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ON GOSPERS FORMULA FOR THE GAMMA FUNCTION

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Abstract. The aim of this paper is to establish a double inequality related to Gosper formula for approximation of big factorials

1. Introduction

It is of general knowledge that one of the most used formula for approximation of the factorial function is the following [4]

$$n! \approx \sqrt{2\pi} \cdot n^{n+1/2} e^{-n},$$

now known as Stirling's formula.

The Euler's gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt\ , \quad x > 0,$$

which can, except for the values $-1,\ -2,\ -3, \ldots$, be continued to the whole complex plane. It is the natural extension of the factorial function, since $\Gamma(n+1) = n!$, for $n = 1, 2, 3, \ldots$. The subject of evaluation of large factorials has a long history which can be traced back to Abraham de Moivre (1667–1754), James Stirling (1692–1770), or Leonhard Euler (1707–1783).

If in probabilities, applied statistics, or statistical physics, such approximation is satisfactory, in pure mathematics, more precise estimates are necessary. As a consequence, there have been a lot of variety of approaches to Stirling's formula, ranging from elementary to advanced methods.

A slightly more accurate estimate than Stirling's formula is the following due to R. W. Gosper [2]

$$n! \sim \sqrt{\pi} \left(\frac{n}{e}\right)^n \sqrt{2n + \frac{1}{3}}. \tag{1.1}$$

We prove in this paper the following double inequality related to the Gosper formula.

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THEOREM 1. *For every* $x \in [1,\infty)$, *we have*

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt{2x+\alpha} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt{2x+\beta},$$

where $\alpha = \frac{1}{3} = 0.33333...$ *and* $\beta = \sqrt[3]{\frac{391}{30}} - 2 = 0.35334...$.

2. The Results

Other much used formula for estimating the large factorials is the following

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}} = \rho_n,$$

now known as Ramanujan formula. Actually, there is the following record in [3, p. 339]:

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6} < \Gamma(x+1) <$$

$$< \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \quad x \geqslant 1. \tag{2.1}$$

For other details, [1, p. 48 (Question 754)] can be consulted. In order to prove our results, we give the following lemmas.

LEMMA 1. *The function* $g : [1,\infty) \to \mathbb{R}$ *given by*

$$g(x) = \left(8x^3+4x^2+x+\frac{1}{30}\right)^{\frac{1}{3}} - 2x$$

is strictly decreasing with $g(1) = \beta$. *In consequence,* $g(x) \leqslant g(1)$, *for every* $x \geqslant 1$.

Proof. By direct computation,

$$g'(x) = \frac{24x^2+8x+1}{3\left(8x^3+4x^2+x+\frac{1}{30}\right)^{\frac{2}{3}}} - 2.$$

As

$$\frac{\left(24x^2+8x+1\right)^3}{27\left(8x^3+4x^2+x+\frac{1}{30}\right)^2} - 8 = -\frac{4\left(14400x^4+4480x^3+240x^2-240x-19\right)}{3\left(240x^3+120x^2+30x+1\right)^2},$$

it results $g'(x) < 0$, for every $x \geqslant 1$, so g is strictly decreasing. $\square$

LEMMA 2. *The function $h:[1,\infty)\to\mathbb{R}$ given by*

$$h(x)=\left(8x^3+4x^2+x+\frac{1}{100}\right)^{\frac{1}{3}}-2x$$

is strictly decreasing with $h(\infty)=\alpha$. In consequence, $h(x)\leqslant h(1)$, for every $x\geqslant 1$.

Proof. By direct computation,

$$h'(x)=\frac{24x^2+8x+1}{3\left(8x^3+4x^2+x+\frac{1}{100}\right)^{\frac{2}{3}}}-2.$$

We have

$$\frac{\left(24x^2+8x+1\right)^3}{27\left(8x^3+4x^2+x+\frac{1}{100}\right)^2}-8=-\frac{8P(x)}{27\left(800x^3+400x^2+100x+1\right)^2},$$

where $P(x)=720000x^4+123200x^3-38\,400x^2-24\,600x-1223$. All the coefficients of the polynomial $P(x+1)$ are positive, so $P(x)>0$, for every $x\geqslant 1$.

It results $h'(x)<0$, for every $x\geqslant 1$, so h is strictly decreasing. □

Now we are in the position to prove our main result.

Proof of Theorem 1. Using the right-hand side inequality (2.1), we get

$$\frac{\Gamma(x+1)}{\sqrt{\pi}\left(\frac{x}{e}\right)^x}<\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}=\sqrt{2x+g(x)}\leqslant\sqrt{2x+g(1)},$$

where the last inequality follows by the monotonicity of the function g.

Finally, we use the left-hand side inequality (2.1), to get

$$\frac{\Gamma(x+1)}{\sqrt{\pi}\left(\frac{x}{e}\right)^x}>\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6}=\sqrt{2x+h(x)}\geqslant\sqrt{2x+h(\infty)},$$

where the last inequality follows by the monotonicity of the function h. □

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