant 1,$$

is the following theorem (see [4]):

THEOREM 1. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geqslant 1$. Then, for $x\in[0,1]$ and $m\in\mathbb{N}$, we have*

$$\int_0^1 f(t)dt = \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_n(x) + \frac{1}{n!\cdot m^n}\int_0^1 B_n^*(x-mt)\,df^{(n-1)}(t), \qquad (8)$$

where

$$T_n(x) = \sum_{j=1}^{n}\frac{B_j(x)}{j!\cdot m^j}\left[f^{(j-1)}(1)-f^{(j-1)}(0)\right].$$

Setting $x=0$ in (8) and using that $B_1=-1/2$, $B_{2i-1}=0$, $i\geqslant 2$, we write (8), with appropriate assumptions, in a more convenient form:

$$\begin{aligned}\int_0^1 f(t)dt &= \frac{1}{m}\frac{f(0)+f(1)}{2} + \frac{1}{m}\sum_{i=1}^{m-1} f\left(\frac{i}{m}\right)\\ &\quad + \sum_{i=1}^{n}\frac{B_{2i}}{(2i)!m^{2i}}\left[f^{(2i-1)}(1)-f^{(2i-1)}(0)\right]\\ &\quad - \frac{1}{(2n+1)!m^{2n+1}}\int_0^1 B_{2n+1}^*(mt)\,df^{(2n)}(t). \qquad (9)\end{aligned}$$

Affine combinations of (9) with frequencies $m_0 = 1 < m_1 < \cdots < m_s$, $m_i\in\mathbb{N}$, $s\in\mathbb{N}$, and weights $\lambda_0,\ldots,\lambda_s$, $\sum_{i=0}^{s}\lambda_i = 1$ give:

$$\begin{aligned}\int_0^1 f(t)dt &= \frac{f(0)+f(1)}{2}\sum_{j=0}^{s}\frac{\lambda_j}{m_j} + \sum_{j=0}^{s}\frac{\lambda_j}{m_j}\sum_{i=1}^{m_j-1} f\left(\frac{i}{m_j}\right)\\ &\quad + \sum_{i=1}^{n}\frac{B_{2i}}{(2i)!}\left[f^{(2i-1)}(1)-f^{(2i-1)}(0)\right]\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2i}}\\ &\quad - \frac{1}{(2n+1)!}\int_0^1\left(\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n+1}}B_{2n+1}^*(m_jt)\right)df^{(2n)}(t). \qquad (10)\end{aligned}$$

Analogously,

$$\int_0^1 f(t)dt = \frac{f(0)+f(1)}{2}\sum_{j=0}^{s}\frac{\lambda_j}{m_j} + \sum_{j=0}^{s}\frac{\lambda_j}{m_j}\sum_{i=1}^{m_j-1} f\left(\frac{i}{m_j}\right)$$
$$+\sum_{i=1}^{n}\frac{B_{2i}}{(2i)!}\left[f^{(2i-1)}(1)-f^{(2i-1)}(0)\right]\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2i}}$$
$$-\frac{1}{(2n+2)!}\int_0^1\left(\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n+2}}\left(B_{2n+2}^*(m_jt)-B_{2n+2}\right)\right)df^{(2n+1)}(t). \tag{11}$$

It is clear that identities (10) and (11) can be written in the form of identities (4) and (5), respectively. Also, it is easy to see that there are identities of the type (4) and (5) which cannot be of a type (10) and (11), respectively.

Again, as in (6) and (7), to produce quadrature formulae it is natural to impose the following conditions:

$$\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n}} = \sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2(n-1)}} = \cdots = \sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2(n-s+1)}} = 0,\ n \geqslant s, \tag{12}$$

or equivalently:

$$G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2s-1)}(0) = 0, \tag{13}$$

where

$$G_{2n+1}(t) = \sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n+1}}B_{2n+1}^*(m_jt). \tag{14}$$

Now, we can state the following special case of Problem 1.

PROBLEM 2. *Find the distribution of frequencies* $m_0 = 1 < m_1 < m_2 < \cdots < m_s$, $m_i \in \mathbf{N}$, *such that* $G_{2n+1}(t) = \sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n+1}}B_{2n+1}^*(m_jt)$ *has no zeros in* $(0,1/2)$, *if* $\sum_{j=0}^{s}\lambda_j = 1$, $G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2s-1)}(0) = 0$, *where* $n \geqslant s$.

2. Some preliminary considerations

To obtain quadrature formulas based on identities (10) and (11), we determine weights $\lambda_0, \lambda_1, \ldots, \lambda_s$ from the linear system

$$M\lambda = b, \tag{15}$$

where

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{m_1^{2n}} & \cdots & \frac{1}{m_s^{2n}} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \frac{1}{m_1^{2(n-s+1)}} & \cdots & \frac{1}{m_s^{2(n-s+1)}} \end{pmatrix}, \tag{16}$$

$\lambda = (\lambda_0 \ \lambda_1 \ \cdots \ \lambda_s)^T$ and $b = (1 \ 0 \ \cdots \ 0)^T$. It is easy to see that $\operatorname{Det} M \neq 0$ (see also [11]), so the system (15) has a unique solution. Cramer's rule and (14) immediately imply:

$$G_{2n+1}(t) = \sum_{j=0}^{s} \frac{\lambda_j}{m_j^{2n+1}} B^*_{2n+1}(m_j t)$$
$$= \frac{1}{\operatorname{Det} M} \begin{vmatrix} B^*_{2n+1}(t) & \frac{B^*_{2n+1}(m_1 t)}{m_1^{2n+1}} & \cdots & \frac{B^*_{2n+1}(m_s t)}{m_s^{2n+1}} \\ 1 & \frac{1}{m_1^{2n}} & \cdots & \frac{1}{m_s^{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{1}{m_1^{2(n-s+1)}} & \cdots & \frac{1}{m_s^{2(n-s+1)}} \end{vmatrix}, \tag{17}$$

which gives

$$G_{2n+1}(t) = \frac{(-1)^s}{(m_1 \cdots m_s)^{2n+1} \operatorname{Det} M} \begin{vmatrix} 1 & 1 & \cdots & 1 & B^*_{2n+1}(t) \\ m_1 & m_1^3 & \cdots & m_1^{2s-1} & B^*_{2n+1}(m_1 t) \\ m_2 & m_2^3 & \cdots & m_2^{2s-1} & B^*_{2n+1}(m_2 t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_s & m_s^3 & \cdots & m_s^{2s-1} & B^*_{2n+1}(m_s t) \end{vmatrix}. \tag{18}$$

Define

$$H_{2n+1}(t) = \begin{vmatrix} 1 & 1 & \cdots & 1 & B^*_{2n+1}(t) \\ m_1 & m_1^3 & \cdots & m_1^{2s-1} & B^*_{2n+1}(m_1 t) \\ m_2 & m_2^3 & \cdots & m_2^{2s-1} & B^*_{2n+1}(m_2 t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_s & m_s^3 & \cdots & m_s^{2s-1} & B^*_{2n+1}(m_s t) \end{vmatrix}. \tag{19}$$

In this way Problem 2 is equivalent to the following problem.

PROBLEM 3. *Find the distribution of frequencies such that $H_{2n+1}(t)$ has no zeros in $(0,1/2)$ for $n \geqslant s$.*

EXAMPLE 1. Suppose that $m_0 = 1 < m_1 = 3 < m_2 = 4$, $n = s = 2$. Using Wolfram's Mathematica, for $H_5(t) = \begin{vmatrix} 1 & 1 & B_5^*(t) \\ 3 & 3^3 & B_5^*(3t) \\ 4 & 4^3 & B_5(4t) \end{vmatrix}$, $H_5(0.45) = 1.11285$ and $H_5(0.3) = -3.3996$, so H_5 has zeros in $(0,1/2)$.

For a given sequence of functions $a_0, \ldots, a_n$ defined on some real interval I and given sequence $x_0, \ldots, x_n$ in I, we introduce

$$D\begin{pmatrix} a_0 & \cdots & a_{n-1} & a_n \\ x_0 & \cdots & x_{n-1} & x_n \end{pmatrix} = \begin{vmatrix} a_0(x_0) & a_1(x_0) & \cdots & a_n(x_0) \\ a_0(x_1) & a_1(x_1) & \cdots & a_n(x_1) \\ \vdots & \vdots & \cdots & \vdots \\ a_0(x_n) & a_1(x_n) & \cdots & a_n(x_n) \end{vmatrix},$$

and, if $a_0,\dots,a_n$ are sufficiently smooth, we denote by $W(a_0,\dots,a_n)(x)$ the Wronskian of the sequence $a_0,\dots,a_n$ at $x\in I$.

To transform the functions H_{2n+1} in a more suitable form, the following General Mean Value theorem from [6] appears to be useful.

THEOREM 2. *Let $a_0,\dots,a_n$ be a sequence of real functions of a real variable x, possessing derivatives up to the order n, and further such that the Wronskians $W(a_0,\dots,a_k)$, $k=0,1,\dots,n$, do not vanish on a certain interval I. Let $f(x)$ be a function possessing derivatives up to the order n in I. Finally let $x_0,x_1,\dots,x_n$ be a system of $(n+1)$ values of x in I. There exists at least one value ξ in I such that*

$$\frac{D\begin{pmatrix} a_0 & \cdots & a_{n-1} & f\\ x_0 & \cdots & x_{n-1} & x_n\end{pmatrix}}{D\begin{pmatrix} a_0 & \cdots & a_{n-1} & a_n\\ x_0 & \cdots & x_{n-1} & x_n\end{pmatrix}}=\frac{W\left(a_0,\dots,a_{n-1},f\right)(\xi)}{W\left(a_0,\dots,a_{n-1},a_n\right)(\xi)}. \tag{20}$$

To apply Theorem 2 we set:

$$a_0(x)=x, a_1(x)=x^3,\dots,a_s(x)=x^{2s+1},$$

$$f(x)=B^*_{2n+1}(xt)=g(xt),$$

$$x_0=1, x_1=m_1,\dots,x_s=m_s,\ I=[1,m_s].$$

Assumptions of Theorem 2 are obviously satisfied, so there is an $\xi\in[1,m_s]$ such that

$$\begin{aligned} H_{2n+1}(t)&=D\begin{pmatrix} x & x^3 & \cdots & x^{2s-1} & g(xt)\\ 1 & m_1 & \cdots & m_{s-1} & m_s\end{pmatrix}\\ &=\frac{D\begin{pmatrix} x & x^3 & \cdots & x^{2s-1} & x^{2s+1}\\ 1 & m_1 & \cdots & m_{s-1} & m_s\end{pmatrix}}{W\left(x,x^3,\dots,x^{2s+1}\right)(\xi)}\\ &\cdot\begin{vmatrix} \xi & \xi^3 & \cdots & \xi^{2s-1} & g(\xi t)\\ 1 & 3\xi^2 & \cdots & (2s-1)\xi^{2s-2} & tg'(\xi t)\\ 0 & 3!\xi & \cdots & (2s-1)(2s-2)\xi^{2s-3} & t^2g''(\xi t)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & \cdots & \cdots & \frac{(2s-1)!}{(s-1)!}\xi^{s-1} & t^sg^{(s)}(\xi t)\end{vmatrix}\end{aligned}$$

Denote the last determinant in (21) by $\widetilde{H}_{2n+1}(t,\xi)$. Multiplying the kth row of this determinant by ξ^{k-1}, $k=2,\dots,s$, then extracting from the lth column ξ^{2l-1}, $l=1,\dots,s$, we have

$$\widetilde{H}_{2n+1}(t,\xi) = \xi^{\frac{s(s+1)}{2}} \begin{vmatrix} 1 & 1 & \cdots & 1 & f(\xi t) \\ 1 & 3 & \cdots & 2s-1 & \xi t f'(\xi t) \\ 0 & 3! & \cdots & (2s-1)(2s-2) & (\xi t)^2 f''(\xi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \frac{(2s-1)!}{(s-1)!} & (\xi t)^s f^{(s)}(\xi t) \end{vmatrix}$$

$$= \xi^{\frac{s(s+1)}{2}} \begin{vmatrix} a_0(1) & a_1(1) & \cdots & a_{s-1}(1) & f(u) \\ a_0'(1) & a_1'(1) & \cdots & a_{s-1}'(1) & uf'(u) \\ a_0''(1) & a_1''(1) & \cdots & a_{s-1}''(1) & u^2 f''(u) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_0^{(s)}(1) & a_1^{(s)}(1) & \cdots & a_{s-1}^{(s)}(1) & u^s f^{(s)}(u) \end{vmatrix}, \tag{21}$$

where $u = \xi t$. Note that $0 < u < m_s/2$ for $0 < u < 1/2$. It remains to investigate the sign of the function given by the last determinant in (21). Using the Laplace expansion of determinants, this function is up to the sign equal to the function

$$F(u) = \sum_{j=0}^{s} (-i)^j \mathrm{Ch}^{(j)} u^j f^{(j)}(u), \tag{22}$$

where $\mathrm{Ch}^{(j)}$ (Ch stands for Chebyshev) is the determinant of the matrix obtained from the $(s+1) \times s$ matrix

$$\mathrm{Ch} = \begin{pmatrix} a_0(1) & a_1(1) & \cdots & a_{s-1}(1) \\ a_0'(1) & a_1'(1) & \cdots & a_{s-1}'(1) \\ \vdots & \vdots & \cdots & \vdots \\ a_0^{(s)}(1) & a_1^{(s)}(1) & \cdots & a_{s-1}^{(s)}(1) \end{pmatrix}$$

by deleting the $(j+1)$th row. The sequence of functions $a_0, a_1, \ldots, a_s$ can be obtained by using the universal construction of Chebyshev systems from [7]. Take $\omega_0(x) = x$, $\omega_1(x) = 2x, \ldots,$ $\omega_s(x) = 2sx$. Then

$$a_0(x) = \omega_0(x),\ a_1(x) = \omega_0(x) \int_0^x \omega_1(t_1)\,dt_1,$$

$$a_2(x) = \omega_0(x) \int_0^x \omega_1(t_1) \int_0^{t_1} \omega_2(t_2)\,dt_2\,dt_1, \ldots,$$

$$a_s(x) = \omega_0(x) \int_0^x \omega_1(t_1) \int_0^{t_1} \omega_2(t_2) \cdots \int_0^{t_{s-1}} \omega_s(t_s)\,dt_s \cdots dt_2\,dt_1.$$

Using this and properties of determinants (manipulating with columns of $\mathrm{Ch}^{(j)}$), straightforward calculation reveals that

$$\mathrm{Ch}^{(j)} = \mathrm{Ch}^{(s)} \cdot \frac{(2s-j)!}{j!(2s-2j)!!} = 2^{s-1} \cdot 4^{s-2} \cdots (2s-4)^2 \cdot (2s-2) \cdot \frac{(2s-j)!}{j!(2s-2j)!!}$$

(for the case $j = s$ see [7]).

LEMMA 1. *Let* $(F_k)_{k=0}^s$ *be the sequence of functions defined by:*

$$F_k(u) = \sum_{j=k}^{s-1}(-1)^j \mathrm{Ch}_k^{(j)} u^{j-k} f^{(j+k)}(u) + (-1)^s \mathrm{Ch}^{(s)} u^{s-k} f^{(s+k)}(u),\ u \in \mathbb{R}, \tag{23}$$

where

$$\mathrm{Ch}_k^{(j)} = \frac{j!}{(j-k)!}\frac{(2s-j-k)!}{(2s-j)!}\mathrm{Ch}^{(j)},\ j = k,\dots,s. \tag{24}$$

Then

1. $F_0(u) = F(u)$ *where* $F(u)$ *is given by (22)*
2. $F_s(u) = (-1)^s \mathrm{Ch}^{(s)} f^{(2s)}(u)$,
3. $F_k'(u) = uF_{k+1}(u)$, $k = 0,\dots s-1$.

Proof. The first two properties are obvious. Let us prove the third property. Simple rearranging gives:

$$\begin{aligned} F_k'(u) = {} & (-1)^k\left[\mathrm{Ch}_k^{(k)} - \mathrm{Ch}_k^{(k+1)}\right] f^{(2k+1)}(u) \\ & + \sum_{j=k+1}^{s-1}(-1)^j\left[\mathrm{Ch}_k^{(j)} - (j+1-k)\mathrm{Ch}_k^{(j+1)}\right] u^{j-k} f^{(j+k+1)}(u) \\ & + (-1)^s \mathrm{Ch}^{(s)} u^{s-k} f^{(s+k+1)}(u). \end{aligned} \tag{25}$$

It is obvious that $\mathrm{Ch}_k^{(k)} = \mathrm{Ch}_k^{(k+1)} = (2s-2k-1)!!$. It remains to show that $\mathrm{Ch}_k^{(j)} - (j+1-k)\mathrm{Ch}_k^{(j+1)} = \mathrm{Ch}_{k+1}^{(j)}$. Using (24) and that $\mathrm{Ch}^{(j+1)} = \frac{2s-2j}{(j+1)(2s-j)}\mathrm{Ch}^{(j)}$ we have:

$$\begin{aligned} & \mathrm{Ch}_k^{(j)} - (j+1-k)\mathrm{Ch}_k^{(j+1)} \\ & = \frac{j!}{(j-k)!}\frac{(2s-j-k)!}{(2s-j)!}\mathrm{Ch}^{(j)} - \frac{(j+1)!}{(j-k)!}\frac{(2s-j-k-1)!}{(2s-j-1)!}\mathrm{Ch}^{(j+1)} \\ & = \mathrm{Ch}^{(j)}\left[\frac{j!}{(j-k)!}\frac{(2s-j-k)!}{(2s-j)!} - \frac{(j+1)!}{(j-k)!}\frac{(2s-j-k-1)!}{(2s-j-1)!}\frac{2s-2j}{(j+1)(2s-j)}\right] \\ & = \mathrm{Ch}^{(j)}\frac{j!}{(j-k-1)!}\frac{(2s-j-k-1)!}{(2s-j)!} = \mathrm{Ch}_{k+1}^{(j)}. \end{aligned}$$

We can write

$$\begin{aligned} F_k'(u) & = u\left[\sum_{j=k+1}^{s-1}(-1)^j \mathrm{Ch}_{k+1}^{(j)} u^{j-k-1} f^{(j+k+1)}(u) + (-1)^s \mathrm{Ch}^{(s)} u^{s-k-1} f^{(s+k+1)}(u)\right] \\ & = uF_{k+1}(u). \qquad \square \end{aligned}$$

THEOREM 3. *Suppose that* $m_0 = 1 < m_1 < m_2 < \cdots < m_s$, $m_i \in \mathbb{N}$ *and* $\sum_{j=0}^s \lambda_j = 1$, $\lambda_j \in \mathbb{R}$. *Then the function* $G_{2n+1}(t) = \sum_{j=0}^s \frac{\lambda_j}{m_j^{2n+1}} B_{2n+1}^*(m_j t)$ *such that* $G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2s-1)}(0) = 0$, $s \leqslant n$, *has no zeros in* $\left(0, \frac{1}{2m_s}\right]$.

Proof. Suppose that $0 < t \leqslant \frac{1}{2m_s}$. Then $0 < u = \xi t \leqslant \frac{1}{2}$ (see the discussion and notation below Theorem 2). The claim follows from previous reductions and because Lemma 25 implies

$$F(u) = F_0(u) = \int_0^u t_1 \int_0^{t_1} t_2 \cdots t_{s-1} \int_0^{t_{s-1}} t_s (-1)^s \mathrm{Ch}^{(s)} B^*_{2n+1-2s}(t_s) dt_s \cdots dt_2 dt_1. \quad (26)$$

□

REMARK 1. Notice that $F_k(u) = (-1)^k \mathrm{Ch}_k^{(k)} f^{(2k)}(u) + u \cdot [\cdots]$. Consequently $F_k(0) = 0$ for functions for which $f^{(2k)}(0) = 0$, $k = 0, \ldots, s$. From Lemma 25 follows:

$$F(u) = F_0(u) = \int_0^u t_1 \int_0^{t_1} t_2 \cdots \int_0^{t_{s-1}} t_s (-1)^s \mathrm{Ch}^{(s)} f^{(2s)}(t_s) dt_s \cdots dt_2 dt_1. \quad (27)$$

3. Case $m_i = m^i$

In the previous section we proved that the function $G_{2n+1}(t)$, defined in (14) such that conditions (13) hold and $\sum_{j=0}^{s} \lambda_j \neq 0$, has no zeros on $(0, \frac{1}{2m_s}]$.

In the present section we give the complete answer for the case $m_i = m^i$, $i = 0, \ldots s$, $m \geqslant 2$, $m \in \mathbb{N}$, in the sense that we prove that the function H_{2n+1} defined in (19), using frequencies $1, m, m^2, \ldots, m^s$, has no zeros on $(0, 1/2)$.

THEOREM 4. *Let $s \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geqslant 2$. Then the function*

$$K_s(t; f) = \begin{vmatrix} 1 & 1 & \cdots & 1 & f(t) \\ m & m^3 & \cdots & m^{2s-1} & f(mt) \\ m^2 & \left(m^2\right)^3 & \cdots & \left(m^2\right)^{2s-1} & f\left(m^2 t\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m^s & (m^s)^3 & \cdots & (m^s)^{2s-1} & f(m^s t) \end{vmatrix} \quad (28)$$

has no zeros on $(0, 1/2)$, for any odd function $f : \mathbb{R} \to \mathbb{R}$ which is periodic with period $T = 1$, such that $f^{(2s-2)}$ is continuous on $\mathbb{R}$ and strictly concave (convex) on $(0, 1/2)$.

Proof. The proof is by induction. Suppose that f is continuous on $\mathbb{R}$ and strictly concave on $(0, 1/2)$. We shall prove that $K_1(t; f) = f(mt) - mf(t)$ is strictly negative for $t \in (0, 1/2)$. Using strict concavity $f(mt) < mf(t)$ for $0 < t \leqslant \frac{1}{2m}$. Now, we split the proof into two cases.

Case $m = 2k+1$: Suppose that $\frac{1}{2} - \frac{1}{2m} \leqslant t < \frac{1}{2}$. Set $g(x) = f(\frac{1}{2} - x)$. Obviously g is strictly concave on $(0, 1/2)$ and $g(0) = 0$. This implies $g(mx) < mg(x)$ for $x = \frac{1}{2} - t$, which gives $f(-k + mt) = f(mt) < mf(t)$. In this way we conclude that $f(mt) < mf(t)$ for $t \in (0, \frac{1}{2m}] \cup [\frac{1}{2} - \frac{1}{2m}, \frac{1}{2})$.

Set $M = \max_{t \in [0, 1/2]} f(t)$. There is $t_1 \in (0, \frac{1}{2m})$ such that $f(mt_1) = M$, and there is $t_2 \in \left(\frac{1}{2} - \frac{1}{2m}, \frac{1}{2}\right)$ such that $f(mt_2) = M$. Suppose that $t \in \left(\frac{1}{2m}, \frac{1}{2} - \frac{1}{2m}\right)$ is arbitrary. Then there is $\lambda \in (0, 1)$ such that $t = \lambda t_1 + (1 - \lambda) t_2$. Finally:

$$f(mt) \leqslant M = \lambda f(mt_1) + (1 - \lambda) f(mt_2) < f\left(m(\lambda t_1 + (1 - \lambda) t_2)\right) = f(mt).$$

Case $m=2k$: Notice that $f(mt)<0$ for $\frac{1}{2}-\frac{1}{2m}<t<\frac{1}{2}$. Arguing as in the final step of the proof for the case $m=2k+1$, it is enough to prove that $f\left(m\left(t-\frac{1}{2m}\right)\right)<mf(t)$ for $\frac{1}{2}-\frac{1}{2m}\leqslant t<\frac{1}{2}$. Set again $g(x)=f(\frac{1}{2}-x)$. Obviously g is strictly concave on $(0,1/2)$ and $g(0)=0$, so $g(mx)<mg(x)$ for $0<x\leqslant\frac{1}{2m}$. This implies for $x=\frac{1}{2}-t$ that $g(\frac{m}{2}-mt)<mg(\frac{1}{2}-t)$, so $f(mt+\frac{1}{2}-k)=f(mt-\frac{1}{2})<mf(t)$.

The proof when f is convex is analogous.

To prove the inductive step we use the Sylvester identity for determinants with the first and the last row and with the two last columns. It follows (where we denote by $\mathrm{V}[\alpha_1,\ldots,\alpha_n]$ the Vandermonde determinant):

$$
\begin{aligned}
K_s(t;f)\cdot&\begin{vmatrix} m & m^3 & \cdots & m^{2s-3}\\ m^2 & (m^2)^3 & \cdots & (m^2)^{2s-3}\\ \vdots & \vdots & \cdots & \vdots\\ m^{s-1} & \left(m^{s-1}\right)^3 & \cdots & \left(m^{s-1}\right)^{2s-3}\end{vmatrix}\\
&=\begin{vmatrix}
\begin{vmatrix} 1 & 1 \cdots 1\\ m & \cdots m^{2s-1}\\ \vdots & \vdots \cdots \vdots\\ m^{s-1} & \cdots \left(m^{s-1}\right)^{2s-1}\end{vmatrix} &
\begin{vmatrix} 1 & 1 & \cdots 1 & f(t)\\ m & m^3 & \cdots m^{2s-3} & f(mt)\\ \vdots & \vdots & \cdots \vdots & \vdots\\ m^{s-1} & \left(m^{s-1}\right)^3 & \cdots \left(m^{s-1}\right)^{2s-3} & f(m^{s-1}t)\end{vmatrix}\\
\begin{vmatrix} m & \cdots m^{2s-1}\\ m^2 & \cdots (m^2)^{2s-1}\\ \vdots & \vdots \cdots \vdots\\ m^s & \cdots (m^s)^{2s-1}\end{vmatrix} &
\begin{vmatrix} m & m^3 & \cdots m^{2s-3} & f(mt)\\ m^2 & (m^2)^3 & \cdots (m^2)^{2s-3} & f(m^2t)\\ \vdots & \vdots & \cdots \vdots & \vdots\\ m^s & (m^s)^3 & \cdots (m^s)^{2s-3} & f(m^st)\end{vmatrix}
\end{vmatrix}\\
&=\begin{vmatrix} m\cdot m^2\cdots m^{s-1}\mathrm{V}[1,m^2,\ldots,m^{2(s-1)}] & K_{s-1}(t;f)\\ m\cdot m^2\cdots m^s\mathrm{V}[m^2,m^4,\ldots m^{2s}] & m\cdot m^3\cdots m^{2s-3}K_{s-1}(mt;f)\end{vmatrix}\\
&=\begin{vmatrix} m^{\frac{(s-1)s}{2}}\mathrm{V}[1,m^2,\ldots,m^{2(s-1)}] & K_{s-1}(t;f)\\ m^{\frac{s(s+1)}{2}}\cdot m^{(s-1)s}\mathrm{V}\left[1,m^2,\ldots,m^{2(s-1)}\right] & m^{(s-1)^2}K_{s-1}(mt;f)\end{vmatrix}\\
&=m^{\frac{(s-1)(3s-2)}{2}}\mathrm{V}\left[1,m^2,\ldots,m^{2(s-1)}\right]\begin{vmatrix} 1 & K_{s-1}(t;f)\\ m^{2s-1} & K_{s-1}(mt;f)\end{vmatrix},
\end{aligned}
$$

which gives

$$
\begin{aligned}
K_s(t;f)&=m^{\frac{(s-1)(s+2)}{2}}\frac{\mathrm{V}\left[1,m^2,\ldots,m^{2(s-1)}\right]}{\mathrm{V}\left[1,m^2,\ldots,m^{2(s-2)}\right]}\begin{vmatrix} 1 & K_{s-1}(t;f)\\ m^{2s-1} & K_{s-1}(mt;f)\end{vmatrix}\\
&=C\begin{vmatrix} 1 & 1 & \cdots & 1 & f(mt)-m^{2s-1}f(t)\\ m & m^3 & \cdots & m^{2s-3} & f(m^2t)-m^{2s-1}f(mt)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ m^{s-1} & \left(m^{s-1}\right)^3 & \cdots & \left(m^{s-1}\right)^{2s-3} & f(m^st)-m^{2s-1}f(m^{s-1}t)\end{vmatrix}\\
&=CK_{s-1}(t;g),
\end{aligned}
\tag{29}
$$

where $g(t) = f(mt) - m^{2s-1} f(t)$. To use the inductive assumption we only have to prove that $g^{(2s-4)}$ is strictly concave or strictly convex on $(0,1/2)$. We have

$$g^{(2s-4)}(t) = m^{2s-4}\left[f^{(2s-4)}(mt) - m^3 f^{(2s-4)}(t)\right].$$

Set $h(t) = f^{(2s-4)}(mt) - m^3 f^{(2s-4)}(t)$. Since $h''(t) = m^2\left[f^{(2s-2)}(mt) - mf^{(2s-2)}(t)\right]$, using assumption ($f^{(2s-2)}$ is strictly concave or strictly convex on $(0,1/2)$) and the basis of induction, we conclude that h'' has no zeros on $(0,1/2)$. Since $f^{(2s-2)}$ is continuous, h'' has constant sign on $(0,1/2)$. It follows that h, and consequently $g^{(2s-4)}$, is strictly concave or strictly convex on $(0,1/2)$. Using inductive assumption, $K_{s-1}(t;g)$ has no zeros on $(0,1/2)$, so by (29), $K_s(t;f)$ has no zeros on $(0,1/2)$. The proof is complete. □

Obvious examples of the functions which satisfy conditions in the previous theorem are $f(t) = B^*_{2n+1}(t)$ and $f(t) = \sin 2\pi t$.

EXAMPLE 2. The Boole and Simpson formula can be easily deduced using above procedure.

4. Using the Fourier expansion of the periodic Bernoulli functions

In this section we present yet another method to study zeros of the functions defined as the function H_{2n+1}. This method is motivated by the Fourier expansion of the periodic Bernoulli functions given by

$$B^*_{2n+1}(t) = \frac{(-1)^{n+1}(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi t)}{k^{2n+1}},\ n \geqslant 1, x \in \mathbb{R},\ n = 0,\ x \neq k. \tag{30}$$

Recall that we reduced the problem of zeros of the function G_{2n+1} to the one of the function H_{2n+1}. Using (30) we can write

$$H_{2n+1}(t) = C_n \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \begin{vmatrix} 1 & 1 & \cdots & 1 & \sin(2k\pi t) \\ m_1 & m_1^3 & \cdots & m_1^{2s-1} & \sin(2km_1\pi t) \\ m_2 & m_2^3 & \cdots & m_2^{2s-1} & \sin(2km_2\pi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_s & m_s^3 & \cdots & m_s^{2s-1} & \sin(2km_s\pi t) \end{vmatrix}. \tag{31}$$

We consider the case with no gaps in frequencies i.e. case with $(s-1)$-nontrivial frequencies $m_1 = 2 < m_2 = 3 < \cdots < m_{s-1} = s$. In that case we have

$$H_{2n+1}(t) = C_n \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \begin{vmatrix} 1 & 1 & \cdots & 1 & \sin(2k\pi t) \\ 2 & 2^3 & \cdots & 2^{2s-3} & \sin(4k\pi t) \\ 3 & 3^3 & \cdots & 3^{2s-3} & \sin(6k\pi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & \sin(2sk\pi t) \end{vmatrix}. \tag{32}$$

To simplify terms in the above expansion set:

$$S(\alpha)=\begin{vmatrix} 1 & 1 & \cdots & 1 & \sin\alpha \\ 2 & 2^3 & \cdots & 2^{2s-3} & \sin 2\alpha \\ 3 & 3^3 & \cdots & 3^{2s-3} & \sin 3\alpha \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & \sin s\alpha \end{vmatrix}=\sin\alpha\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^3 & \cdots & 2^{2s-3} & \frac{\sin 2\alpha}{\sin\alpha} \\ 3 & 3^3 & \cdots & 3^{2s-3} & \frac{\sin 3\alpha}{\sin\alpha} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & \frac{\sin s\alpha}{\sin\alpha} \end{vmatrix}.$$

Recall that the Chebyshev polynomials of the second kind are defined by

$$U_n(x)=\frac{\sin(n+1)\alpha}{\sin\alpha},\ \alpha=\arccos x,\ n=0,1,\dots.$$

We will need the following properties of the Chebyshev polynomials U_n:

1. $U_n(1)=n+1$
2. $U_n^{(k)}(1)=\frac{(n+k+1)!}{(2k+1)!!(n-k)!}\Leftarrow U_n^{(k)}(x)=xU_{n-1}^{(k)}(x)+(k+n)U_{n-1}^{(k-1)}(x)$
3. $|U_n(x)|\leqslant n+1$

Set:

$$\overline{S}(x)=\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^3 & \cdots & 2^{2s-3} & U_1(x) \\ 3 & 3^3 & \cdots & 3^{2s-3} & U_2(x) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & U_{s-1}(x) \end{vmatrix}.$$

Obviously $\overline{S}(1)=0$, $\overline{S}^{(s-1)}(1)=(s-1)!\mathrm{V}[1,2^2,\dots,(s-1)^2]$. We want to prove that $\overline{S}^{(k)}(1)=0$ for $k=1,\dots,s-2$. We have:

$$\overline{S}'(x)=\begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ 2 & 2^3 & \cdots & 2^{2s-3} & U_1'(x) \\ 3 & 3^3 & \cdots & 3^{2s-3} & U_2'(x) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & U_{s-1}'(x) \end{vmatrix}.$$

We have $U_l'(1)=\frac{(l+2)!}{3!!(l-1)!}$. Multiplying the first column with -1 and adding to the second column, we have in the $(l+1)$th row:

$$(l+1)^3-(l+1)=(l+1)(l+2)l=\frac{(l+2)!}{(l-1)!},$$

which obviously implies that $\overline{S}'(1)=0$. To make a general argument we compare $U_{l-1}^{(k)}(1)$, $k\leqslant l-1$, with the lth row in the $(k+1)$th column after reducing the first

$k+1$ columns on the lower trapezoid form. Using properties of the polynomials U_n we have:

$$U_{l-1}^{(k)}(1) = \frac{(l+k)!}{(2k+1)!!(l-k-1)!}.$$

After reducing the first $k+1$ columns on the lower trapezoid form in the l th row and the $(k+1)$ th column, using inductive property of the Vandermonde determinant applied on determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^3 & \cdots & 2^{2k+1} \\ \vdots & \vdots & \cdots & \vdots \\ k & k^3 & \cdots & k^{2k+1} \\ l & l^3 & \cdots & l^{2k+1} \end{vmatrix},$$

we obtain

$$\begin{aligned} & l(l^2-k^2)(l^2-(k-1)^2 \cdots (l^2-2^2)(l^2-1) \\ = & \; l(l-k)(l+k)(l+k-1)(l-k+1)\cdots(l+2)(l-2))(l+1)(l-1) \\ = & \; \frac{(l+k)!}{(l-k-1)!}. \end{aligned}$$

This finishes the proof that $\overline{S}^{(k)}(1) = 0$ for $k = 0,1,\dots,s-2$.

Since $\overline{S}$ is the polynomial of degree $s-1$, we conclude

$$\overline{S}(x) = \mathrm{V}[1,2^2,\dots,(s-1)^2](x-1)^{s-1}.$$

It follows

$$\begin{aligned} (\alpha) &= \sin\alpha \mathrm{V}[1,2^2,\dots,(s-1)^2](x-1)^{s-1} \\ &= (-1)^{s-1}\mathrm{V}[1,2^2,\dots,(s-1)^2]\sin\alpha(1-\cos\alpha)^{s-1}. \end{aligned}$$

Finally, we can write

$$H_{2n+1}(t) = D_n \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sin(2k\pi t)(1-\cos(2k\pi t))^{s-1}.$$

To illustrate how this expression helps in proving that H_{2n+1} has no zeros in $(0,1/2)$, we will prove that

$$\sin(2\pi t)(1-\cos(2\pi t))^{s-1} > -\sum_{k=2}^{\infty} \frac{1}{k^{2n+1}} \sin(2k\pi t)(1-\cos(2k\pi t))^{s-1},\ s \leqslant n.$$

Rearranging this is obviously equivalent to inequality

$$-\sum_{k=2}^{\infty} \frac{1}{k^{2n+1}} \frac{\sin(2k\pi t)}{\sin(2\pi t)} \left(\frac{\sin(k\pi t)}{\sin \pi t}\right)^{2s-2} < 1.$$

Since $\left|\frac{\sin k\alpha}{\sin\alpha}\right| \leqslant k$, it is enough to prove that

$$\sum_{k=2}^{\infty} \frac{1}{k^{2n+1}} k^{2s-1} = \sum_{k=2}^{\infty} \frac{1}{k^{2(n-s)+2}} < 1.$$

Recall that $s \leqslant n$, so it is enough to prove that

$$\sum_{k=2}^{\infty} \frac{1}{k^2} < 1,$$

and this is obvious since the LHS is equal to $\pi^2/6 - 1$.

The method of this section can give more information in negative direction. Let us consider Example 1 from Section 2 i.e.

$$H_5(t) = \begin{vmatrix} 1 & 1 & B_5^*(t) \\ 3 & 3^3 & B_5^*(3t) \\ 4 & 4^3 & B_5^*(4t) \end{vmatrix} = C_n \sum_{k=1}^{\infty} \frac{1}{k^5} \begin{vmatrix} 1 & 1 & \sin(2k\pi t) \\ 3 & 3^3 & \sin(6k\pi t) \\ 4 & 4^3 & \sin(8k\pi t) \end{vmatrix},$$

and consider the first term

$$\begin{vmatrix} 1 & 1 & \sin(2\pi t) \\ 3 & 3^3 & \sin(6\pi t) \\ 4 & 4^3 & \sin(8\pi t) \end{vmatrix} = \sin(2\pi t) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3^3 & U_2(x) \\ 4 & 4^3 & U_3(x) \end{vmatrix} = \sin(2\pi t)\overline{S}(x).$$

It can be shown easily that

$$\overline{S}(x) = (x-1)^2(144 + 192x),$$

which implies

$$\begin{vmatrix} 1 & 1 & \sin(2\pi t) \\ 3 & 3^3 & \sin(6\pi t) \\ 4 & 4^3 & \sin(8\pi t) \end{vmatrix} = \sin(2\pi t)(1-\cos(2\pi t))^2(144 + 192\cos(2\pi t)).$$

It can be shown that terms with higher frequencies (and small amplitudes) cannot remove the zeros in the basic term.

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