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THE OPTIMAL POWER MEAN BOUNDS FOR TWO CONVEX COMBINATIONS OF $A-G-H$ MEANS

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Abstract. For $p \in \mathbb{R}$, let $M_p(a,b)$ denote the usual power mean of order p of positive real numbers a and b, and let $A = M_1$, $G = M_0$ and $H = M_{-1}$. We prove that the inequalities $M_0(a,b) \leqslant \frac{1}{3}[A(a,b)+G(a,b)+H(a,b)] \leqslant M_{\frac{\ln 2}{\ln 6}}(a,b)$ and $M_{-\frac{1}{6}}(a,b) \leqslant \frac{1}{2}[He(a,b)+H(a,b)] \leqslant M_{\frac{\ln 2}{\ln 6}}(a,b)$ hold for all positive real numbers a and b, with strict inequality for $a \neq b$, and that the orders of power means involved are optimal.

1. Introduction

Let a and b be positive real numbers. For $p \in \mathbb{R}$, the power mean of order p of numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

In particular, for $p = 1$, $p = 0$, and $p = -1$ we respectively get the arithmetic, the geometric and the harmonic mean of a and b,

$$A(a,b) = M_1(a,b) = \frac{a+b}{2}, \quad G(a,b) = M_0(a,b) = \lim_{p\to 0} M_p(a,b) = \sqrt{ab}$$

and

$$H(a,b) = M_{-1}(a,b) = \frac{2ab}{a+b}.$$

We also consider the Heronian mean of a and b, defined as

$$He(a,b) = \frac{1}{3}(a+\sqrt{ab}+b) = \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b).$$

It is obvious that $M_p(a,a) = He(a,a) = a$, for all $p \in \mathbb{R}$ and $a \in \mathbb{R}^+$, and it is well-known that the function $p \mapsto M_p(a,b)$ is strictly increasing on $\mathbb{R}$ for any fixed

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$a,b \in \mathbb{R}^+$, $a \neq b$. Moreover, $\min\{a,b\} \leqslant M_p(a,b) \leqslant \max\{a,b\}$ holds for all $p \in \mathbb{R}$ and $a,b \in \mathbb{R}^+$, with equality only if $a = b$. Furthermore, in [1], Alzer and Janous proved that

$$M_{\frac{\ln 2}{\ln 3}}(a,b) \leqslant He(a,b) \leqslant M_{\frac{2}{3}}(a,b), \quad a,b \in \mathbb{R}^+, \tag{1.1}$$

Neuman and Sándor [6] obtained the double inequality

$$He(a,b) \leqslant M_{\frac{2}{3}}(a,b) \leqslant \frac{3}{2\sqrt{2}} He(a,b), \quad a,b \in \mathbb{R}^+, \tag{1.2}$$

while the inequalities

$$M_{-\frac{1}{3}}(a,b) \leqslant \frac{2}{3} G(a,b) + \frac{1}{3} H(a,b) \leqslant M_0(a,b), \quad a,b \in \mathbb{R}^+, \tag{1.3}$$

and

$$M_{-\frac{2}{3}}(a,b) \leqslant \frac{1}{3} G(a,b) + \frac{2}{3} H(a,b) \leqslant M_0(a,b), \quad a,b \in \mathbb{R}^+, \tag{1.4}$$

are due to Chu and Xia [3]. All inequalities (1.1) – (1.4) are sharp, that is, with equality only for $a = b$, and the orders of power means appearing on their respective left-hand and right-hand sides are the best (greatest or least) possible.

In this paper, we obtain the optimal lower and upper bounds for the bivariate means $\frac{1}{3}[A(a,b) + G(a,b) + H(a,b)]$ and $\frac{1}{2}[He(a,b) + H(a,b)]$ in terms of a power mean. More precisely, we find the greatest real values of indices q and s and the least real values of indices p and r, such that the inequalities

$$M_q(a,b) \leqslant \frac{1}{3}[A(a,b) + G(a,b) + H(a,b)] \leqslant M_p(a,b) \tag{1.5}$$

and

$$M_s(a,b) \leqslant \frac{1}{2}[He(a,b) + H(a,b)] \leqslant M_r(a,b) \tag{1.6}$$

hold for all positive real numbers a and b.

2. Arithmetic mean of $A(a,b)$, $G(a,b)$, and $H(a,b)$

We start with the optimal lower and upper bound for the convex combination $\frac{1}{3}[A(a,b) + G(a,b) + H(a,b)]$, that is, we find the greatest value $q \in \mathbb{R}$ and the least value $p \in \mathbb{R}$, such that the double inequality (1.5) holds for all positive real numbers a and b.

In order to get this result, we make use of the following technical lemma.

LEMMA 2.1. *Let $p = \frac{\ln 2}{\ln 6}$ and the function $g : \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$g(x) = \frac{x^5 + x^4 + 2x^3 + 2x^2 + 5x + 1}{x^5 + 5x^4 + 2x^3 + 2x^2 + x + 1} - x^{2p-1}. \tag{2.1}$$

Then there exists a unique $x_0 \in (0,1)$, such that $g(x_0) = 0$, $g(x) < 0$ for $x \in (0,x_0)$, and $g(x) > 0$ for $x \in (x_0,1)$.

Proof. First, notice that both expressions $g_1(x) = x^5 + x^4 + 2x^3 + 2x^2 + 5x + 1$ and $g_2(x) = x^5 + 5x^4 + 2x^3 + 2x^2 + x + 1$ take positive values for $x \in \mathbb{R}^+$, so we can define the function $h : \mathbb{R}^+ \to \mathbb{R}$, $h(x) = \ln(g_1(x)) - \ln(g_2(x)) - (2p-1)\ln x$. Then $h'(x) = \frac{k(x)}{x \cdot g_1(x) \cdot g_2(x)}$, where k is a polynomial with real coefficients, given by

$$\begin{aligned} k(x) = {} & (1-2p)x^{10} + 2(5-6p)x^9 + 9(1-2p)x^8 + 8(1-4p)x^7 \\ & -2(22p+5)x^6 - 36(2p+1)x^5 - 2(22p+5)x^4 + 8(1-4p)x^3 \\ & +9(1-2p)x^2 + 2(5-6p)x + 1 - 2p. \end{aligned}$$

For $p = \frac{\ln 2}{\ln 6}$, we respectively have $1-2p \approx 0.23$, $2(5-6p) \approx 5.36$, $9(1-2p) \approx 2.04$, $8(1-4p) \approx -4.38$, $-2(22p+5) \approx -27.02$, and $-36(2p+1) \approx -63.85$, so the number of sign changes between consecutive coefficients of k is 2. According to Descartes' rule of signs, the polynomial k has two or zero positive roots. Since $k(0) = 1-2p \approx 0.23 > 0$, $k(1) = -288p \approx -111.41 < 0$, and $k(3) = 315\,904 - 596\,576p \approx 85\,116.9 > 0$, we conclude that k has exactly two positive roots, one less than 1 and the other greater than 1. Moreover, if $x_1 \in (0,1)$ is such that $k(x_1) = 0$, then $k(x) > 0$ for $x \in (0, x_1)$, $k(x) < 0$ for $x \in (x_1, 1)$, and, consequently, $h'(x) > 0$ for $x \in (0, x_1)$, $h'(x_1) = 0$, and $h'(x) < 0$ for $x \in (x_1, 1)$. Hence, the function h is strictly increasing on $(0, x_1]$, strictly decreasing on $[x_1, 1)$, and $\max_{x \in (0,1)} h(x) = h(x_1)$. This, together with $\lim_{x \to 0+} h(x) = -\infty$ and $h(1) = 0$, implies that $h(x_1) > 0$ and that there exists a unique $x_0 \in (0, x_1)$, such that $h(x_0) = 0$, $h(x) < 0$ for $x \in (0, x_0)$, and $h(x) > 0$ for $x \in (x_0, 1)$. The lemma follows from the monotonicity of the exponential function $t \mapsto e^t$. $\square$

Now, we are ready to state and prove the main result in this section.

THEOREM 2.1. *Inequality*

$$M_0(a,b) \leqslant \frac{1}{3}[A(a,b) + G(a,b) + H(a,b)] \leqslant M_{\frac{\ln 2}{\ln 6}}(a,b) \tag{2.2}$$

holds for all $a, b \in \mathbb{R}^+$, with strict inequality for $a \neq b$. The constants 0 and $\frac{\ln 2}{\ln 6}$ are the best possible.

Proof. Denote $x = \sqrt{\frac{a}{b}}$. Then

$$\begin{aligned} \frac{1}{3}[A(a,b) + G(a,b) + H(a,b)] - M_0(a,b) &= \frac{1}{3}[A(a,b) - 2G(a,b) + H(a,b)] \\ = b\left[\frac{x^2+1}{6} - \frac{2x}{3} + \frac{2x^2}{3(x^2+1)}\right] &= \frac{b(x-1)^4}{6(x^2+1)} \geqslant 0, \end{aligned} \tag{2.3}$$

so the first inequality in (2.2) is proved. Obviously, equality in (2.3) holds only for $x = 1$, that is, for $a = b$. In that case $A(a,a) = G(a,a) = H(a,a) = a$.

Next, we prove that the parameter $q = 0$ is the best possible, that is, that it cannot be replaced with any larger parameter. For arbitrary $\varepsilon > 0$ and $0 < |t| < 1$ we have

$$\begin{aligned}
&\left[M_\varepsilon((1+t)^2,1)\right]^\varepsilon - \left[\frac{1}{3}A((1+t)^2,1) + \frac{1}{3}G((1+t)^2,1) + \frac{1}{3}H((1+t)^2,1)\right]^\varepsilon \\
&= \frac{(1+t)^{2\varepsilon}+1}{2} - \left(\frac{1+2t+\frac{3t^2}{2}+\frac{t^3}{2}+\frac{t^4}{12}}{1+t+\frac{t^2}{2}}\right)^\varepsilon = \frac{h_\varepsilon(t)}{2\left(1+t+\frac{t^2}{2}\right)^\varepsilon},
\end{aligned} \tag{2.4}$$

where

$$h_\varepsilon(t) = \left(1+t+\frac{t^2}{2}\right)^\varepsilon \left[1+(1+t)^{2\varepsilon}\right] - 2\left(1+2t+\frac{3t^2}{2}+\frac{t^3}{2}+\frac{t^4}{12}\right)^\varepsilon.$$

If $t \longrightarrow 0$, by writing the binomial series we obtain

$$\begin{aligned}
h_\varepsilon(t) &= \left[1+\varepsilon t+\frac{\varepsilon^2}{2}t^2+o(t^2)\right]\cdot\left[2+2\varepsilon t+\varepsilon(2\varepsilon-1)t^2+o(t^2)\right] \\
&\quad -2\left[1+2\varepsilon t+\frac{\varepsilon(16\varepsilon-5)}{2}t^2+o(t^2)\right] = \varepsilon(4-11\varepsilon)t^2+o(t^2).
\end{aligned} \tag{2.5}$$

Therefore, (2.4) and (2.5) yield that for any $\varepsilon \in \left(0, \frac{4}{11}\right)$ there exists $\delta(\varepsilon) \in (0,1)$ such that

$$M_\varepsilon((1+t)^2,1) > \frac{1}{3}\left[A((1+t)^2,1)+G((1+t)^2,1)+H((1+t)^2,1)\right]$$

for $t \in (0,\delta(\varepsilon))$. Thus, the parameter $q = 0$ cannot be enlarged so that the first inequality in (1.5) still holds.

Now, we prove the second inequality in (2.2). For any $p \in \mathbb{R},\ p \neq 0$, we have

$$\begin{aligned}
&\frac{1}{3}[A(a,b)+G(a,b)+H(a,b)] - M_p(a,b) \\
&= b\left[\frac{x^2+1}{6}+\frac{x}{3}+\frac{2x^2}{3(x^2+1)}\right] - b\left(\frac{x^{2p}+1}{2}\right)^{\frac{1}{p}} \\
&= b\left[\frac{x^4+2x^3+6x^2+2x+1}{6(x^2+1)} - \left(\frac{x^{2p}+1}{2}\right)^{\frac{1}{p}}\right].
\end{aligned} \tag{2.6}$$

Let $p = \frac{\ln 2}{\ln 6}$. To get the desired result, we need to prove that the function from the last line of (2.6) takes only negative values on $\mathbb{R}^+$. Since all bivariate power means are symmetric, without loss of generality we can assume that $a \leqslant b$, that is, that $x \in (0,1]$. For $x \in [0,\infty)$, define

$$f(x) = \ln(x^4+2x^3+6x^2+2x+1) - \ln(x^2+1) - \frac{1}{p}\ln(x^{2p}+1) + \frac{1}{p}\ln 2 - \ln 6.$$

Then $f'(x) = g(x) \cdot l(x)$, where the function g is given by (2.1) and

$$l(x) = \frac{2(x^5+5x^4+2x^3+2x^2+x+1)}{(x^4+2x^3+6x^2+2x+1)(x^2+1)(x^{2p}+1)} > 0, \ \ x \in \mathbb{R}^+.$$

Lemma 2.1 implies that there exists a unique $x_0 \in (0,1)$, such that $f'(x_0) = 0$, $f'(x) < 0$ for $x \in (0,x_0)$, and $f'(x) > 0$ for $x \in (x_0,1)$. Hence, the function f is strictly decreasing on $(0,x_0]$ and strictly increasing on $[x_0,1)$, so from $f(0) = f(1) = 0$ we obtain that $f(x) < 0$ for $x \in (0,1)$. Finally, the monotonicity of the exponential function $t \mapsto e^t$ gives that (2.6) is less than or equal to 0 for all $a,b \in \mathbb{R}^+$, with equality only for $x = 1$, that is, for $a = b$. In that case, both-hand sides of the second inequality in (2.2) are equal to a.

Finally, notice that $p = \frac{\ln 2}{\ln 6} \approx 0.39 > 0$ and

$$\lim_{t\to 0+} \left\{ \frac{1}{3}[A(t^2,1)+G(t^2,1)+H(t^2,1)] - M_p(t^2,1) \right\} = 0,$$

while for any $\varepsilon \in (0,p)$ we have

$$\begin{aligned} &\lim_{t\to 0+} \left\{ \frac{1}{3}[A(t^2,1)+G(t^2,1)+H(t^2,1)] - M_{p-\varepsilon}(t^2,1) \right\} \\ &= \lim_{t\to 0+} \left[\frac{t^4+2t^3+6t^2+2t+1}{6(t^2+1)} - 2^{\frac{1}{\varepsilon-p}} \left(t^{2(p-\varepsilon)}+1 \right)^{\frac{1}{p-\varepsilon}} \right] \\ &= \frac{1}{6} - 2^{\frac{1}{\varepsilon-p}} > 0. \end{aligned}$$

Therefore, the parameter $\frac{\ln 2}{\ln 6}$ cannot be diminished, that is, it is optimal for the second inequality in (1.5). $\square$

3. Arithmetic mean of $He(a,b)$ and $H(a,b)$

Now, we derive the best possible lower and upper bound for the arithmetic mean of the Heronian mean and the harmonic mean of two positive real numbers in terms of a power mean, that is, we obtain the greatest value $s \in \mathbb{R}$ and the least value $r \in \mathbb{R}$, such that the double inequality (1.6) holds for all positive real numbers a and b.

Similarly to the previous section, the following lemma provides an important argument for the proof of our result.

LEMMA 3.1. *Let $r = \frac{\ln 2}{\ln 6}$ and the function $g : \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$g(x) = \frac{2x^5+x^4+4x^3+2x^2+14x+1}{x^5+14x^4+2x^3+4x^2+x+2} - x^{2r-1}. \tag{3.1}$$

Then there exists a unique $x_* \in (0,1)$, such that $g(x_*) = 0$, $g(x) < 0$ for $x \in (0,x_*)$, and $g(x) > 0$ for $x \in (x_*,1)$.

Proof. Observe that $g_1(x) = 2x^5 + x^4 + 4x^3 + 2x^2 + 14x + 1 > 0$ and $g_2(x) = x^5 + 14x^4 + 2x^3 + 4x^2 + x + 2 > 0$ for $x \in \mathbb{R}^+$, so the function $h : \mathbb{R}^+ \to \mathbb{R}$, $h(x) = \ln(g_1(x)) - \ln(g_2(x)) - (2r-1)\ln x$ is well-defined. Calculating its derivative, we get $h'(x) = \frac{2k(x)}{x \cdot g_1(x) \cdot g_2(x)}$, where k is a polynomial with real coefficients, given by

$$\begin{aligned} k(x) = \ & (1-2r)x^{10} + (28-29r)x^9 + 11(1-2r)x^8 + 4(4-17r)x^7 \\ & -4(14r+5)x^6 - 6(37r+28)x^5 - 4(14r+5)x^4 + 4(4-17r)x^3 \\ & +11(1-2r)x^2 + (28-29r)x + 1 - 2r. \end{aligned}$$

For $r = \frac{\ln 2}{\ln 6}$, we respectively have $1-2r \approx 0.23$, $28-29r \approx 16.78$, $11(1-2r) \approx 2.49$, $4(4-17r) \approx -10.31$, $-4(14r+5) \approx -41.66$, and $-6(37r+28) \approx -253.88$, so there are two sign changes between consecutive coefficients of k. Therefore, Descartes' rule of signs gives that the polynomial k has two or zero positive roots. Since $k(0) = 1 - 2r \approx 0.23 > 0$, $k(1) = -576r - 96 \approx -318.83 < 0$, and $k(3) = 660\,928 - 1\,083\,392r \approx 241\,814.8 > 0$, we see that the number of positive roots of k is exactly 2 and that one them is less than 1 while the other one is greater than 1. Moreover, if $\tilde{x} \in (0,1)$ is a positive root of k, then $k(x) > 0$ for $x \in (0,\tilde{x})$, $k(x) < 0$ for $x \in (\tilde{x},1)$, and, consequently, $h'(x) > 0$ for $x \in (0,\tilde{x})$, $h'(\tilde{x}) = 0$, and $h'(x) < 0$ for $x \in (\tilde{x},1)$. Thus, the function h is strictly increasing on $(0,\tilde{x}]$, strictly decreasing on $[\tilde{x},1)$, and $\max_{x\in(0,1)} h(x) = h(\tilde{x})$. Considering that $\lim_{x\to 0+} h(x) = -\infty$ and $h(1) = 0$, we conclude that $h(\tilde{x}) > 0$ and that there is a unique $x_* \in (0,\tilde{x})$, such that $h(x_*) = 0$, $h(x) < 0$ for $x \in (0,x_*)$, and $h(x) > 0$ for $x \in (x_*,1)$. The proof is completed by taking into account the monotonicity of the exponential function $t \mapsto e^t$. $\square$

The result announced above is given in the following theorem.

THEOREM 3.1. *Inequality*

$$M_{-\frac{1}{6}}(a,b) \leqslant \frac{1}{2}[He(a,b) + H(a,b)] \leqslant M_{\frac{\ln 2}{\ln 6}}(a,b) \tag{3.2}$$

holds for all $a,b \in \mathbb{R}^+$, with strict inequality for $a \neq b$. The constants 0 *and* $\frac{\ln 2}{\ln 6}$ *are the best possible.*

Proof. First, observe that $\frac{1}{2}[He(a,b)+H(a,b)] = \frac{1}{3}A(a,b) + \frac{1}{6}G(a,b) + \frac{1}{2}H(a,b)$, for all $a,b \in \mathbb{R}^+$. If $y = \sqrt[6]{\frac{a}{b}}$, we have

$$\begin{aligned} & \frac{1}{2}[He(a,b) + H(a,b)] - M_{-\frac{1}{6}}(a,b) \\ & = \frac{b}{6}\left[(y^6+1) + y^3 + \frac{6y^6}{y^6+1} - \frac{384y^6}{(y+1)^6}\right] = \frac{b(y-1)^4}{6(y^6+1)(y+1)^6}\big(y^{14} + 10y^{13} \\ & \quad + 49y^{12} + 161y^{11} + 410y^{10} + 881y^9 + 1\,304y^8 + 1\,472y^7 + 1\,304y^6 + 881y^5 \\ & \quad + 410y^4 + 161y^3 + 49y^2 + 10y + 1\big) \geqslant 0 \end{aligned}$$

for all $y \in \mathbb{R}^+$, with equality if and only if $y=1$, that is, $a=b$, so the first inequality in (3.2) is proved. In the case $a=b$, both-hand sides of the first inequality in (3.2) are equal to a.

To prove that the parameter $s=-\frac{1}{6}$ is optimal, take arbitrary $\varepsilon \in \left(0,\frac{1}{6}\right)$ and $0<|t|<1$. Then

$$\begin{aligned}
&\left[M_{-\frac{1}{6}+\varepsilon}((1+t)^2,1)\right]^{\frac{1}{6}-\varepsilon}-\left[\frac{1}{2}He((1+t)^2,1)+\frac{1}{2}H((1+t)^2,1)\right]^{\frac{1}{6}-\varepsilon}\\
&\quad=\frac{2(1+t)^{\frac{1}{3}-2\varepsilon}}{1+(1+t)^{\frac{1}{3}-2\varepsilon}}-\left(\frac{1+2t+\frac{17t^2}{12}+\frac{t^3}{2}+\frac{t^4}{12}}{1+t+\frac{t^2}{2}}\right)^{\frac{1}{6}-\varepsilon}\\
&\quad=\frac{h_\varepsilon(t)}{\left(1+t+\frac{t^2}{2}\right)^{\frac{1}{6}-\varepsilon}\left[1+(1+t)^{\frac{1}{3}-2\varepsilon}\right]},
\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}
h_\varepsilon(t)&=2(1+t)^{\frac{1}{3}-2\varepsilon}\left(1+t+\frac{t^2}{2}\right)^{\frac{1}{6}-\varepsilon}\\
&\quad-\left[1+(1+t)^{\frac{1}{3}-2\varepsilon}\right]\left(1+2t+\frac{17t^2}{12}+\frac{t^3}{2}+\frac{t^4}{12}\right)^{\frac{1}{6}-\varepsilon}.
\end{aligned}$$

By writing the binomial series, for $t\to 0$, we get

$$\begin{aligned}
h_\varepsilon(t)&=2\left[1+\frac{1-6\varepsilon}{3}t+\frac{(6\varepsilon-1)(3\varepsilon+1)}{9}t^2+o(t^2)\right]\\
&\quad\times\left[1+\frac{1-6\varepsilon}{6}t+\frac{(1-6\varepsilon)^2}{72}t^2+o(t^2)\right]\\
&\quad-2\left[1+\frac{1-6\varepsilon}{6}t+\frac{(6\varepsilon-1)(3\varepsilon+1)}{18}t^2+o(t^2)\right]\\
&\quad\times\left[1+\frac{1-6\varepsilon}{3}t+\frac{(6\varepsilon-1)(8\varepsilon+1)}{24}t^2+o(t^2)\right]=\frac{(1-6\varepsilon)\varepsilon}{6}t^2+o(t^2),
\end{aligned}\tag{3.4}$$

so (3.3) and (3.4) imply that for any $\varepsilon\in\left(0,\frac{1}{6}\right)$ there exists $\delta(\varepsilon)\in(0,1)$ such that

$$M_{-\frac{1}{6}+\varepsilon}((1+t)^2,1)>\frac{1}{2}\left[He((1+t)^2,1)+H((1+t)^2,1)\right]$$

for $t\in(0,\delta(\varepsilon))$. This means that the parameter $s=-\frac{1}{6}$ cannot be enlarged so that the first inequality in (1.6) still holds.

It is left to prove the second inequality in (3.2). Let $x=\sqrt{\frac{a}{b}}$. Then for any $r\in\mathbb{R}$,

$r \neq 0$, we have

$$
\begin{aligned}
&\frac{1}{2}[He(a,b)+H(a,b)]-M_r(a,b)\\
&\quad = b\left[\frac{x^2+1}{6}+\frac{x}{6}+\frac{x^2}{x^2+1}\right]-b\left(\frac{x^{2r}+1}{2}\right)^{\frac{1}{r}}\\
&\quad = b\left[\frac{x^4+x^3+8x^2+x+1}{6(x^2+1)}-\left(\frac{x^{2r}+1}{2}\right)^{\frac{1}{r}}\right]. \qquad (3.5)
\end{aligned}
$$

We need to prove that the last line of (3.5) is negative on $\mathbb{R}^+$ for $r=\frac{\ln 2}{\ln 6}$. Without loss of generality, assume that $a \leqslant b$, that is, $x \in (0,1]$, and define

$$f(x)=\ln(x^4+x^3+8x^2+x+1)-\ln(x^2+1)-\frac{1}{r}\ln(x^{2r}+1)+\frac{1}{r}\ln 2-\ln 6.$$

Then $f'(x)=g(x)\cdot l(x)$, where the function g is given by (3.1) and

$$l(x)=\frac{x^5+14x^4+2x^3+4x^2+x+2}{(x^4+x^3+8x^2+x+1)(x^2+1)(x^{2r}+1)}>0, \quad x \in \mathbb{R}^+.$$

According to Lemma 3.1, there exists a unique $x_* \in (0,1)$, such that $f'(x_*)=0$, $f'(x)<0$ for $x \in (0,x_*)$, and $f'(x)>0$ for $x \in (x_*,1)$. Therefore, the function f is strictly decreasing on $(0,x_*]$ and strictly increasing on $[x_*,1)$, so $f(0)=f(1)=0$ provides that $f(x)<0$ for $x \in (0,1)$. The monotonicity of the exponential function $t \mapsto e^t$ implies that (3.5) is less than or equal to 0 for all $a,b \in \mathbb{R}^+$, with equality only for $x=1$, that is, for $a=b$. In that case, both-hand sides of the second inequality in (3.2) are equal to a.

Finally, since for $r=\frac{\ln 2}{\ln 6}$ and any $\varepsilon \in (0,p)$ we have

$$\lim_{t\to 0+}\left\{\frac{1}{2}[He(t^2,1)+H(t^2,1)]-M_r(t^2,1)\right\}=0$$

and

$$
\begin{aligned}
&\lim_{t\to 0+}\left\{\frac{1}{2}[He(t^2,1)+H(t^2,1)]-M_{r-\varepsilon}(t^2,1)\right\}\\
&\quad =\lim_{t\to 0+}\left[\frac{t^4+t^3+8t^2+t+1}{6(t^2+1)}-2^{\frac{1}{\varepsilon-r}}\left(t^{2(r-\varepsilon)}+1\right)^{\frac{1}{r-\varepsilon}}\right]\\
&\quad =\frac{1}{6}-2^{\frac{1}{\varepsilon-r}}>0,
\end{aligned}
$$

the parameter $\frac{\ln 2}{\ln 6}$ is the best possible for the second inequality in (1.6). □

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