Journal of
Mathematical
Inequalities
Volume 5, Number 4 (2011), 601–609

INEQUALITIES FOR THE SCHWAB–BORCHARDT MEAN AND THEIR APPLICATIONS

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(*Communicated by G. Toader*)

Abstract. Inequalities for the Schwab-Borchardt mean are obtained. They contain known results for the trigonometric and hyperbolic functions including those obtained by J. Wilker [15] and C. Huygens [5]. The main results of this paper can also be utilized to obtain new inequalities for some bivariate means including the logarithmic mean and two means introduced by Seiffert.

1. Introduction

The following trigonometric inequalities

$$2 < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \tag{1.1}$$

and

$$3 < 2\frac{\sin x}{x} + \frac{\tan x}{x} \tag{1.2}$$

$(0 < |x| < \frac{\pi}{2})$ have been discovered respectively by J.B. Wilker [15] and C. Huygens [5]. They attracted attention of many researchers. Several proofs of (1.1) and (1.2) can be found in mathematical literature (see [4, 7, 12, 16, 17, 18]).

The hyperbolic counterparts of (1.1) and (1.2) have been obtained. They read as follows

$$2 < \left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} \tag{1.3}$$

and

$$3 < 2\frac{\sinh x}{x} + \frac{\tanh x}{x} \tag{1.4}$$

$(x \neq 0)$. For the proofs of these results the interested reader is referred to [19] and [12], respectively. Several generalizations and extensions of these results have been obtained recently. For instance, refinements of (1.1) and (1.2) read as follows

$$2 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \tag{1.5}$$

Mathematics subject classification (2010): Primary: 26D05, 26D07, 33B10.
Keywords and phrases: The Schwab-Borchardt mean, trigonometric functions, hyperbolic functions, logarithmic mean, Seiffert means, inequalities.

Paper JMI-05-52

and

$$3 < 2\frac{x}{\sin x} + \frac{x}{\tan x} < 2\frac{\sin x}{x} + \frac{\tan x}{x}. \tag{1.6}$$

Proofs of (1.5) and (1.6) can be found in [17, 12, 16]. The hyperbolic counterparts of (1.5) and (1.6) are obtained in [12].

The goal of this paper is to demonstrate that the inequalities (1.1)–(1.4) are special cases of some inequalities for the Schwab-Borchardt mean. This little known mean is discussed in [1], [3], [10] and [11]. This paper, which can be regarded as a continuation of the research presented in [12] and [9], is organized as follows. Notation and definitions, used in the subsequent parts of this work, are introduced in Section 2. The next section contains some lemmas employed in Section 4. Therein the main results of this paper are obtained. Applications to some bivariate means are also mentioned. Refinements of the inequalities of D.D. Adamović, D.S. Mitrinović, and I. Lazarević are obtained in Section 5. Refinements of inequalities (1.1)–(1.4) are also obtained in this section.

2. Notation and Definitions

The geometic, arithmetic, and the root-mean square means of $x > 0$ and $y > 0$ will be denoted by G, A, and Q, respectively, and they are defined as follows

$$G \equiv G(x,y) = \sqrt{xy}, \;\; A \equiv A(x,y) = \frac{x+y}{2}, \;\; Q \equiv Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}.$$

Other bivariate means used in the subsequent sections include the logarithmic mean

$$L = \frac{z}{\tanh^{-1} z} A, \tag{2.1}$$

the first and second Seiffert means P and T, where

$$P = \frac{z}{\sin^{-1} z} A \tag{2.2}$$

and

$$T = \frac{z}{\tan^{-1} z} A \tag{2.3}$$

(see [13], [14], [10], [8]). Here

$$z = \frac{x-y}{x+y} \tag{2.4}$$

$(x \neq y)$. Another mean which is also of interest has been introduced in [10, (2.6)] and is defined as

$$M = \frac{z}{\sinh^{-1} z} A. \tag{2.5}$$

It is known that

$$G < L < P < A < M < T < Q$$

(see [10, (2.10)]). The means L, P, T and M are special cases of the Schwab-Borchardt mean. For $x \geqslant 0$ and $y > 0$ the mean under discussion will be denoted by $SB(x,y)$ or simply by SB. This mean is the iterative mean, i.e.,

$$SB = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n\,,$$

where

$$x_0 = x,\ \ y_0 = y,\ \ x_{n+1} = \frac{x_n+y_n}{2},\ \ y_{n+1} = \sqrt{x_{n+1}y_n} \tag{2.6}$$

$(n = 0,1,\ldots)$. It is known that

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\cos^{-1}(x/y)} & \text{if } x<y, \\ \dfrac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)} & \text{if } y<x \end{cases} \tag{2.7}$$

(see [1, Theorem 8.4], [3, (2.3)]). The mean SB is nonsymmetric, homogeneous of degree 1 and strictly increasing it its variables. It has been established in [10] that

$$L = SB(A,G),\ \ P = SB(G,A),\ \ T = SB(A,Q),\ \ M = SB(Q,A). \tag{2.8}$$

Using (2.7) one can easily verify that

$$SB(\cos x,1) = \frac{\sin x}{x}, \qquad SB\left(\sqrt{1-x^2},1\right) = \frac{x}{\sin^{-1}x}, \tag{2.9}$$

$$SB(\cosh x,1) = \frac{\sinh x}{x}, \qquad SB\left(\sqrt{1+x^2},1\right) = \frac{x}{\sinh^{-1}x}, \tag{2.10}$$

$$SB(1,\sec x) = \frac{\tan x}{x}, \qquad SB\left(1,\sqrt{1+x^2}\right) = \frac{x}{\tan^{-1}x}, \tag{2.11}$$

$$SB(1,\operatorname{sech} x) = \frac{\tanh x}{x}, \qquad SB\left(1,\sqrt{1-x^2}\right) = \frac{x}{\tanh^{-1}x}, \tag{2.12}$$

3. Lemmas

In what follows we will assume that u and v are positive and unqual numbers. The following lemmas will be utilized in the subsequent sections of this paper.

LEMMA 3.1. ([12]) *If* $uv > 1$, *then*

$$\frac{1}{u}+\frac{1}{v} < u+v.$$

LEMMA 3.2. ([9]) *Let* $\alpha > 0$, $\beta > 0$ *with* $\alpha+\beta = 1$. *If*

$$1 < \alpha\frac{1}{u}+\beta\frac{1}{v} < \alpha u+\beta v, \tag{3.1}$$

then

$$1 < \alpha\frac{1}{u^p} + \beta\frac{1}{v^p} < \alpha u^p + \beta v^p. \tag{3.2}$$

The first inequality in (3.2) *is valid for* $p \geqslant 1$ *while the second one holds true provided* $p > 0$.

4. Main Results

The goal of this section is to establish inequalities for the Schwab-Borchardt mean. These results can be used to obtain refinements and generalizations of the Wilker-type and Huygens-type inequalities for the trigonometric and hyperbolic functions. Applications of the obtained results to inequalities involving means L, P, T and M are also included.

In what follows the symbols $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$ will stand for sequences defined in (2.6). Also, SB will denote the Schwab-Borchardt mean of $x > 0$ and $y > 0$.

Our first result reads as follows.

THEOREM 4.1. *Let* $n = 0, 1, \ldots$. *Then*

$$2 < \left(\frac{y_n}{SB}\right)^{2p} + \left(\frac{x_n}{SB}\right)^{p} < \left(\frac{SB}{y_n}\right)^{2p} + \left(\frac{SB}{x_n}\right)^{p}. \tag{4.1}$$

The first inequality in (4.1) *holds true for* $p \geqslant 1$ *while the second one is valid if* $p > 0$.

Proof. The following two-sided inequality

$$(x_n y_n^2)^{1/3} < SB < \frac{x_n + 2y_n}{3} \tag{4.2}$$

($n = 0, 1, \ldots$) has been obtained in [10]. It follows from the second inequality in (4.2) that

$$x_n > 3SB - 2y_n. \tag{4.3}$$

For the proof of the first inequality in (4.1) we use the inequality for the bivariate power mean of $u > 0$ and $v > 0 : u^p + v^p > 2^{1-p}(u+v)^p$ $(p \geqslant 1)$ together with (4.3) to obtain

$$\begin{aligned}
\left(\frac{y_n}{SB}\right)^{2p} + \left(\frac{x_n}{SB}\right)^{p} &> 2^{1-p}\left[\left(\frac{y_n}{SB}\right)^{2} + \frac{x_n}{SB}\right]^{p} \\
&> 2^{1-p}\left[\left(\frac{y_n}{SB}\right)^{2} + \frac{3SB - 2y_n}{SB}\right]^{p} \\
&= 2^{1-p}\left[2 + \left(\frac{y_n}{SB} - 1\right)^{2}\right]^{p} > 2.
\end{aligned}$$

We shall establish now the second inequality in (4.1). To this aim we write the left inequality in (4.2) as

$$1 < \left(\frac{SB}{y_n}\right)^{2}\frac{SB}{x_n}.$$

Application of Lemma 3.1 with $u = \left(\frac{SB}{y_n}\right)^2$ and $v = \frac{SB}{x_n}$ together with the use of the first inequality in (4.1) yields

$$2 < \left(\frac{y_n}{SB}\right)^2 + \frac{x_n}{SB} < \left(\frac{SB}{y_n}\right)^2 + \frac{SB}{x_n}.$$

Dividing both sides by 2 and next using Lemma 3.2 with $\alpha = \beta = 1/2$ and u and v as defined above gives the asserted result. $\square$

Special cases of (4.1) appear in mathematical literature (see [16], [12], [7]). One can obtain these results by letting in (4.1) $n = 0$, $y_0 = 1$ and $x_0 = \cos x$ or $x_0 = \cosh x$. Using first formulas in (2.9) and (2.10) one has $SB = \sin x/x$ and $SB = \sinh x/x$.

We are in a position to state and prove the following.

THEOREM 4.2. *Let n be a nonnegative integer. Then*

$$3 < 2\left(\frac{y_n}{SB}\right)^p + \left(\frac{x_n}{SB}\right)^p < 2\left(\frac{SB}{y_n}\right)^p + \left(\frac{SB}{x_n}\right)^p, \tag{4.4}$$

where the first inequality holds true provided $p \geqslant 1$ while the second one is valid for all $p > 0$.

Proof. We shall establish first the two-sided inequality (4.4) when $p = 1$. In this case the first inequality follows immediately from the right inequality in (4.2). For the proof of the second inequality in (4.4) we introduce quantities $a = SB/y_n$ and $c = x_n/y_n$. Then the inequality in question can be written as

$$2\frac{1}{a} + \frac{c}{a} < 2a + \frac{a}{c}$$

or what is the same that

$$a^2 > \frac{c(2+c)}{2c+1}. \tag{4.5}$$

In order to prove (4.5) we use the invariance property of the Schwab-Borchardt mean

$$SB = SB(x_{n+1}, y_{n+1})$$

$(n = 0, 1, \ldots)$. Applicaiton of the first inequality in (4.2) gives

$$SB > (x_{n+1}y_{n+1}^2)^{1/3} = (A^2 y_n)^{1/3},$$

where $A := x_{n+1} = \dfrac{x_n + y_n}{2}$ and $y_{n+1} = \sqrt{Ay_n}$ (see (2.6)). Hence

$$\left(\frac{SB}{y_n}\right)^3 > \left(\frac{A}{y_n}\right)^2.$$

Since $\frac{A}{y_n} = \frac{1+c}{2}$, the last inequality can be written as

$$a^2 > \left(\frac{1+c}{2}\right)^{4/3}. \tag{4.6}$$

It has been shown in the proof of Theorem 2.6 in [12] that

$$\left(\frac{1+c}{2}\right)^{4/3} > \frac{c(2+c)}{2c+1}$$

holds for all $c > 0$. This in conjunction with (4.6) gives the desired result (4.5). This completes the proof of (4.4) when $p = 1$. Application of Lemma 3.2, with $\alpha = 2/3$, $\beta = 1/3$, $u = SB/y_n$ and $v = SB/x_n$, gives the desired result when $p > 1$. The proof is complete. $\square$

We close this section with some inequalities for bivariate means defined in Section 2. Let x and y be positive and unequal numbers and let A and G be the arithmetic and geometric means of x and y. It follows from (2.8) that $SB(A,G) = L(x,y)$. Letting in (4.1) and (4.4), $n = 0$, $x_0 = A$, and $y_0 = G$ we obtain the inequalities

$$2 < \left(\frac{G}{L}\right)^{2p} + \left(\frac{A}{L}\right)^{p} < \left(\frac{L}{G}\right)^{2p} + \left(\frac{L}{A}\right)^{p} \tag{4.7}$$

and

$$3 < 2\left(\frac{G}{L}\right)^{p} + \left(\frac{A}{L}\right)^{p} < 2\left(\frac{L}{G}\right)^{p} + \left(\frac{L}{A}\right)^{p} \tag{4.8}$$

which are valid for values of p as stated in Theorem 4.1 and Theorem 4.2. Taking into account that $P = SB(G,A)$ (see (2.8)) one can derive two inequalities for Seiffert's first mean P:

$$2 < \left(\frac{A}{P}\right)^{2p} + \left(\frac{G}{P}\right)^{p} < \left(\frac{P}{A}\right)^{2p} + \left(\frac{P}{G}\right)^{p}$$

and

$$3 < 2\left(\frac{A}{P}\right)^{p} + \left(\frac{G}{P}\right)^{p} < 2\left(\frac{P}{A}\right)^{p} + \left(\frac{P}{G}\right)^{p}$$

by interchanging in (4.7)–(4.8) A with G and by replacing L by P. In a similar fashion one can obtain two pairs of inequalities involving means T and M. We omit further details.

5. Refinements of Certain Inequalities

The purpose of this section is to obtain refinements of the Adamović and Mitrinović inequality [6]

$$(\cos x)^{1/3} < \frac{\sin x}{x} \tag{5.1}$$

$(0 < |x| < \frac{\pi}{2})$ and the Lazarević inequality [2]

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} \tag{5.2}$$

$(x \neq 0)$. These results will be employed to obtain refinements of the Wilker and Huygens inequalities (1.1)–(1.2) and also to obtain refinements of (1.3) and (1.4). Other refinements of inequalities (5.1) and (5.2) are obtained in [16].

In what follows, L will stand for the logarithmic mean $L(1+\sin x, 1-\sin x)$. Similarly the letter T will be used to denote the second Seiffert mean $T(1+\sinh x, 1-\sinh x)$, $x \in D$, where $D = \left[\ln(\sqrt{2}-1), \ln(\sqrt{2}+1)\right]$.

Our first result reads as follows.

THEOREM 5.1. *Let* $0 < |x| < \frac{\pi}{2}$. *Then*

$$(\cos x)^{1/3} < L^{1/2} < \frac{\sin x}{x}. \tag{5.3}$$

Proof. Let $u = 1 + \sin x$ and $v = 1 - \sin x$. Then $A \equiv A(u,v) = 1$ and $G \equiv G(u,v) = \cos x$. Making use of the first inequality in (4.2) with $x_0 = A$, $y_0 = G$ followed by application of (2.8) we obtain $(AG^2)^{1/3} < L$ or what is the same as

$$(\cos x)^{2/3} < L \tag{5.4}$$

which is equivalent with the first inequality in (5.3). For the proof of the second inequality in (5.3) we use the following one

$$y SB(y,x) < SB^2(x,y) \tag{5.5}$$

(see [11, (3.1)]) with $x := \cos x$ and $y = 1$ to obtain

$$SB(1, \cos x) < SB^2(\cos x, 1). \tag{5.6}$$

The first Schwab-Borchardt mean in (5.6) can be expressed in terms of L. We have

$$SB(1,\cos x) = SB(1, \sqrt{1-\sin^2 x}) = \frac{\sin x}{\tanh^{-1}(\sin x)} = L,$$

where in the last two steps we have used a second formula in (2.12) and (2.1). Since $SB(\cos x, 1) = \frac{\sin x}{x}$ (see (2.9)), the assertion follows from (5.5). The proof is complete. $\square$

COROLLARY 5.2. *The following inequalities*

$$2 < \left(\frac{\sin x}{x}\right)^2 + \frac{1}{L} < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \tag{5.7}$$

and

$$3 < 2\frac{\sin x}{x} + \frac{1}{L} < 2\frac{\sin x}{x} + \frac{\tan x}{x} \tag{5.8}$$

are valid provided $0 < |x| < \frac{\pi}{2}$.

Proof. The first inequalities in (5.7) and (5.8) can be obtained with the aid of

$$1 < \left(\frac{\sin x}{x}\right)^2 \frac{1}{L}, \tag{5.9}$$

which is an obvious consequence of the second inequality in (5.3). Application of the inequality of the arithmetic and geometric means first with equal weights followed by use of this inequality with weights 2/3 and 1/3 gives the asserted results. For the proof of the second inequalities in (5.7) and (5.8) it suffices to show that

$$\frac{x}{\tan x} < L.$$

To obtain the last inequality we write (5.1) as $\frac{x}{\sin x} < (\cos x)^{-1/3}$. Multiplying both sides by $\cos x$ and next using (5.4) we obtain the desired result. $\square$

The hyperbolic counterpart of Theorem 5.1 is contained in the following.

THEOREM 5.3. *Let $x \in D$. Then*

$$(\cosh x)^{1/3} < T^{1/2} < \frac{\sinh x}{x}. \tag{5.10}$$

Proof. The assumption $x \in D$ implies that $|\sinh x| \leqslant 1$. Let $u = 1 + \sinh x$ and $v = 1 - \sinh x$. Then $A \equiv A(u,v) = 1$ and $Q \equiv Q(u,v) = \cosh x$. The first inequality in (5.10) follows from $(AQ^2)^{1/3} < T$ which is a special case of the left inequality in (4.2) when $n = 0$, $x_0 = A$ and $y_0 = Q$ and the formula $SB(A,Q) = T$ (see (2.8)). For the proof of the second inequality in (5.10) we let in (5.5) $x := \cosh x$ and $y = 1$ to obtain

$$SB(1, \cosh x) < SB^2(\cosh x, 1). \tag{5.11}$$

The left side of (5.11) is equal to T and this follows from

$$SB(1, \cosh x) = SB(1, \sqrt{1 + \sinh^2 x}) = \frac{\sinh x}{\tan^{-1}(\sinh x)} = T$$

where in the last two steps we have used the second part of (2.11) and (2.3). Taking into account that $SB(\cosh x, 1) = \frac{\sinh x}{x}$ (see this first formula in (2.10)) we obtain the assertion. This completes the proof. $\square$

COROLLARY 5.4. *Refinements of inequalities* (1.3) *and* (1.4)

$$2 < \left(\frac{\sinh x}{x}\right)^2 + \frac{1}{T} < \left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} \tag{5.12}$$

and

$$3 < 2\frac{\sinh x}{x} + \frac{1}{T} < 2\frac{\sinh x}{x} + \frac{\tanh x}{x} \tag{5.13}$$

are valid provided $x \in D$.

Since the proof of the last two inequalities is very similar to the proof of (5.7) and (5.8), it is not included here.

We close this section with a remark that the functions $\frac{\tan x}{x}$ and $\frac{\tanh x}{x}$ are bounded from above by $M^2(1+\sin x, 1-\sin x)$ and $P^2(1+\tanh x, 1-\tanh x)$, respectively, where the means M and P are defined in (2.5) and (2.2), respectively. These bounds are obtained using (5.5) with $x := \sec x$, $y = 1$ and $x := \operatorname{sech} x$, $y = 1$, respectively. The appropriate formulas from Section 2 should be applied to obtain the desired results. We omit further details.

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(Received April 4, 2011)

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