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ON p–TH ORDER OF A FUNCTION OF SEVERAL COMPLEX VARIABLES ANALYTIC IN THE UNIT POLYDISC

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Abstract. This paper is concerned with the study of the maximum modulus and the co-efficients of the power series expansion of a function of several complex variables analytic in the unit polydisc.

1. Introduction and Definitions

Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be analytic in the unit disc $U=\{z:|z|<1\}$ and $M(r)=M(r,f)$ be the maximum of $|f(z)|$ on $|z|=r$.

In 1968 Sons [9] introduced the following definition of the order ρ and the lower order λ as

$$\begin{matrix}\rho\\ \lambda\end{matrix}=\lim_{r\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\,\log M(r,f)}{-\,\log(1-r)}.$$

Maclane [7] and Kapoor [6] proved the following results which are the characterization of order and lower order of a function f analytic in U, in terms of the coefficients c_n.

THEOREM 1.1. [7] *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be analytic in U, having order ρ $(0\leqslant\rho\leqslant\infty)$. Then*

$$\frac{\rho}{1+\rho}=\lim\sup_{n\to\infty}\frac{\log^+\,\log^+|c_n|}{\log\,n}.$$

THEOREM 1.2. [6] *Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be analytic in U, having lower order λ $(0\leqslant\lambda\leqslant\infty)$. Then*

$$\frac{\lambda}{1+\lambda}\geqslant\lim\inf_{n\to\infty}\frac{\log^+\,\log^+|c_n|}{\log\,n}.$$

NOTATION 1.3. [8] $\log^{[0]}x=x$, $\exp^{[0]}x=x$ and for positive integer m, $\log^{[m]}x=\log(\log^{[m-1]}x)$, $\exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

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NOTATION 1.4. [1] For $0 < x < \infty$ we write $\log^{*(0)} x = x$, $\log^{*(1)} x = \log(1+x)$, $\log^{*(2)} x = \log(1+\log(1+x))$, $\log^{*(3)} x = \log(1+\log(1+\log(1+x)))$ etc.

In a paper [5] Juneja and Kapoor introduced the definition of p-th order and lower p-th order and in 2005 Banerjee [1] generalized Theorem 1.1 and Theorem 1.2 for p-th order and lower p-th order respectively.

DEFINITION 1.5. [5] If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U, its p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{-\log(1-r)}, \quad p \geqslant 2.$$

THEOREM 1.6. [1] *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *be analytic in* U *and having p-th order* ρ_p $(0 \leqslant \rho_p \leqslant \infty)$. *Then*

$$\frac{\rho_p}{1+\rho_p} = \lim_{n\to\infty} \sup \frac{\log^{+[p]} |c_n|}{\log n}.$$

THEOREM 1.7. [1] *Let* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *be analytic in* U *and having lower p-th order* λ_p $(0 \leqslant \lambda_p \leqslant \infty)$. *Then*

$$\frac{\lambda_p}{1+\lambda_p} \geqslant \lim_{n\to\infty} \inf \frac{\log^{+[p]} |c_n|}{\log n}.$$

In 2008 Banerjee and Dutta [2] introduced the following definition.

DEFINITION 1.8. Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc

$$P : \{(z_1, z_2) : |z_j| \leqslant 1; j = 1, 2\}$$

then order of f is denoted by ρ and is defined by

$$\rho = \inf\left\{\mu > 0 : F(r_1, r_2) < \exp\left(\frac{1}{1-r_1} \cdot \frac{1}{1-r_2}\right)^{\mu} ; \text{for all } 0 < r_0(\mu) < r_1, r_2 < 1\right\}.$$

Equivalent formula for ρ is

$$\rho = \lim_{r_1, r_2 \to 1} \sup \frac{\log\log F(r_1, r_2)}{-\log(1-r_1)(1-r_2)}.$$

In a recent paper [3] Banerjee and Dutta introduce the definition of p-th order and lower p-th order of functions of two complex variables analytic in the unit polydisc and generalized the above results for functions of two complex variables analytic in the unit polydisc.

DEFINITION 1.9. Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty}c_{mn}z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U=\{(z_1,z_2):|z_j|\leqslant 1;\ j=1,2\}$$

and

$$F(r_1,r_2)=\max\{|f(z_1,z_2)|:|z_j|\leqslant r_j;\ j=1,2\},$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p\\ \lambda_p\end{matrix}=\lim_{r_1,r_2\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}F(r_1,r_2)}{-\log(1-r_1)(1-r_2)},\ p\geqslant 2.$$

When $p=2$ Definition 1.9 coincides with Definition 1.8.

THEOREM 1.10. *Let $f(z_1,z_2)$ be analytic in U and having p-th order ρ_p $(0\leqslant\rho_p\leqslant\infty)$. Then*

$$\frac{\rho_p}{1+\rho_p}=\lim\sup_{m,n\to\infty}\frac{\log^{+[p]}|c_{mn}|}{\log mn}.$$

THEOREM 1.11. *Let $f(z_1,z_2)$ be analytic in U and having lower p-th order λ_p $(0\leqslant\lambda_p\leqslant\infty)$. Then*

$$\frac{\lambda_p}{1+\lambda_p}\geqslant\lim\inf_{m,n\to\infty}\frac{\log^{+[p]}|c_{mn}|}{\log mn}.$$

In a recent paper [4] Dutta introduce the definition of order and lower order of functions of several complex variables analytic in the unit polydisc and generalized the above results for functions of several complex variables analytic in the unit polydisc.

DEFINITION 1.12. Let $f(z_1,z_2,...,z_n)=\sum_{m_1,m_2,...,m_n=0}^{\infty}c_{m_1m_2...m_n}z_1^{m_1}z_2^{m_2}...z_n^{m_n}$ be a function of n complex variables z_1, z_2,..., z_n holomorphic in the unit polydisc

$$U=\{(z_1,z_2,...,z_n):|z_j|\leqslant 1;\quad j=1,2,...,n\}$$

and

$$F(r_1,r_2,...,r_n)=\max\{|f(z_1,z_2,...,z_n)|:|z_j|\leqslant r_j;\quad j=1,2,...,n\},$$

be its maximum modulus. Then the order ρ and lower order λ are defined as

$$\begin{matrix}\rho\\ \lambda\end{matrix}=\lim_{r_1,r_2,...,r_n\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log F(r_1,r_2,...,r_n)}{-\log(1-r_1)(1-r_2)...(1-r_n)}.$$

When $n=2$ Definition 1.12 coincides with Definition 1.8.

THEOREM 1.13. *Let $f(z_1,z_2,...,z_n)$ be analytic in U and having order ρ $(0 \leqslant \rho \leqslant \infty)$. Then*

$$\frac{\rho}{1+\rho} = \lim \sup_{m_1,m_2,...,m_n\to\infty} \frac{\log^+\log^+|c_{m_1m_2...m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

THEOREM 1.14. *Let $f(z_1,z_2,...,z_n)$ be analytic in U and having lower order λ $(0 \leqslant \lambda \leqslant \infty)$.*
Then

$$\frac{\lambda}{1+\lambda} \geqslant \lim \inf_{m_1,m_2,...,m_n\to\infty} \frac{\log^+\log^+|c_{m_1m_2...m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

In this paper we consider a more general situation in the case of analytic functions of several complex variables in the unit polydisc and for which we introduce the following definition.

DEFINITION 1.15. Let $f(z_1,z_2,...,z_n) = \sum_{m_1,m_2,...,m_n=0}^{\infty} c_{m_1m_2...m_n} z_1^{m_1} z_2^{m_2} ... z_n^{m_n}$ be a function of n complex variables z_1, z_2,..., z_n holomorphic in the unit polydisc

$$U = \{(z_1,z_2,...,z_n) : |z_j| \leqslant 1; \quad j = 1,2,...,n\}$$

and

$$F(r_1,r_2,...,r_n) = \max\{|f(z_1,z_2,...,z_n)| : |z_j| \leqslant r_j; \quad j = 1,2,...,n\},$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r_1,r_2,...,r_n\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} F(r_1,r_2,...,r_n)}{-\log(1-r_1)(1-r_2)...(1-r_n)}, \ p \geqslant 2.$$

When $n = 2$ Definition 1.15 coincides with Definition 1.9 also when $p = 2$ Definition 1.15 coincides with Definition 1.12.

In this paper we find a similar analytic expression for ρ_p and λ_p in terms of the co-efficient $c_{m_1m_2...\,m_n}$ for several complex variables.

Here $f(z_1,z_2,...,z_n) = \sum_{m_1,m_2,...,m_n=0}^{\infty} c_{m_1m_2...m_n} z_1^{m_1} z_2^{m_2} ... z_n^{m_n}$ will denote a function analytic in the unit polydisc.

2. Lemmas

The following lemmas will be needed in the sequel.

LEMMA 2.1. *Let the maximum modulus $F(r_1,r_2,...,r_n)$ of a function $f(z_1,z_2,...,z_n)$ analytic in U, satisfy*

$$\log^{[p-1]} F(r_1,r_2,...,r_n) < A\left\{\prod_{j=1}^{n}(1-r_j)\right\}^{-B} \tag{2.1}$$

$1 < A,B < \infty$ for all r_j such that $r_0(A,B) < r_j < 1$; $j = 1,2,...,n$.
Then for all $m_j > m_{j_0}(A,B) > 1$; $j = 1,2,...,n$.

$$\log^{[p-1]} |c_{m_1 m_2 \ \dots \ m_n}| \leqslant S(A,B) \left(\prod_{j=1}^{n} m_j \right)^{\frac{B}{B+1}}$$

where

$$S(A,B) = (1+2B) \left(\frac{A}{B^B} \right)^{\frac{1}{1+B}}.$$

Proof. At first define n sequences $\{r_{jm_j}\}$ by

$$(1 - r_{jm_j})^{-1} = \left(\frac{m_j}{AB} \right)^{\frac{1}{B+1}}; \quad j = 1,2,...,n.$$

Then $r_{jm_j} \to 1$ as $m_j \to \infty$ for all $j = 1,2,...,n$. Also,

$$\begin{aligned}
|c_{m_1 m_2 \dots m_n}| &= \frac{1}{\prod_{j=1}^{n}(m_j!)} \left| \frac{\partial^{m_1+m_2+\dots+m_n} f(0,0,...,0)}{\partial z_1^{m_1}\ \partial z_2^{m_2} \dots \partial z_n^{m_n}} \right| \\
&= \left| \frac{1}{(2\pi i)^n} \int_{|z_1|=r_1} \int_{|z_2|=r_2} \cdots \int_{|z_n|=r_n} \frac{f(z_1,z_2,...,z_n) dz_1 dz_2 \dots\ dz_n}{z_1^{m_1+1}\ z_2^{m_2+1} \dots\ z_n^{m_n+1}} \right| \\
&\leqslant \frac{F(r_1,r_2,...,r_n)}{r_1^{m_1} r_2^{m_2} \dots r_n^{m_n}} \\
&= \frac{F(r_1,r_2,...,r_n)}{\prod_{j=1}^{n} r_j^{m_j}}. \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2) we have for all $m_j > m_{j_0}(A,B) > 1$; $j = 1,2,...,n$

$$\begin{aligned}
\log |c_{m_1 m_2 \dots m_n}| &\leqslant \log F(r_{1m_1}, r_{2m_2}, ..., r_{nm_n}) - \sum_{j=1}^{n} m_j \quad \log\ r_{jm_j} \\
&< \exp^{[p-2]} \left[A \left\{ \prod_{j=1}^{n} (1 - r_{jm_j}) \right\}^{-B} \right] + \left[\sum_{j=1}^{n} m_j (1 - r_{jm_j}) \right] [1 + O(1)] \\
&= \exp^{[p-2]} \left[A \left(\frac{\prod_{j=1}^{n} m_j}{(AB)^n} \right)^{\frac{B}{B+1}} \right] + \left[\sum_{j=1}^{n} m_j \left(\frac{AB}{m_j} \right)^{\frac{1}{B+1}} \right] [1 + O(1)]
\end{aligned}$$

$$\leqslant \exp^{[p-2]}\left[A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}\right]+\prod_{j=1}^{n} m_j\left(\frac{AB}{\prod_{j=1}^{n} m_j}\right)^{\frac{1}{B+1}}[1+O(1)]$$

$$=\exp^{[p-2]}\left[A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}\right]+A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}} B[1+O(1)]$$

$$\leqslant \exp^{[p-2]}\left[A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}\right]+B\exp^{[p-2]}\left[A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}\right][1+O(1)]$$

$$=[1+B\{1+O(1)\}]\exp^{[p-2]}\left\{A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}\right\}$$

$$\leqslant \exp^{[p-2]}\left[\left\{(1+2B)\left(\frac{A}{B^B}\right)^{\frac{1}{B+1}}\right\}\left(\prod_{j=1}^{n} m_j\right)^{\frac{B}{B+1}}\right].$$

Therefore

$$\log^{[p-1]}|c_{m_1 m_2 \ldots m_n}| \leqslant S(A,B)\left(\prod_{j=1}^{n} m_j\right)^{\frac{B}{B+1}}$$

where

$$S(A,B)=(1+2B)\left(\frac{A}{B^B}\right)^{\frac{1}{1+B}}.$$

This proves the lemma. □

LEMMA 2.2. *Let $f(z_1,z_2,\ldots,z_n)$ be analytic in U and satisfy*

$$|c_{m_1 m_2 \ldots m_n}| < \prod_{j=1}^{n} \exp^{[p-1]}\left(Cm_j^D\right),\ \ 0<C<\infty,\ 0<D<1$$

for all $m_j > m_{j_0}(C,D)$; $j=1,2,\ldots,n$. Then for all r_j such that $r_{j0}(C,D)<r_j<1$; $j=1,2,\ldots,n$

$$\log^{[p-1]} F(r_1,r_2,\ldots,r_n) < T(C,D)\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)^{\frac{-D}{1-D}},$$

where

$$T(C,D) = C^{\frac{n}{1-D}} D^{\frac{nD}{1-D}} [1+o(1)].$$

Proof. For all $m_j > m_{j_0}(C,D);\ j = 1,2,...,n,$

$$|c_{m_1 m_2 ... m_n}| < \prod_{j=1}^{n} \exp^{[p-1]}\left(C m_j^D\right).$$

Now for $|z_j| = r_j < 1; j = 1,2,...,n$

$$\begin{aligned}
F(r_1,r_2,...,r_n) &< \sum_{m_1,m_2,...,m_n=0}^{\infty} |c_{m_1 m_2 ... m_n}|\, r_1^{m_1} r_2^{m_2} ... r_n^{m_n} \\
&< K(m_{1_0}, m_{2_0}, ..., m_{n_0}) + \sum_{\substack{m_1 = m_{1_0}+1 \\ m_2 = m_{2_0}+1 \\ \vdots \\ m_n = m_{n_0}+1}}^{\infty} \left\{ \prod_{j=1}^{n} \exp^{[p-1]}\left(C m_j^D\right) r_j^{m_j} \right\} \\
&\leqslant K(m_{1_0}, m_{2_0}, ..., m_{n_0}) + \prod_{j=1}^{n} \left[\sum_{m_j = m_{j_0}+1}^{\infty} \exp^{[p-1]}\left(C m_j^{\frac{B}{B+1}}\right) r_j^{m_j} \right],
\end{aligned}$$

where $B = \frac{D}{1-D}$. Choose

$$M_j = M(r_j) = \left[\left(\frac{2^{2p-3} C}{\log^{*(p-2)}\left(\log \frac{1}{r_j}\right)} \right)^{B+1} \right]; \quad j = 1,2,...,n$$

where $[x]$ denotes the greatest integer not greater then x.

Clearly $M(r_j) \to \infty$ as $r_j \to 1$ for $j = 1,2,...,n.$

The above estimate of $F(r_1,r_2,...,r_n)$ for all $r_j;\ j = 1,2,...,n$ sufficiently close to 1 gives,

$$F(r_1,r_2,...,r_n) < K(m_{1_0}, m_{2_0}, ..., m_{n_0}) + \prod_{j=1}^{n} \left[M(r_j) H(r_j) + \sum_{m_j = M_j+1}^{\infty} r_j^{m_j/2} \right] \tag{2.3}$$

where

$$H(r_j) = \max_{m_j} \left\{ \exp^{[p-1]}\left(C\, m_j^{\frac{B}{B+1}}\right) r_j^{m_j} \right\}; \quad j = 1,2,...,n$$

for if $m_j \geqslant M_j + 1,$ then

$$
\begin{aligned}
m_j &> \left(\frac{2^{2p-3}C}{\log^{*(p-2)}\left(\log\frac{1}{r_j}\right)}\right)^{B+1}.\\
\therefore \qquad C m_j^{\frac{B}{B+1}} &< \frac{m_j}{2^{2p-3}}\ \log^{*(p-2)}\left(\log\frac{1}{r_j}\right)\\
&= \log\left[1+\log^{*(p-3)}\left(\log\frac{1}{r_j}\right)\right]^{\frac{m_j}{2^{2p-3}}}\\
&\leqslant \log\left[1+\frac{m_j}{2^{2p-4}}\ \log^{*(p-3)}\left(\log\frac{1}{r_j}\right)\right].\\
\therefore \qquad \exp\left(C m_j^{\frac{B}{B+1}}\right) &\leqslant 1+\frac{m_j}{2^{2p-4}}\ \log^{*(p-3)}\left(\log\frac{1}{r_j}\right)\\
&\leqslant \frac{m_j}{2^{2p-5}}\ \log^{*(p-3)}\left(\log\frac{1}{r_j}\right)\\
&\leqslant \log\left[1+\frac{m_j}{2^{2p-6}}\ \log^{*(p-4)}\left(\log\frac{1}{r_j}\right)\right].\\
\therefore \qquad \exp^{[2]}\left(C m_j^{\frac{B}{B+1}}\right) &\leqslant 1+\frac{m_j}{2^{2p-6}}\ \log^{*(p-4)}\left(\log\frac{1}{r_j}\right)\\
&\leqslant \frac{m_j}{2^{2p-7}}\ \log^{*(p-4)}\left(\log\frac{1}{r_j}\right).
\end{aligned}
$$

Taking repeated exponential, we obtain

$$
\begin{aligned}
\exp^{[p-2]}\left(C m_j^{\frac{B}{B+1}}\right) &< \frac{m_j}{2}\ \log\frac{1}{r_j}\\
\text{i.e.}\ \ \exp^{[p-1]}\left(C m_j^{\frac{B}{B+1}}\right) r_j^{m_j} &< r_j^{\frac{m_j}{2}}
\end{aligned}
$$

for all $j = 1,2,...,n$.

Therefore the infinite series $\sum_{m_j=M_j+1}^{\infty} r_j^{\frac{m_j}{2}}$ in (2.3) is bounded by $r_j^{\frac{M_j+1}{2}}\left(\frac{1}{1-r_j^{\frac{1}{2}}}\right)$ for all $j = 1,2,...,n$.

Since $B > 0$, we have

$$
\begin{aligned}
&-\frac{M_j+1}{2}\ \log\frac{1}{r_j}-\log\left(1-r_j^{\frac{1}{2}}\right)\\
&< -\frac{1}{2}\left(\frac{2^{2p-3}C}{\log^{*(p-2)}\left(\log\frac{1}{r_j}\right)}\right)^{B+1}\log\frac{1}{r_j}-\log(1-r_j)+\log\left(1+r_j^{\frac{1}{2}}\right)
\end{aligned}
$$

$$< -\frac{1}{2}\left(\frac{2^{2p-3}C}{\log\frac{1}{r_j}}\right)^{B+1}\log\frac{1}{r_j} - \log(1-r_j) + \log\left(1+r_j^{\frac{1}{2}}\right)$$
$$\to -\infty \text{ as } r_j \to 1.$$

Thus for r_j sufficiently close to 1,

$$\sum_{m_j=M_j+1}^{\infty} r_j^{m_j/2} = o(1)$$

for all $j = 1,2,...,n$.

The maximum of $\exp^{[p-1]}\left(Cm_j^{\frac{B}{B+1}}\right) r_j^{m_j}$ assume at the point $m_j = \left(\frac{BC}{(B+1)\,\log\frac{1}{r_j}}\right)^{B+1}$ and $H(r_j)$ is given by

$$\begin{aligned}\log H(r_j) &= \exp^{[p-2]}\left(Cm_j^{\frac{B}{B+1}}\right) + m_j\,\log\,r_j\\ &= \exp^{[p-2]}\left\{\frac{C.B^B.C^B}{(B+1)^B\left(\log\frac{1}{r_j}\right)^B}\right\} - \left(\frac{(BC)^{B+1}\log\frac{1}{r_j}}{(B+1)^{B+1}\left(\log\frac{1}{r_j}\right)^{B+1}}\right)\\ &\leqslant \exp^{[p-2]}\left\{\frac{C^{B+1}.B^B}{(B+1)^B\left(\log\frac{1}{r_j}\right)^B}\right\}.\end{aligned}\tag{2.4}$$

Thus for $r_j;\ j = 1,2,...,n$ sufficiently close to 1, from (2.3)

$$\begin{aligned}F(r_1,r_2,...,r_n) &< \prod_{j=1}^{n}[M(r_j)H(r_j)+o(1)]\left[1+\frac{K(m_{1_0},m_{2_0},...,m_{n_0})}{\prod_{j=1}^{n}[M(r_j)H(r_j)+o(1)]}\right]\\ &= \prod_{j=1}^{n}[M(r_j)H(r_j)+o(1)]\,[1+O(1)].\end{aligned}$$

$$\begin{aligned}\therefore\ \log F(r_1,r_2,...,r_n) &< \sum_{j=1}^{n}\{\log M(r_j)+\log H(r_j)\}+O(1)\\ &\leqslant \sum_{j=1}^{n}\left\{-(B+1)\log^{[p]}\frac{1}{r_j}+\exp^{[p-2]}\left(\frac{C^{B+1}.B^B}{(B+1)^B\left(\log\frac{1}{r_j}\right)^B}\right)\right\}+O(1)\\ &\qquad\text{[from (2.4)]}\\ &\leqslant \sum_{j=1}^{n}\exp^{[p-2]}\left\{\frac{C^{B+1}.B^B}{(B+1)^B\ \left(\log\frac{1}{r_j}\right)^B}\right\}+O(1)\end{aligned}$$

$$\leqslant \exp^{[p-2]}\left\{\frac{C^{n(B+1)}.B^{nB}}{(B+1)^{nB}}\prod_{j=1}^{n}\left(\log\frac{1}{r_j}\right)^{-B}\right\}+O(1)$$

$$\leqslant \exp^{[p-2]}\left[C^{\frac{n}{1-D}}D^{\frac{nD}{1-D}}\prod_{j=1}^{n}\left(\log\frac{1}{r_j}\right)^{-B}\right][1+o(1)].$$

Therefore

$$\log^{[p-1]}F(r_1,r_2,...,r_n)\leqslant C^{\frac{n}{1-D}}D^{\frac{nD}{1-D}}[1+o(1)]\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)^{\frac{-D}{1-D}}$$

$$=T(C,D)\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)^{\frac{-D}{1-D}}$$

where

$$T(C,D)=C^{\frac{n}{1-D}}D^{\frac{nD}{1-D}}[1+o(1)].$$

This proves the lemma. □

3. Theorem

We prove the following theorems.

THEOREM 3.1. *Let $f(z_1,z_2,...,z_n)$ be analytic in U and having p-th order ρ_p $(0\leqslant\rho_p\leqslant\infty)$. Then*

$$\frac{\rho_p}{1+\rho_p}=\lim\sup_{m_1,m_2,...,m_n\to\infty}\frac{\log^{+[p]}|c_{m_1m_2...m_n}|}{\log\left(\prod_{j=1}^{n}m_j\right)}. \tag{3.1}$$

Proof. If $|c_{m_1m_2...m_n}|$ is bounded by K for all m_j; $j=1,2,...,n$ then $\sum_{m_1,m_2,...,m_n=0}^{\infty}c_{m_1m_2...m_n}z_1^{m_1}z_2^{m_2}...z_n^{m_n}$ is bounded by $\dfrac{K}{\prod_{j=1}^{n}(1-r_j)}$.

Therefore

$$F(r_1,r_2,...,r_n)\leqslant\sum_{m_1,m_2,...,m_n=0}^{\infty}|c_{m_1m_2...m_n}|r_1^{m_1}r_2^{m_2}...r_n^{m_n}$$

$$\leqslant\frac{K}{\prod_{j=1}^{n}(1-r_j)}$$

$$<\exp^{[p-1]}\left(\frac{1}{\prod_{j=1}^{n}(1-r_j)}\right)^{\varepsilon}$$

for any $0<\varepsilon<1$ and for all $r_j;\ j=1,2,...,n$ sufficiently close to 1.

Therefore

$$\rho_p=\lim\sup_{r_1,r_2,...,r_n\to 1}\frac{\log^{[p]}F(r_1,r_2,...,r_n)}{-\log\left(\prod_{j=1}^{n}(1-r_j)\right)}\leqslant\varepsilon$$

since $0<\varepsilon<1$ arbitrary, $\rho_p=0$ and so (3.1) is satisfied.

Thus we need to consider only the case

$$\lim\sup_{m_1,m_2,...,m_n\to\infty}|c_{m_1m_2...m_n}|=\infty.$$

In this case all the $\log^+$ in (3.1) may be replaced by $\log$. First let $0<\rho_p<\infty$ and $\rho_p^{'}>\rho_p$.

Then for all $r_j;\ j=1,2,...,n$ sufficiently close to 1,

$$\log^{[p-1]}F(r_1,r_2,...,r_n)<\left\{\prod_{j=1}^{n}(1-r_j)\right\}^{-\rho_p^{'}}.$$

Using Lemma 2.1 with $A=1,\ B=\rho_p^{'}$ it follows from the above inequality that for $m_j>m_{j_0}(\rho_p^{'});\ j=1,2,...,n.$

$$\log^{[p-1]}|c_{m_1m_2...m_n}|\leqslant(1+2\rho_p^{'})\left(\frac{1}{\rho_p^{'}\rho_p^{'}}\right)^{\frac{1}{1+\rho_p^{'}}}\left(\prod_{j=1}^{n}m_j\right)^{\frac{\rho_p^{'}}{\rho_p^{'}+1}}.$$

Therefore

$$\lim\sup_{m_1,m_2,...,m_n\to\infty}\frac{\log^{[p]}|c_{m_1m_2...m_n}|}{\log\left(\prod_{j=1}^{n}m_j\right)}\leqslant\frac{\rho_p^{'}}{1+\rho_p^{'}}.$$

Since $\rho_p^{'}>\rho_p$ is arbitrary, it follows that

$$\lim\sup_{m_1,m_2,...,m_n\to\infty}\frac{\log^{[p]}|c_{m_1m_2...m_n}|}{\log\left(\prod_{j=1}^{n}m_j\right)}\leqslant\frac{\rho_p}{1+\rho_p}. \tag{3.2}$$

Since $f(z_1,z_2,...,z_n)$ is analytic in U, the above inequality is trivially true if $\rho_p=\infty$ and the right hand side is interpreted as 1 in this case.

Conversely, if

$$\theta=\lim\sup_{m_1,m_2,...,m_n\to\infty}\frac{\log^{[p]}|c_{m_1m_2...m_n}|}{\log\left(\prod_{j=1}^{n}m_j\right)}$$

then $0 \leqslant \theta \leqslant 1$.

First let $\theta < 1$ and choose $\theta < \theta' < 1$.

Then for all sufficiently large m_j; $j = 1, 2, ..., n$,

$$\log |c_{m_1 m_2 \ldots m_n}| < \left(\prod_{j=1}^{n} m_j\right)^{\theta'}.$$

Using Lemma 2.2 with $C = 1$, $D = \theta'$ it follows from the above inequality that for all r_j such that $r_{j_0}(\theta') < r_j < 1$; $j = 1, 2, ..., n$,

$$\log^{[p-1]} F(r_1, r_2, ..., r_n) < \theta'^{\frac{n\theta'}{1-\theta'}} \left(\prod_{j=1}^{n} \log \frac{1}{r_j}\right)^{\frac{-\theta'}{1-\theta'}} [1 + o(1)].$$

$$\therefore \quad \log^{[p]} F(r_1, r_2, ..., r_n) < \frac{n\theta'}{1-\theta'} \log(\theta') + \frac{-\theta'}{1-\theta'} \log\left(\prod_{j=1}^{n} \log \frac{1}{r_j}\right) + \log[1 + o(1)]$$

that is

$$\lim \sup_{r_1, r_2, ..., r_n \to 1} \frac{\log^{[p]} F(r_1, r_2, ..., r_n)}{-\log\left(\prod_{j=1}^{n}(1 - r_j)\right)} \leqslant -\frac{\theta'}{1-\theta'} \lim \sup_{r_1, r_2, ..., r_n \to 1} \frac{\log\left(\prod_{j=1}^{n} \log \frac{1}{r_j}\right)}{-\log\left(\prod_{j=1}^{n}(1 - r_j)\right)}.$$

$$\therefore \quad \rho_p < \frac{\theta'}{1-\theta'}.$$

Since $\theta' > \theta$ is arbitrary, it follows that

$$\frac{\rho_p}{1+\rho_p} \leqslant \theta = \lim \sup_{m_1, m_2, ..., m_n \to \infty} \frac{\log^{[p]} |c_{m_1 m_2 \ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}. \tag{3.3}$$

If $\theta = 1$, the above inequality is obviously true.

Inequality (3.2) and (3.3) together gives (3.1) when $\lim \sup_{m_1, m_2, ..., m_n \to \infty} |c_{m_1 m_2 \ldots m_n}| = \infty$.

This proves the theorem. □

THEOREM 3.2. *Let $f(z_1, z_2, ..., z_n)$ be analytic in U and having lower p-th order λ_p $(0 \leqslant \lambda_p \leqslant \infty)$.*

Then

$$\frac{\lambda_p}{1+\lambda_p} \geqslant \lim \inf_{m_1, m_2, ..., m_n \to \infty} \frac{\log^{+[p]} |c_{m_1 m_2 \ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

Proof. Let

$$\lim \inf_{m_1,m_2,\ldots,m_n\to\infty} \frac{\log^{+[p]}|c_{m_1m_2\ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)} = A. \tag{3.4}$$

First suppose that $0 < A < 1$.

From (3.4), for $0 < \varepsilon < A < 1$,

$$\log|c_{m_1m_2\ldots m_n}| > \exp^{[p-2]}\left(\prod_{j=1}^{n} m_j\right)^{A-\varepsilon}$$

for $m_j > M_j = M_j(\varepsilon); j = 1,2,\ldots,n$.

Also

$$|c_{m_1m_2\ldots m_n}| \leqslant \frac{F(r_1,r_2,\ldots,r_n)}{\prod_{j=1}^{n} r_j^{m_j}}.$$

$$\therefore \quad \log F(r_1,r_2,\ldots,r_n) \geqslant \log|c_{m_1m_2\ldots m_n}| + \sum_{j=1}^{n} m_j \log r_j. \tag{3.5}$$

Choose

$$m_j = \left(\log\frac{1}{r_j}\right)^{\frac{1}{A-\varepsilon-1}} \text{ where } j = 1,2,\ldots,n.$$

Then from (3.5)

$$\begin{aligned}
\log F(r_1,r_2,\ldots,r_n) &> \exp^{[p-2]}\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)^{\frac{A-\varepsilon}{A-\varepsilon-1}} - \sum_{j=1}^{n}\left(\log\frac{1}{r_j}\right)^{\frac{1}{A-\varepsilon-1}}\log\frac{1}{r_j}\\
&= \exp^{[p-2]}\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)^{\frac{A-\varepsilon}{A-\varepsilon-1}} - \sum_{j=1}^{n}\left(\log\frac{1}{r_j}\right)^{\frac{A-\varepsilon}{A-\varepsilon-1}}\\
&> \frac{1}{k}\exp^{[p-2]}\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)^{\frac{A-\varepsilon}{A-\varepsilon-1}}, \text{ where } k \text{ is a suitable constant.}
\end{aligned}$$

$$\therefore \quad \log^{[p]} F(r_1,r_2,\ldots,r_n) > \frac{A-\varepsilon}{A-\varepsilon-1}\log\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right) + O(1).$$

$$\therefore \quad \frac{\log^{[p]} F(r_1,r_2,\ldots,r_n)}{-\log\left(\prod_{j=1}^{n}(1-r_j)\right)} > \frac{A-\varepsilon}{A-\varepsilon-1}\frac{\log\left(\prod_{j=1}^{n}\log\frac{1}{r_j}\right)}{-\log\left(\prod_{j=1}^{n}(1-r_j)\right)} + O(1).$$

Therefore

$$\lambda_p = \lim \inf_{r_1,r_2,\dots,r_n \to 1} \frac{\log^{[p]} F(r_1,r_2,\dots,r_n)}{-\log\left(\prod_{j=1}^{n}(1-r_j)\right)} \geqslant \frac{A-\varepsilon}{1-A+\varepsilon}.$$

Since $0 < \varepsilon < A < 1$ is arbitrary,

$$\lambda_p \geqslant \frac{A}{1-A}.$$

Which implies

$$\frac{\lambda_p}{1+\lambda_p} \geqslant A.$$

This inequality holds obviously when $A = 0$. For $A = 1$ the above arguments with a number K arbitrarily near to 1 in place of $A-\varepsilon$, give

$$\frac{\lambda_p}{1+\lambda_p} = 1.$$

This proves the theorem. □

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