

THE OPTIMAL POWER MEAN BOUNDS FOR TWO CONVEX COMBINATIONS OF A - G - H MEANS

ALEKSANDRA ČIŽMEŠIJA

(Communicated by J. Pečarić)

Abstract. For $p \in \mathbb{R}$, let $M_p(a, b)$ denote the usual power mean of order p of positive real numbers a and b , and let $A = M_1$, $G = M_0$ and $H = M_{-1}$. We prove that the inequalities $M_0(a, b) \leq \frac{1}{3}[A(a, b) + G(a, b) + H(a, b)] \leq M_{\frac{2}{\ln 6}}(a, b)$ and $M_{-\frac{1}{6}}(a, b) \leq \frac{1}{2}[He(a, b) + H(a, b)] \leq M_{\frac{2}{\ln 6}}(a, b)$ hold for all positive real numbers a and b , with strict inequality for $a \neq b$, and that the orders of power means involved are optimal.

1. Introduction

Let a and b be positive real numbers. For $p \in \mathbb{R}$, the power mean of order p of numbers a and b is defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

In particular, for $p = 1$, $p = 0$, and $p = -1$ we respectively get the arithmetic, the geometric and the harmonic mean of a and b ,

$$A(a, b) = M_1(a, b) = \frac{a+b}{2}, \quad G(a, b) = M_0(a, b) = \lim_{p \rightarrow 0} M_p(a, b) = \sqrt{ab}$$

and

$$H(a, b) = M_{-1}(a, b) = \frac{2ab}{a+b}.$$

We also consider the Heronian mean of a and b , defined as

$$He(a, b) = \frac{1}{3}(a + \sqrt{ab} + b) = \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b).$$

It is obvious that $M_p(a, a) = He(a, a) = a$, for all $p \in \mathbb{R}$ and $a \in \mathbb{R}^+$, and it is well-known that the function $p \mapsto M_p(a, b)$ is strictly increasing on \mathbb{R} for any fixed

Mathematics subject classification (2010): 26E60, 26D15.

Keywords and phrases: Arithmetic mean, geometric mean, harmonic mean, power mean, Heronian mean, sharp inequality.

$a, b \in \mathbb{R}^+$, $a \neq b$. Moreover, $\min\{a, b\} \leq M_p(a, b) \leq \max\{a, b\}$ holds for all $p \in \mathbb{R}$ and $a, b \in \mathbb{R}^+$, with equality only if $a = b$. Furthermore, in [1], Alzer and Janous proved that

$$M_{\frac{\ln 2}{\ln 3}}(a, b) \leq He(a, b) \leq M_{\frac{2}{3}}(a, b), \quad a, b \in \mathbb{R}^+, \quad (1.1)$$

Neuman and Sándor [6] obtained the double inequality

$$He(a, b) \leq M_{\frac{2}{3}}(a, b) \leq \frac{3}{2\sqrt{2}}He(a, b), \quad a, b \in \mathbb{R}^+, \quad (1.2)$$

while the inequalities

$$M_{-\frac{1}{3}}(a, b) \leq \frac{2}{3}G(a, b) + \frac{1}{3}H(a, b) \leq M_0(a, b), \quad a, b \in \mathbb{R}^+, \quad (1.3)$$

and

$$M_{-\frac{2}{3}}(a, b) \leq \frac{1}{3}G(a, b) + \frac{2}{3}H(a, b) \leq M_0(a, b), \quad a, b \in \mathbb{R}^+, \quad (1.4)$$

are due to Chu and Xia [3]. All inequalities (1.1) – (1.4) are sharp, that is, with equality only for $a = b$, and the orders of power means appearing on their respective left-hand and right-hand sides are the best (greatest or least) possible.

In this paper, we obtain the optimal lower and upper bounds for the bivariate means $\frac{1}{3}[A(a, b) + G(a, b) + H(a, b)]$ and $\frac{1}{2}[He(a, b) + H(a, b)]$ in terms of a power mean. More precisely, we find the greatest real values of indices q and s and the least real values of indices p and r , such that the inequalities

$$M_q(a, b) \leq \frac{1}{3}[A(a, b) + G(a, b) + H(a, b)] \leq M_p(a, b) \quad (1.5)$$

and

$$M_s(a, b) \leq \frac{1}{2}[He(a, b) + H(a, b)] \leq M_r(a, b) \quad (1.6)$$

hold for all positive real numbers a and b .

2. Arithmetic mean of $A(a, b)$, $G(a, b)$, and $H(a, b)$

We start with the optimal lower and upper bound for the convex combination $\frac{1}{3}[A(a, b) + G(a, b) + H(a, b)]$, that is, we find the greatest value $q \in \mathbb{R}$ and the least value $p \in \mathbb{R}$, such that the double inequality (1.5) holds for all positive real numbers a and b .

In order to get this result, we make use of the following technical lemma.

LEMMA 2.1. Let $p = \frac{\ln 2}{\ln 6}$ and the function $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined by

$$g(x) = \frac{x^5 + x^4 + 2x^3 + 2x^2 + 5x + 1}{x^5 + 5x^4 + 2x^3 + 2x^2 + x + 1} - x^{2p-1}. \quad (2.1)$$

Then there exists a unique $x_0 \in (0, 1)$, such that $g(x_0) = 0$, $g(x) < 0$ for $x \in (0, x_0)$, and $g(x) > 0$ for $x \in (x_0, 1)$.

Proof. First, notice that both expressions $g_1(x) = x^5 + x^4 + 2x^3 + 2x^2 + 5x + 1$ and $g_2(x) = x^5 + 5x^4 + 2x^3 + 2x^2 + x + 1$ take positive values for $x \in \mathbb{R}^+$, so we can define the function $h : \mathbb{R}^+ \rightarrow \mathbb{R}$, $h(x) = \ln(g_1(x)) - \ln(g_2(x)) - (2p - 1)\ln x$. Then $h'(x) = \frac{k(x)}{x \cdot g_1(x) \cdot g_2(x)}$, where k is a polynomial with real coefficients, given by

$$\begin{aligned} k(x) = & (1 - 2p)x^{10} + 2(5 - 6p)x^9 + 9(1 - 2p)x^8 + 8(1 - 4p)x^7 \\ & - 2(22p + 5)x^6 - 36(2p + 1)x^5 - 2(22p + 5)x^4 + 8(1 - 4p)x^3 \\ & + 9(1 - 2p)x^2 + 2(5 - 6p)x + 1 - 2p. \end{aligned}$$

For $p = \frac{\ln 2}{\ln 6}$, we respectively have $1 - 2p \approx 0.23$, $2(5 - 6p) \approx 5.36$, $9(1 - 2p) \approx 2.04$, $8(1 - 4p) \approx -4.38$, $-2(22p + 5) \approx -27.02$, and $-36(2p + 1) \approx -63.85$, so the number of sign changes between consecutive coefficients of k is 2. According to Descartes' rule of signs, the polynomial k has two or zero positive roots. Since $k(0) = 1 - 2p \approx 0.23 > 0$, $k(1) = -288p \approx -111.41 < 0$, and $k(3) = 315904 - 596576p \approx 85116.9 > 0$, we conclude that k has exactly two positive roots, one less than 1 and the other greater than 1. Moreover, if $x_1 \in (0, 1)$ is such that $k(x_1) = 0$, then $k(x) > 0$ for $x \in (0, x_1)$, $k(x) < 0$ for $x \in (x_1, 1)$, and, consequently, $h'(x) > 0$ for $x \in (0, x_1)$, $h'(x_1) = 0$, and $h'(x) < 0$ for $x \in (x_1, 1)$. Hence, the function h is strictly increasing on $(0, x_1]$, strictly decreasing on $[x_1, 1)$, and $\max_{x \in (0, 1)} h(x) = h(x_1)$. This, together with $\lim_{x \rightarrow 0^+} h(x) = -\infty$ and $h(1) = 0$, implies that $h(x_1) > 0$ and that there exists a unique $x_0 \in (0, x_1)$, such that $h(x_0) = 0$, $h(x) < 0$ for $x \in (0, x_0)$, and $h(x) > 0$ for $x \in (x_0, 1)$. The lemma follows from the monotonicity of the exponential function $t \mapsto e^t$. \square

Now, we are ready to state and prove the main result in this section.

THEOREM 2.1. *Inequality*

$$M_0(a, b) \leq \frac{1}{3}[A(a, b) + G(a, b) + H(a, b)] \leq M_{\frac{\ln 2}{\ln 6}}(a, b) \quad (2.2)$$

holds for all $a, b \in \mathbb{R}^+$, with strict inequality for $a \neq b$. The constants 0 and $\frac{\ln 2}{\ln 6}$ are the best possible.

Proof. Denote $x = \sqrt{\frac{a}{b}}$. Then

$$\begin{aligned} \frac{1}{3}[A(a, b) + G(a, b) + H(a, b)] - M_0(a, b) &= \frac{1}{3}[A(a, b) - 2G(a, b) + H(a, b)] \\ &= b \left[\frac{x^2 + 1}{6} - \frac{2x}{3} + \frac{2x^2}{3(x^2 + 1)} \right] = \frac{b(x - 1)^4}{6(x^2 + 1)} \geq 0, \end{aligned} \quad (2.3)$$

so the first inequality in (2.2) is proved. Obviously, equality in (2.3) holds only for $x = 1$, that is, for $a = b$. In that case $A(a, a) = G(a, a) = H(a, a) = a$.

Next, we prove that the parameter $q = 0$ is the best possible, that is, that it cannot

be replaced with any larger parameter. For arbitrary $\varepsilon > 0$ and $0 < |t| < 1$ we have

$$\begin{aligned} & [M_\varepsilon((1+t)^2, 1)]^\varepsilon - \left[\frac{1}{3}A((1+t)^2, 1) + \frac{1}{3}G((1+t)^2, 1) + \frac{1}{3}H((1+t)^2, 1) \right]^\varepsilon \\ &= \frac{(1+t)^{2\varepsilon} + 1}{2} - \left(\frac{1 + 2t + \frac{3t^2}{2} + \frac{t^3}{2} + \frac{t^4}{12}}{1 + t + \frac{t^2}{2}} \right)^\varepsilon = \frac{h_\varepsilon(t)}{2 \left(1 + t + \frac{t^2}{2} \right)^\varepsilon}, \end{aligned} \quad (2.4)$$

where

$$h_\varepsilon(t) = \left(1 + t + \frac{t^2}{2} \right)^\varepsilon [1 + (1+t)^{2\varepsilon}] - 2 \left(1 + 2t + \frac{3t^2}{2} + \frac{t^3}{2} + \frac{t^4}{12} \right)^\varepsilon.$$

If $t \rightarrow 0$, by writing the binomial series we obtain

$$\begin{aligned} h_\varepsilon(t) &= \left[1 + \varepsilon t + \frac{\varepsilon^2}{2}t^2 + o(t^2) \right] \cdot [2 + 2\varepsilon t + \varepsilon(2\varepsilon - 1)t^2 + o(t^2)] \\ &\quad - 2 \left[1 + 2\varepsilon t + \frac{\varepsilon(16\varepsilon - 5)}{2}t^2 + o(t^2) \right] = \varepsilon(4 - 11\varepsilon)t^2 + o(t^2). \end{aligned} \quad (2.5)$$

Therefore, (2.4) and (2.5) yield that for any $\varepsilon \in (0, \frac{4}{11})$ there exists $\delta(\varepsilon) \in (0, 1)$ such that

$$M_\varepsilon((1+t)^2, 1) > \frac{1}{3} [A((1+t)^2, 1) + G((1+t)^2, 1) + H((1+t)^2, 1)]$$

for $t \in (0, \delta(\varepsilon))$. Thus, the parameter $q = 0$ cannot be enlarged so that the first inequality in (1.5) still holds.

Now, we prove the second inequality in (2.2). For any $p \in \mathbb{R}$, $p \neq 0$, we have

$$\begin{aligned} & \frac{1}{3} [A(a, b) + G(a, b) + H(a, b)] - M_p(a, b) \\ &= b \left[\frac{x^2 + 1}{6} + \frac{x}{3} + \frac{2x^2}{3(x^2 + 1)} \right] - b \left(\frac{x^{2p} + 1}{2} \right)^{\frac{1}{p}} \\ &= b \left[\frac{x^4 + 2x^3 + 6x^2 + 2x + 1}{6(x^2 + 1)} - \left(\frac{x^{2p} + 1}{2} \right)^{\frac{1}{p}} \right]. \end{aligned} \quad (2.6)$$

Let $p = \frac{\ln 2}{\ln 6}$. To get the desired result, we need to prove that the function from the last line of (2.6) takes only negative values on \mathbb{R}^+ . Since all bivariate power means are symmetric, without loss of generality we can assume that $a \leq b$, that is, that $x \in (0, 1]$. For $x \in [0, \infty)$, define

$$f(x) = \ln(x^4 + 2x^3 + 6x^2 + 2x + 1) - \ln(x^2 + 1) - \frac{1}{p} \ln(x^{2p} + 1) + \frac{1}{p} \ln 2 - \ln 6.$$

Then $f'(x) = g(x) \cdot l(x)$, where the function g is given by (2.1) and

$$l(x) = \frac{2(x^5 + 5x^4 + 2x^3 + 2x^2 + x + 1)}{(x^4 + 2x^3 + 6x^2 + 2x + 1)(x^2 + 1)(x^{2p} + 1)} > 0, \quad x \in \mathbb{R}^+.$$

Lemma 2.1 implies that there exists a unique $x_0 \in (0, 1)$, such that $f'(x_0) = 0$, $f'(x) < 0$ for $x \in (0, x_0)$, and $f'(x) > 0$ for $x \in (x_0, 1)$. Hence, the function f is strictly decreasing on $(0, x_0]$ and strictly increasing on $[x_0, 1)$, so from $f(0) = f(1) = 0$ we obtain that $f(x) < 0$ for $x \in (0, 1)$. Finally, the monotonicity of the exponential function $t \mapsto e^t$ gives that (2.6) is less than or equal to 0 for all $a, b \in \mathbb{R}^+$, with equality only for $x = 1$, that is, for $a = b$. In that case, both-hand sides of the second inequality in (2.2) are equal to a .

Finally, notice that $p = \frac{\ln 2}{\ln 6} \approx 0.39 > 0$ and

$$\lim_{t \rightarrow 0^+} \left\{ \frac{1}{3} [A(t^2, 1) + G(t^2, 1) + H(t^2, 1)] - M_p(t^2, 1) \right\} = 0,$$

while for any $\varepsilon \in (0, p)$ we have

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \left\{ \frac{1}{3} [A(t^2, 1) + G(t^2, 1) + H(t^2, 1)] - M_{p-\varepsilon}(t^2, 1) \right\} \\ &= \lim_{t \rightarrow 0^+} \left[\frac{t^4 + 2t^3 + 6t^2 + 2t + 1}{6(t^2 + 1)} - 2^{\frac{1}{\varepsilon-p}} \left(t^{2(p-\varepsilon)} + 1 \right)^{\frac{1}{p-\varepsilon}} \right] \\ &= \frac{1}{6} - 2^{\frac{1}{\varepsilon-p}} > 0. \end{aligned}$$

Therefore, the parameter $\frac{\ln 2}{\ln 6}$ cannot be diminished, that is, it is optimal for the second inequality in (1.5). \square

3. Arithmetic mean of $He(a, b)$ and $H(a, b)$

Now, we derive the best possible lower and upper bound for the arithmetic mean of the Heronian mean and the harmonic mean of two positive real numbers in terms of a power mean, that is, we obtain the greatest value $s \in \mathbb{R}$ and the least value $r \in \mathbb{R}$, such that the double inequality (1.6) holds for all positive real numbers a and b .

Similarly to the previous section, the following lemma provides an important argument for the proof of our result.

LEMMA 3.1. *Let $r = \frac{\ln 2}{\ln 6}$ and the function $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined by*

$$g(x) = \frac{2x^5 + x^4 + 4x^3 + 2x^2 + 14x + 1}{x^5 + 14x^4 + 2x^3 + 4x^2 + x + 2} - x^{2r-1}. \tag{3.1}$$

Then there exists a unique $x_ \in (0, 1)$, such that $g(x_*) = 0$, $g(x) < 0$ for $x \in (0, x_*)$, and $g(x) > 0$ for $x \in (x_*, 1)$.*

Proof. Observe that $g_1(x) = 2x^5 + x^4 + 4x^3 + 2x^2 + 14x + 1 > 0$ and $g_2(x) = x^5 + 14x^4 + 2x^3 + 4x^2 + x + 2 > 0$ for $x \in \mathbb{R}^+$, so the function $h: \mathbb{R}^+ \rightarrow \mathbb{R}$, $h(x) = \ln(g_1(x)) - \ln(g_2(x)) - (2r-1)\ln x$ is well-defined. Calculating its derivative, we get $h'(x) = \frac{2k(x)}{x \cdot g_1(x) \cdot g_2(x)}$, where k is a polynomial with real coefficients, given by

$$\begin{aligned} k(x) = & (1-2r)x^{10} + (28-29r)x^9 + 11(1-2r)x^8 + 4(4-17r)x^7 \\ & - 4(14r+5)x^6 - 6(37r+28)x^5 - 4(14r+5)x^4 + 4(4-17r)x^3 \\ & + 11(1-2r)x^2 + (28-29r)x + 1 - 2r. \end{aligned}$$

For $r = \frac{\ln 2}{\ln 6}$, we respectively have $1-2r \approx 0.23$, $28-29r \approx 16.78$, $11(1-2r) \approx 2.49$, $4(4-17r) \approx -10.31$, $-4(14r+5) \approx -41.66$, and $-6(37r+28) \approx -253.88$, so there are two sign changes between consecutive coefficients of k . Therefore, Descartes' rule of signs gives that the polynomial k has two or zero positive roots. Since $k(0) = 1-2r \approx 0.23 > 0$, $k(1) = -576r - 96 \approx -318.83 < 0$, and $k(3) = 660928 - 1083392r \approx 241814.8 > 0$, we see that the number of positive roots of k is exactly 2 and that one of them is less than 1 while the other one is greater than 1. Moreover, if $\tilde{x} \in (0, 1)$ is a positive root of k , then $k(x) > 0$ for $x \in (0, \tilde{x})$, $k(x) < 0$ for $x \in (\tilde{x}, 1)$, and, consequently, $h'(x) > 0$ for $x \in (0, \tilde{x})$, $h'(\tilde{x}) = 0$, and $h'(x) < 0$ for $x \in (\tilde{x}, 1)$. Thus, the function h is strictly increasing on $(0, \tilde{x}]$, strictly decreasing on $[\tilde{x}, 1)$, and $\max_{x \in (0, 1)} h(x) = h(\tilde{x})$. Considering that $\lim_{x \rightarrow 0^+} h(x) = -\infty$ and $h(1) = 0$, we conclude that $h(\tilde{x}) > 0$ and that there is a unique $x_* \in (0, \tilde{x})$, such that $h(x_*) = 0$, $h(x) < 0$ for $x \in (0, x_*)$, and $h(x) > 0$ for $x \in (x_*, 1)$. The proof is completed by taking into account the monotonicity of the exponential function $t \mapsto e^t$. \square

The result announced above is given in the following theorem.

THEOREM 3.1. *Inequality*

$$M_{-\frac{1}{6}}(a, b) \leq \frac{1}{2}[He(a, b) + H(a, b)] \leq M_{\frac{\ln 2}{\ln 6}}(a, b) \quad (3.2)$$

holds for all $a, b \in \mathbb{R}^+$, with strict inequality for $a \neq b$. The constants 0 and $\frac{\ln 2}{\ln 6}$ are the best possible.

Proof. First, observe that $\frac{1}{2}[He(a, b) + H(a, b)] = \frac{1}{3}A(a, b) + \frac{1}{6}G(a, b) + \frac{1}{2}H(a, b)$, for all $a, b \in \mathbb{R}^+$. If $y = \sqrt[6]{\frac{a}{b}}$, we have

$$\begin{aligned} & \frac{1}{2}[He(a, b) + H(a, b)] - M_{-\frac{1}{6}}(a, b) \\ &= \frac{b}{6} \left[(y^6 + 1) + y^3 + \frac{6y^6}{y^6 + 1} - \frac{384y^6}{(y+1)^6} \right] = \frac{b(y-1)^4}{6(y^6+1)(y+1)^6} (y^{14} + 10y^{13} \\ & \quad + 49y^{12} + 161y^{11} + 410y^{10} + 881y^9 + 1304y^8 + 1472y^7 + 1304y^6 + 881y^5 \\ & \quad + 410y^4 + 161y^3 + 49y^2 + 10y + 1) \geq 0 \end{aligned}$$

for all $y \in \mathbb{R}^+$, with equality if and only if $y = 1$, that is, $a = b$, so the first inequality in (3.2) is proved. In the case $a = b$, both-hand sides of the first inequality in (3.2) are equal to a .

To prove that the parameter $s = -\frac{1}{6}$ is optimal, take arbitrary $\varepsilon \in (0, \frac{1}{6})$ and $0 < |t| < 1$. Then

$$\begin{aligned} & \left[M_{-\frac{1}{6}+\varepsilon}((1+t)^2, 1) \right]^{\frac{1}{6}-\varepsilon} - \left[\frac{1}{2}He((1+t)^2, 1) + \frac{1}{2}H((1+t)^2, 1) \right]^{\frac{1}{6}-\varepsilon} \\ &= \frac{2(1+t)^{\frac{1}{3}-2\varepsilon}}{1+(1+t)^{\frac{1}{3}-2\varepsilon}} - \left(\frac{1+2t+\frac{17t^2}{12}+\frac{t^3}{2}+\frac{t^4}{12}}{1+t+\frac{t^2}{2}} \right)^{\frac{1}{6}-\varepsilon} \\ &= \frac{h_\varepsilon(t)}{\left(1+t+\frac{t^2}{2}\right)^{\frac{1}{6}-\varepsilon} \left[1+(1+t)^{\frac{1}{3}-2\varepsilon}\right]}, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} h_\varepsilon(t) &= 2(1+t)^{\frac{1}{3}-2\varepsilon} \left(1+t+\frac{t^2}{2}\right)^{\frac{1}{6}-\varepsilon} \\ &\quad - \left[1+(1+t)^{\frac{1}{3}-2\varepsilon}\right] \left(1+2t+\frac{17t^2}{12}+\frac{t^3}{2}+\frac{t^4}{12}\right)^{\frac{1}{6}-\varepsilon}. \end{aligned}$$

By writing the binomial series, for $t \rightarrow 0$, we get

$$\begin{aligned} h_\varepsilon(t) &= 2 \left[1 + \frac{1-6\varepsilon}{3}t + \frac{(6\varepsilon-1)(3\varepsilon+1)}{9}t^2 + o(t^2) \right] \\ &\quad \times \left[1 + \frac{1-6\varepsilon}{6}t + \frac{(1-6\varepsilon)^2}{72}t^2 + o(t^2) \right] \\ &\quad - 2 \left[1 + \frac{1-6\varepsilon}{6}t + \frac{(6\varepsilon-1)(3\varepsilon+1)}{18}t^2 + o(t^2) \right] \\ &\quad \times \left[1 + \frac{1-6\varepsilon}{3}t + \frac{(6\varepsilon-1)(8\varepsilon+1)}{24}t^2 + o(t^2) \right] = \frac{(1-6\varepsilon)\varepsilon}{6}t^2 + o(t^2), \end{aligned} \quad (3.4)$$

so (3.3) and (3.4) imply that for any $\varepsilon \in (0, \frac{1}{6})$ there exists $\delta(\varepsilon) \in (0, 1)$ such that

$$M_{-\frac{1}{6}+\varepsilon}((1+t)^2, 1) > \frac{1}{2} [He((1+t)^2, 1) + H((1+t)^2, 1)]$$

for $t \in (0, \delta(\varepsilon))$. This means that the parameter $s = -\frac{1}{6}$ cannot be enlarged so that the first inequality in (1.6) still holds.

It is left to prove the second inequality in (3.2). Let $x = \sqrt{\frac{a}{b}}$. Then for any $r \in \mathbb{R}$,

$r \neq 0$, we have

$$\begin{aligned} & \frac{1}{2}[He(a,b) + H(a,b)] - M_r(a,b) \\ &= b \left[\frac{x^2+1}{6} + \frac{x}{6} + \frac{x^2}{x^2+1} \right] - b \left(\frac{x^{2r}+1}{2} \right)^{\frac{1}{r}} \\ &= b \left[\frac{x^4+x^3+8x^2+x+1}{6(x^2+1)} - \left(\frac{x^{2r}+1}{2} \right)^{\frac{1}{r}} \right]. \end{aligned} \quad (3.5)$$

We need to prove that the last line of (3.5) is negative on \mathbb{R}^+ for $r = \frac{\ln 2}{\ln 6}$. Without loss of generality, assume that $a \leq b$, that is, $x \in (0, 1]$, and define

$$f(x) = \ln(x^4 + x^3 + 8x^2 + x + 1) - \ln(x^2 + 1) - \frac{1}{r} \ln(x^{2r} + 1) + \frac{1}{r} \ln 2 - \ln 6.$$

Then $f'(x) = g(x) \cdot l(x)$, where the function g is given by (3.1) and

$$l(x) = \frac{x^5 + 14x^4 + 2x^3 + 4x^2 + x + 2}{(x^4 + x^3 + 8x^2 + x + 1)(x^2 + 1)(x^{2r} + 1)} > 0, \quad x \in \mathbb{R}^+.$$

According to Lemma 3.1, there exists a unique $x_* \in (0, 1)$, such that $f'(x_*) = 0$, $f'(x) < 0$ for $x \in (0, x_*)$, and $f'(x) > 0$ for $x \in (x_*, 1)$. Therefore, the function f is strictly decreasing on $(0, x_*)$ and strictly increasing on $[x_*, 1)$, so $f(0) = f(1) = 0$ provides that $f(x) < 0$ for $x \in (0, 1)$. The monotonicity of the exponential function $t \mapsto e^t$ implies that (3.5) is less than or equal to 0 for all $a, b \in \mathbb{R}^+$, with equality only for $x = 1$, that is, for $a = b$. In that case, both-hand sides of the second inequality in (3.2) are equal to a .

Finally, since for $r = \frac{\ln 2}{\ln 6}$ and any $\varepsilon \in (0, p)$ we have

$$\lim_{t \rightarrow 0^+} \left\{ \frac{1}{2}[He(t^2, 1) + H(t^2, 1)] - M_r(t^2, 1) \right\} = 0$$

and

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \left\{ \frac{1}{2}[He(t^2, 1) + H(t^2, 1)] - M_{r-\varepsilon}(t^2, 1) \right\} \\ &= \lim_{t \rightarrow 0^+} \left[\frac{t^4 + t^3 + 8t^2 + t + 1}{6(t^2 + 1)} - 2^{\frac{1}{\varepsilon-r}} \left(t^{2(r-\varepsilon)} + 1 \right)^{\frac{1}{r-\varepsilon}} \right] \\ &= \frac{1}{6} - 2^{\frac{1}{\varepsilon-r}} > 0, \end{aligned}$$

the parameter $\frac{\ln 2}{\ln 6}$ is the best possible for the second inequality in (1.6). \square

Acknowledgements

This research was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant 058-1170889-1050.

REFERENCES

- [1] H. ALZER AND W. JANOUS, *Solution of problem 8**, Crux Math. **13** (1987), 173–178.
- [2] P. S. BULLEN, *Handbook of means and their inequalities*, revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, Reidel, Dordrecht]; Mathematics and its Applications, 560. Kluwer Academic Publishers Group, Dordrecht, 2003.
- [3] Y.-M. CHU AND W.-F. XIA, *Two sharp inequalities for power mean, geometric mean, and harmonic mean*, J. Inequal. Appl. **2009**, Article ID 741923, 6 pages (electronic).
- [4] W. JANOUS, *A note on generalized Heronian means*, Math. Inequal. Appl. **4**, 3 (2001), 369–375.
- [5] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Academic Publishers Group, Dordrecht, 1993.
- [6] E. NEUMAN AND J. SÁNDOR, *Companion inequalities for certain bivariate means*, Appl. Anal. Discrete Math. **3** (2009), 46–51.

(Received August 23, 2011)

Aleksandra Čižmešija
Department of Mathematics, University of Zagreb
Bijenička cesta 30
10000 Zagreb, Croatia
e-mail: cizmesij@math.hr