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LOG–CONVEXITY AND CAUCHY MEANS RELATED TO BERWALD'S INEQUALITY

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Abstract. In this paper, we investigate the famous Berwald's inequality. More precisely, we study the Berwald's difference in non-weighted and weighted case. We prove an interesting property of log-convexity of this difference which allows us to deduce Lyapunov's type inequality for these differences. Cauchy type means in this setting are also studied.

1. Introduction and preliminaries

Let Ω be a set equipped with a normalized measure μ. Then for a strictly monotonic continuous function g, the quasi-arithmetic mean $M_g(f,\mu)$ is defined as follows:

$$M_g(f,\mu) := g^{-1}\left(\int_\Omega g(f(u))\,d\mu(u)\right). \tag{1}$$

From (1) we can deduce integral power means. Indeed, for $r \in \mathbb{R}$, the integral power mean is defined as follows:

$$M_r(f,\mu) := \begin{cases} \left(\int_\Omega f^r(u)\,d\mu(u)\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\int_\Omega \log f(u)\,d\mu(u)\right), & r = 0. \end{cases} \tag{2}$$

If $r < s$, then the well-known inequality for means

$$M_r(f,\mu) \leqslant M_s(f,\mu), \tag{3}$$

is valid (see [2]). The well-known Berwald's inequality states:

THEOREM 1.1. *Let f be a non-negative concave function on $[a,b] \subset \mathbb{R}$. If $s > q > 0$, we have*

$$\left(\frac{q+1}{b-a}\int_a^b f^q(x)\,dx\right)^{\frac{1}{q}} \geqslant \left(\frac{s+1}{b-a}\int_a^b f^s(x)\,dx\right)^{\frac{1}{s}}. \tag{4}$$

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Note that (4) can be considered as a reverse of (3). Theorem 1.1 can be obtained from the following result, also obtained by Berwald (cf. [5, p. 214]).

THEOREM 1.2. *Let f be a non-negative continuous concave function, not identically zero on $[a,b]$, and ψ be a continuous strictly monotonic function on $[0,y_0]$, where y_0 is sufficiently large. If $\overline{z}$ is the unique positive root of the equation*

$$\frac{1}{\overline{z}}\int_0^{\overline{z}} \psi(y)\,dy = \frac{1}{b-a}\int_a^b \psi(f(x))\,dx, \tag{5}$$

then for every function $\phi : [0,y_0] \to \mathbb{R}$ which is convex with respect to ψ, we have

$$\frac{1}{\overline{z}}\int_0^{\overline{z}} \phi(y)\,dy \geqslant \frac{1}{b-a}\int_a^b \phi(f(x))\,dx. \tag{6}$$

In [1], Matloob Anwar and J. Pečarić defined new Cauchy's means as follows:

$$M_{r,l}^{s}(f,\mu) := \left(\frac{l(l-s)}{r(r-s)}\frac{M_r^r(f,\mu)-M_s^r(f,\mu)}{M_l^l(f,\mu)-M_s^l(f,\mu)}\right)^{\frac{1}{r-l}},\ l \neq r \neq s, l, r \neq 0. \tag{7}$$

In the remaining cases, $M_{r,l}^{s}$ is defined by the limit procedure. They proved that if $t < v$, $r < u$, then

$$M_{t,r}^{s}(f,\mu) \leqslant M_{v,u}^{s}(f,\mu). \tag{8}$$

In this paper, we define Cauchy's means motivated by Berwald's inequality (4) and we prove results related to (8). We need the following definitions and lemmas from log-convexity theory (cf. [3]).

DEFINITION 1.3. It is said that a positive function f is $\log$-convex in the Jensen sense on some interval $I \subseteq \mathbb{R}$ if

$$f(s)f(t) \geqslant f^2\left(\frac{s+t}{2}\right)$$

holds for every $s,t \in I$.

LEMMA 1.4. *A positive function f is* $\log$*-convex in the Jensen sense on an interval $I \subseteq \mathbb{R}$ if and only if the relation*

$$u^2 f(s) + 2uw f\left(\frac{s+t}{2}\right) + w^2 f(t) \geqslant 0 \tag{9}$$

holds for each real u,w and $s,t \in I$.

The following lemma gives a useful characterization of convex functions (cf. [5, p. 2]).

LEMMA 1.5. *A function ϕ is convex on an interval $I \subseteq \mathbb{R}$ if and only if*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geqslant 0 \tag{10}$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

Throughout the paper we will frequently use the following family of functions, convex with respect to $\psi(x) = x^q$ $(q > 0)$ on $(0,\infty)$:

$$\varphi_s(x) := \begin{cases} \frac{q^2}{s(s-q)} x^s, & s \neq 0, q, \\ -q \log x, & s = 0, \\ q x^q \log x, & s = q. \end{cases} \tag{11}$$

2. Berwald's inequality and Berwald's differences

THEOREM 2.1. *Let f be a positive continuous concave function on $[a,b]$, $q > 0$ and*

$$\Upsilon_s(f) := \begin{cases} \frac{q^2}{s(s-q)} \left[\left(\frac{q+1}{b-a} \int_a^b f^q(x)\,dx \right)^{\frac{s}{q}} - \frac{s+1}{b-a} \int_a^b f^s(x)\,dx \right], \; s \neq 0, q, \\ q + \frac{q}{b-a} \int_a^b \log f(x)\,dx - \log\left(\frac{q+1}{b-a} \int_a^b f^q(x)\,dx \right), \; s = 0, \\ \left(\frac{q+1}{b-a} \int_a^b f^q(x)\,dx \right) \log\left(\frac{q+1}{b-a} \int_a^b f^q(x)\,dx \right) \\ -\frac{q}{b-a} \int_a^b f^q(x)\,dx - \frac{q(q+1)}{b-a} \int_a^b f^q(x) \log f(x)\,dx, \; s = q. \end{cases} \tag{12}$$

Then $\Upsilon_s(f)$ is $\log$-convex for $s \geqslant 0$ and the following inequality holds for $0 \leqslant r < s < t < \infty$:

$$\Upsilon_s^{t-r}(f) \leqslant \Upsilon_r^{t-s}(f)\, \Upsilon_t^{s-r}(f). \tag{13}$$

Proof. Let us consider the function defined by

$$\phi(x) = u^2 \varphi_s(x) + 2uw\, \varphi_r(x) + w^2 \varphi_t(x),$$

where $r = \frac{s+t}{2}$, φ_s is defined by (11) and $u, w \in \mathbb{R}$.
Now, we shall show that $\phi(x)$ is convex with respect to $\psi(x) = x^q$ $(q > 0)$.
Set

$$F(x) = \phi(x^{\frac{1}{q}}) = u^2 \varphi_s(x^{\frac{1}{q}}) + 2uw\, \varphi_r(x^{\frac{1}{q}}) + w^2 \varphi_t(x^{\frac{1}{q}}).$$

We have

$$F''(x) = u^2 x^{\frac{s}{q}-2} + 2uw x^{\frac{r}{q}-2} + w^2 x^{\frac{t}{q}-2} = \left(u x^{\frac{s}{2q}-1} + w x^{\frac{t}{2q}-1} \right)^2 \geqslant 0, \; x > 0.$$

Therefore, $\phi(x)$ is convex with respect to $\psi(x) = x^q$ $(q > 0)$ for $x > 0$. Applying Theorem 1.2, we get

$$\frac{1}{\overline{z}} \int_0^{\overline{z}} \phi(y)\,dy \geqslant \frac{1}{b-a} \int_a^b \phi(f(x))\,dx,$$

where,

$$\bar{z} = \left(\frac{q+1}{b-a} \int_a^b f^q(x)\,dx \right)^{\frac{1}{q}}.$$

We have

$$\frac{1}{\bar{z}} \int_0^{\bar{z}} \left[u^2 \varphi_s(y) + 2uw\,\varphi_r(y) + w^2 \varphi_t(y) \right] dy$$
$$- \frac{1}{b-a} \int_a^b \left[u^2 \varphi_s(f(x)) + 2uw\,\varphi_r(f(x)) + w^2 \varphi_t(f(x)) \right] dx \geqslant 0,$$

or equivalently

$$u^2 \left[\frac{1}{\bar{z}} \int_0^{\bar{z}} \varphi_s(y)\,dy - \frac{1}{b-a} \int_a^b \varphi_s(f(x))\,dx \right]$$
$$+ 2uw \left[\frac{1}{\bar{z}} \int_0^{\bar{z}} \varphi_r(y)\,dy - \frac{1}{b-a} \int_a^b \varphi_r(f(x))\,dx \right]$$
$$+ w^2 \left[\frac{1}{\bar{z}} \int_0^{\bar{z}} \varphi_t(y)\,dy - \frac{1}{b-a} \int_a^b \varphi_t(f(x))\,dx \right] \geqslant 0.$$

Since

$$\Upsilon_s(f) := \frac{1}{\bar{z}} \int_0^{\bar{z}} \varphi_s(y)\,dy - \frac{1}{b-a} \int_a^b \varphi_s(f(x))\,dx,$$

we have

$$u^2 \Upsilon_s(f) + 2uw \Upsilon_r(f) + w^2 \Upsilon_t(f) \geqslant 0.$$

By Lemma 1.4, we have that $\Upsilon_s(f)$ is log-convex in the Jensen sense for $s \geqslant 0$.

Note that $\Upsilon_s(f)$ is continuous for $s \geqslant 0$ since

$$\lim_{s \to 0} \Upsilon_s(f) = \Upsilon_0(f) \text{ and } \lim_{s \to q} \Upsilon_s(f) = \Upsilon_q(f),$$

and therefore it is log-convex. Since $\Upsilon_s(f)$ is log-convex, i.e., $s \mapsto \log \Upsilon_s(f)$ is convex, by Lemma 1.5 for $0 \leqslant r < s < t < \infty$, we get

$$\log \Upsilon_s^{t-r}(f) \leqslant \log \Upsilon_r^{t-s}(f) + \log \Upsilon_t^{s-r}(f),$$

which is equivalent to (13). □

THEOREM 2.2. *Let f and $\Upsilon_s(f)$ be defined as in* Theorem 2.1 *and $t,s,u,v \geqslant 0$ such that $s \leqslant u, t \leqslant v, s \neq t, u \neq v$. Then*

$$\left(\frac{\Upsilon_t(f)}{\Upsilon_s(f)} \right)^{\frac{1}{t-s}} \leqslant \left(\frac{\Upsilon_v(f)}{\Upsilon_u(f)} \right)^{\frac{1}{v-u}}. \tag{14}$$

Proof. For a convex function φ, it holds (cf. [5, p. 2])

$$\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1} \leqslant \frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1}, \tag{15}$$

where $x_1 \leqslant y_1, x_2 \leqslant y_2, x_1 \neq x_2, y_1 \neq y_2$. Since by Theorem 2.1, $\Upsilon_s(f)$ is log-convex, we can set in (15): $\varphi(x) = \log \Upsilon_x(f), x_1 = s, x_2 = t, y_1 = u, y_2 = v$ to obtain

$$\frac{\log \Upsilon_t(f) - \log \Upsilon_s(f)}{t-s} \leqslant \frac{\log \Upsilon_v(f) - \log \Upsilon_u(f)}{v-u},$$

where from (14) trivially follows. □

REMARK 2.3. If we substitute $q = 1$ in (12), we get

$$\Upsilon_s(f) := \begin{cases} \frac{1}{s(s-1)}\left[\left(\frac{2}{b-a}\int_a^b f(x)\,dx\right)^s - \frac{s+1}{b-a}\int_a^b f^s(x)\,dx\right], & s \neq 0,1, \\ 1 + \frac{1}{b-a}\int_a^b \log f(x)\,dx - \log\left(\frac{2}{b-a}\int_a^b f(x)\,dx\right), & s = 0, \\ \left(\frac{2}{b-a}\int_a^b f(x)\,dx\right)\log\left(\frac{2}{b-a}\int_a^b f(x)\,dx\right) & \\ \quad -\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{2}{b-a}\int_a^b f(x)\log f(x)\,dx, & s = 1, \end{cases} \tag{16}$$

which is the same as $\Delta_s(f)$ defined in [3] for Favard's inequality.

3. Weighted Berwald's Inequality

The weighted version of Berwald's inequality was obtained by L. Maligranda, J. E. Pečarić, L. E. Persson (cf. [4]).

THEOREM 3.1. *Let φ be a convex function with respect to the strictly increasing function ψ on $[0,\infty)$, i.e., let $\varphi \circ \psi^{-1}$ be convex.*

1. *If f is a positive increasing concave function on $[a,b]$ and if $\overline{z}_i$ is a positive root of the equation*

$$\frac{1}{\overline{z}_i}\int_0^{\overline{z}_i} \psi(y)\, w\left(a + \frac{b-a}{\overline{z}_i} y\right) dy = \frac{1}{b-a}\int_a^b \psi(f(t))\, w(t)\,dt, \tag{17}$$

then

$$\frac{1}{b-a}\int_a^b \varphi(f(t))\, w(t)\,dt \leqslant \int_0^1 \varphi(s\overline{z}_i)\, w[a(1-s)+bs]\,ds. \tag{18}$$

If f is an increasing convex function on $[a,b]$ with $f(a)=0$, then the reverse inequality in (18) *holds.*

2. *If f is a positive decreasing concave function on $[a,b]$ and if $\bar{z}_d$ is a positive root of the equation*

$$\frac{1}{\bar{z}_d}\int_0^{\bar{z}_d}\psi(y)\,w\left(b-\frac{b-a}{\bar{z}_d}y\right)dy=\frac{1}{b-a}\int_a^b\psi\left(f(t)\right)w(t)\,dt, \tag{19}$$

then

$$\frac{1}{b-a}\int_a^b\varphi\left(f(t)\right)w(t)\,dt\leqslant\int_0^1\varphi\left(s\bar{z}_d\right)w\left[as+b(1-s)\right]ds. \tag{20}$$

If f is a decreasing convex function on $[a,b]$ with $f(b)=0$, then the reverse inequality in (20) holds.

THEOREM 3.2.

1. *Let f be a positive increasing concave function on $[a,b]$, $\bar{z}_i$ is a positive root of the equation* (17) *for $\psi(x)=x^q\ (q>0)$ and*

$$\Gamma_s(f):=\int_0^1\varphi_s\left(r\bar{z}_i\right)w\left[a(1-r)+br\right]dr-\frac{1}{b-a}\int_a^b\varphi_s\left(f(t)\right)w(t)\,dt. \tag{21}$$

Then $\Gamma_s(f)$ is log*-convex for $s\geqslant 0$ and the following inequality holds for $0\leqslant r<s<t<\infty$:*

$$\Gamma_s^{t-r}(f)\leqslant\Gamma_r^{t-s}(f)\,\Gamma_t^{s-r}(f).$$

2. *Let f be an increasing convex function on $[a,b]$, $f(a)=0$, $\overline{\Gamma}_s(f):=-\Gamma_s(f)$, where $\Gamma_s(f)$ is defined as in* (21). *Then $\overline{\Gamma}_s(f)$ is* log*-convex for $s\geqslant 0$ and the following inequality holds for $0\leqslant r<s<t<\infty$:*

$$\overline{\Gamma}_s^{t-r}(f)\leqslant\overline{\Gamma}_r^{t-s}(f)\,\overline{\Gamma}_t^{s-r}(f).$$

Proof. Analogous to the proof of Theorem 2.1, only we use Theorem 3.1(1) instead of Theorem 1.2. □

THEOREM 3.3.

1. *Let f and $\Gamma_s(f)$ be defined as in* Theorem 3.2(1) *and $t,s,u,v\geqslant 0$ be such that $s\leqslant u$, $t\leqslant v$, $s\neq t$, $u\neq v$. Then*

$$\left(\frac{\Gamma_t(f)}{\Gamma_s(f)}\right)^{\frac{1}{t-s}}\leqslant\left(\frac{\Gamma_v(f)}{\Gamma_u(f)}\right)^{\frac{1}{v-u}}. \tag{22}$$

2. *Let f and $\overline{\Gamma}_s(f)$ be defined as in* Theorem 3.2(2) *and $t,s,u,v\geqslant 0$ be such that $s\leqslant u$, $t\leqslant v$, $s\neq t$, $u\neq v$. Then*

$$\left(\frac{\overline{\Gamma}_t(f)}{\overline{\Gamma}_s(f)}\right)^{\frac{1}{t-s}}\leqslant\left(\frac{\overline{\Gamma}_v(f)}{\overline{\Gamma}_u(f)}\right)^{\frac{1}{v-u}}. \tag{23}$$

Proof. Analogous to the proof of Theorem 2.2. □

THEOREM 3.4.

1. *Let f be a positive decreasing concave function on $[a,b]$, $\bar{z}_d$ is a positive root of the equation* (19) *for* $\psi(x) = x^q$ $(q > 0)$ *and*

$$\Phi_s(f) := \int_0^1 \varphi_s(r\bar{z}_d)\, w[ar+b(1-r)]\, dr - \frac{1}{b-a}\int_a^b \varphi_s(f(t))\, w(t)\, dt. \quad (24)$$

   Then $\Phi_s(f)$ is log*-convex for $s \geqslant 0$ and the following inequality holds for $0 \leqslant r < s < t < \infty$:*

$$\Phi_s^{t-r}(f) \leqslant \Phi_r^{t-s}(f)\, \Phi_t^{s-r}(f).$$

2. *Let f be a decreasing convex function on $[a,b]$, $f(b) = 0$, $\overline{\Phi}_s(f) := -\Phi_s(f)$, where $\Phi_s(f)$ is defined as in* (24)*. Then $\overline{\Phi}_s(f)$ is* log*-convex for $s \geqslant 0$ and the following inequality holds for $0 \leqslant r < s < t < \infty$:*

$$\overline{\Phi}_s^{t-r}(f) \leqslant \overline{\Phi}_r^{t-s}(f)\, \overline{\Phi}_t^{s-r}(f).$$

Proof. Analogous to the proof of Theorem 2.1, only we use Theorem 3.1(2) instead of Theorem 1.2. □

THEOREM 3.5.

1. *Let f and $\Phi_s(f)$ be defined as in* Theorem 3.4(1) *and $t,s,u,v \geqslant 0$ be such that $s \leqslant u$, $t \leqslant v$, $s \neq t$, $u \neq v$. Then*

$$\left(\frac{\Phi_t(f)}{\Phi_s(f)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\Phi_v(f)}{\Phi_u(f)}\right)^{\frac{1}{v-u}}. \quad (25)$$

2. *Let f and $\overline{\Phi}_s(f)$ be defined as in* Theorem 3.4(2) *and $t,s,u,v \geqslant 0$ be such that $s \leqslant u$, $t \leqslant v$, $s \neq t$, $u \neq v$. Then*

$$\left(\frac{\overline{\Phi}_t(f)}{\overline{\Phi}_s(f)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\overline{\Phi}_v(f)}{\overline{\Phi}_u(f)}\right)^{\frac{1}{v-u}}. \quad (26)$$

Proof. Analogous to the proof of Theorem 2.2. □

REMARK 3.6. Let $0 < q \leqslant s$. If $w \equiv 1$ and f is a positive concave function on $[a,b]$, then the decreasing rearrangement f^* is also concave function on $[a,b]$. By applying Theorem 3.4 to f^*, it follows that $\Phi_s(f^*)$ is log-convex. Equimeasurability of f with f^* gives $\Phi_s(f) = \Phi_s(f^*)$ and we see that Theorem 3.4 recaptures Theorem 2.1.

REMARK 3.7. Let us note that (21) can be given in the following form

$$\Gamma_s(f) := \begin{cases} \frac{q^2}{s(s-q)}\left[\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (t-a)^q w(t)\,dt}\right)^{\frac{s}{q}}\int_a^b (t-a)^s w(t)\,dt - \int_a^b f^s(t)w(t)\,dt\right], & s \neq 0,q, \\ -\log\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (t-a)^q w(t)\,dt}\right)\int_a^b w(t)\,dt - q\int_a^b \log(t-a)w(t)\,dt & \\ \quad + q\int_a^b \log f(t)w(t)\,dt, & s=0, \\ \left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (t-a)^q w(t)\,dt}\right)\log\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (t-a)^q w(t)\,dt}\right)\int_a^b (t-a)^q w(t)\,dt & \\ \quad + q\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (t-a)^q w(t)\,dt}\right)\int_a^b (t-a)^q \log(t-a)w(t)\,dt & \\ \quad - q\int_a^b f^q(t)\log f(t)w(t)\,dt, & s=q, \end{cases} \tag{27}$$

while (24) can be given in the following form

$$\Phi_s(f) := \begin{cases} \frac{q^2}{s(s-q)}\left[\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (b-t)^q w(t)\,dt}\right)^{\frac{s}{q}}\int_a^b (b-t)^s w(t)\,dt - \int_a^b f^s(t)w(t)\,dt\right], & s \neq 0,q, \\ -\log\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (b-t)^q w(t)\,dt}\right)\int_a^b w(t)\,dt - q\int_a^b \log(b-t)w(t)\,dt & \\ \quad + q\int_a^b \log f(t)w(t)\,dt, & s=0, \\ \left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (b-t)^q w(t)\,dt}\right)\log\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (b-t)^q w(t)\,dt}\right)\int_a^b (b-t)^q w(t)\,dt & \\ \quad + q\left(\frac{\int_a^b f^q(t)w(t)\,dt}{\int_a^b (b-t)^q w(t)\,dt}\right)\int_a^b (b-t)^q \log(b-t)w(t)\,dt & \\ \quad - q\int_a^b f^q(t)\log f(t)w(t)\,dt, & s=q. \end{cases} \tag{28}$$

REMARK 3.8. If in Remark 3.7 we take $w(t) \equiv 1$, then $\Gamma_s(f)$ and $\Phi_s(f)$ convert to $\Upsilon_s(f)$.

4. Cauchy Means

Let us note that (14), (22), (23), (25) and (26) have the form of some known inequalities between means (eg. Stolarsky means, Gini means, etc). Here we will prove that expressions in (22) are also means. The proofs in remaining cases are analogous.

LEMMA 4.1. *Let $g,h \in C^2(I)$, $I \subseteq \mathbb{R}^+$, be such that $g'(y) > 0$ for every $y \in I$ and*

$$m \leqslant \frac{g'(y)h''(y) - h'(y)g''(y)}{(g'(y))^3} \leqslant M. \tag{29}$$

Then the functions ϕ_1 *and* ϕ_2 *defined by*

$$\phi_1(x) = \frac{1}{2} M g^2(x) - h(x),$$

and

$$\phi_2(x) = h(x) - \frac{1}{2} m g^2(x),$$

are convex functions with respect to g.

Proof. Set

$$G(x) = \phi_1\left[g^{-1}(x)\right] = \frac{1}{2} M x^2 - h\left[g^{-1}(x)\right].$$

We have

$$G''(x) = M - \frac{g'\left(g^{-1}(x)\right) h''\left(g^{-1}(x)\right) - h'\left(g^{-1}(x)\right) g''\left(g^{-1}(x)\right)}{\left(g'(g^{-1}(x))\right)^3},$$

which shows that ϕ_1 is convex with respect to g.

Similarly, set

$$H(x) = \phi_2\left[g^{-1}(x)\right] = h\left[g^{-1}(x)\right] - \frac{1}{2} m x^2.$$

We have

$$H''(x) = \frac{g'\left(g^{-1}(x)\right) h''\left(g^{-1}(x)\right) - h'\left(g^{-1}(x)\right) g''\left(g^{-1}(x)\right)}{\left(g'\left(g^{-1}(x)\right)\right)^3} - m.$$

This shows that ϕ_2 is convex with respect to g. □

THEOREM 4.2. *Let* w *be a positive integrable function on* $[a,b]$ *with* $\int_a^b w(t)dt = 1$. *Let* f *be a positive increasing concave function on* $[a,b]$, $g \in C^2([0,\infty))$ *and* $h \in C^2([0,\bar{z}_i])$. *Let* $g'(y) > 0$ *for every* $y \in [0,\bar{z}_i]$ *and* $\bar{z}_i$ *is defined as in* (17) *using the function* g. *Then there exists* $\xi \in [0,\bar{z}_i]$ *such that*

$$\begin{aligned} &\int_0^1 h(r\bar{z}_i)\, w\left[a(1-r)+br\right] dr - \frac{1}{b-a}\int_a^b h\left(f(t)\right) w(t)\, dt \\ &\quad = \frac{g'(\xi)\, h''(\xi) - h'(\xi)\, g''(\xi)}{2\left(g'(\xi)\right)^3}\Big[\int_0^1 g^2(r\bar{z}_i)\, w\left[a(1-r)+br\right] dr \\ &\qquad - \frac{1}{b-a}\int_a^b g^2(f(t))\, w(t)\, dt\Big]. \end{aligned} \tag{30}$$

Proof. Set $m = \min_{y\in[0,\bar{z}_i]} \Psi(y)$ and $M = \max_{y\in[0,\bar{z}_i]} \Psi(y)$, where

$$\Psi(y) = \frac{g'(y)\, h''(y) - h'(y)\, g''(y)}{\left(g'(y)\right)^3}.$$

Applying (18) for ϕ_1 and ϕ_2 defined in Lemma 4.1, we have

$$\int_0^1 \phi_1(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr \geqslant \frac{1}{b-a}\int_a^b \phi_1\,(f(t))\, w(t)\, dt$$

and

$$\int_0^1 \phi_2(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr \geqslant \frac{1}{b-a}\int_a^b \phi_2\,(f(t))\, w(t)\, dt,$$

that is,

$$\frac{M}{2}\left[\int_0^1 g^2(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b g^2(f(t))\, w(t)\, dt\right] \tag{31}$$
$$\geqslant \int_0^1 h(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b h\,(f(t))\, w(t)\, dt$$

and

$$\int_0^1 h(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b h\,(f(t))\, w(t)\, dt \tag{32}$$
$$\geqslant \frac{m}{2}\left[\int_0^1 g^2(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b g^2(f(t))\, w(t)\, dt\right].$$

By combining (31) and (32), (30) follows from continuity of $\Psi(y)$. □

THEOREM 4.3. *Let w be a positive integrable function on $[a,b]$ with $\int_a^b w(x)dx = 1$ and f be a positive increasing concave non-linear function on $[a,b]$. If $g \in C^2([0,\infty))$ and $h_1, h_2 \in C^2([0,\bar{z}_i])$ such that $g'(y) > 0$ for every $y \in [0,\bar{z}_i]$ and $\bar{z}_i$ is defined as in* Theorem 4.2 *using the function g, then there exists $\xi \in [0,\bar{z}_i]$ such that*

$$\frac{g'(\xi)\, h_1''(\xi) - h_1'(\xi)\, g''(\xi)}{g'(\xi)\, h_2''(\xi) - h_2'(\xi)\, g''(\xi)} = \frac{\int_0^1 h_1(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b h_1\,(f(t))\, w(t)\, dt}{\int_0^1 h_2(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b h_2\,(f(t))\, w(t)\, dt} \tag{33}$$

provided that $g'(y)\, h_2''(y) - h_2'(y)\, g''(y) \neq 0$ for every $y \in [0,\bar{z}_i]$.

Proof. Define the functional $\Theta : C^2([0,\bar{z}_i]) \to \mathbb{R}$ with:

$$\Theta(h) = \int_0^1 h(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr - \frac{1}{b-a}\int_a^b h\,(f(t))\, w(t)\, dt$$

and set $h_0 = \Theta(h_2)\, h_1 - \Theta(h_1)\, h_2$. Obviously $\Theta(h_0) = 0$. Using Theorem 4.2, there exists $\xi \in [0,\bar{z}_i]$ such that

$$\Theta(h_0) = \frac{g'(\xi)\, h_0''(\xi) - h_0'(\xi)\, g''(\xi)}{2\,(g'(\xi))^3}\left[\int_0^1 g^2(r\bar{z}_i)\, w\,[a(1-r)+br]\, dr \right.$$
$$\left. - \frac{1}{b-a}\int_a^b g^2(f(t))\, w(t)\, dt\right]. \tag{34}$$

We give a proof that the expression in square brackets in (34) is non-zero (actually strictly positive by inequality (18)) for non-linear function f. Suppose that the expression in square brackets in (34) is equal to zero, which is by simple rearrangements equivalent to equality

$$0 = \int_a^b \left[g^2\left(\frac{t-a}{b-a}\bar{z}_i\right) - g^2(f(t)) \right] w(t)\,dt. \tag{35}$$

In [4], it was proved that

$$\int_a^x g\left(\frac{t-a}{b-a}\bar{z}_i\right) w(t)\,dt \leqslant \int_a^x g\left(f(t)\right) w(t)\,dt,\ \ x \in [a,b].$$

Set

$$F(x) = \int_a^x \left[g\left(\frac{t-a}{b-a}\bar{z}_i\right) - g\left(f(t)\right) \right] w(t)\,dt.$$

We have $F(x) \leqslant 0$ and $F(a) = F(b) = 0$. By (35), obvious estimations and integration by parts we have

$$\begin{aligned} 0 &= \int_a^b \left[g^2\left(\frac{t-a}{b-a}\bar{z}_i\right) - g^2(f(t)) \right] w(t)\,dt \\ &\geqslant \int_a^b 2g\left(f(t)\right) \left[g\left(\frac{t-a}{b-a}\bar{z}_i\right) - g\left(f(t)\right) \right] w(t)\,dt \\ &= \int_a^b 2g\left(f(t)\right) dF(t) = -\int_a^b F(t)\, d\left[2g\left(f(t)\right)\right] \geqslant 0. \end{aligned}$$

This implies

$$\int_a^b \left[g^2\left(\frac{t-a}{b-a}\bar{z}_i\right) - g^2\left(f(t)\right) \right] w(t)\,dt = \int_a^b 2g\left(f(t)\right) \left[g\left(\frac{t-a}{b-a}\bar{z}_i\right) - g\left(f(t)\right) \right] w(t)\,dt$$

or equivalently

$$\int_a^b \left(g\left(\frac{t-a}{b-a}\bar{z}_i\right) - g\left(f(t)\right) \right)^2 w(t)\,dt = 0,$$

which implies that f is a linear function.

Since the function f is non-linear, the expression in square brackets of (34) is strictly positive which implies that $g'(\xi)\,h_0''(\xi) - h_0'(\xi)\,g''(\xi) = 0$, and this gives (33). Notice that Theorem 4.2 for $h = h_2$ implies that the denominator of the right-hand side of (33) is non-zero. □

COROLLARY 4.4. *Let w be a positive integrable function with $\int_a^b w(x)dx = 1$. If f is a positive increasing concave non-linear function on $[a,b]$ and $\bar{z}_i$ is defined as in* Theorem 4.3 *for $g(x) = x^q$ $(q > 0)$ or explicitly*

$$\bar{z}_i = (b-a)\left(\frac{\int_a^b f^q(t)\,w(t)\,dt}{\int_a^b (t-a)^q\,w(t)\,dt} \right)^{\frac{1}{q}},$$

then for $0 < s \neq t \neq q \neq s$*, there exists* $\xi \in (0, \bar{z}_i]$ *such that*

$$\xi^{t-s} = \frac{s(s-q)}{t(t-q)} \frac{\int_0^1 (r\bar{z}_i)^t\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^t(r)w(r)\,dr}{\int_0^1 (r\bar{z}_i)^s\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^s(r)w(r)\,dr}. \tag{36}$$

Proof. Set $h_1(x) = x^t$, $h_2(x) = x^s$ and $g(x) = x^q$, $t \neq s \neq 0, q$ in (33). □

REMARK 4.5. Since the function $\xi \to \xi^{t-s}$ is invertible, then from (36) we have

$$0 < \left(\frac{s(s-q)}{t(t-q)} \frac{\int_0^1 (r\bar{z}_i)^t\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^t(r)w(r)\,dr}{\int_0^1 (r\bar{z}_i)^s\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^s(r)w(r)\,dr} \right)^{\frac{1}{t-s}} \leqslant \bar{z}_i. \tag{37}$$

In fact, a similar result can also be given for (33). Namely, suppose that

$$\Lambda(y) = \left(g'(y)h_1''(y) - h_1'(y)g''(y)\right) \Big/ \left(g'(y)h_2''(y) - h_2'(y)g''(y)\right)$$

has an inverse function. Then from (33) we have

$$\xi = \Lambda^{-1}\left(\frac{\int_0^1 h_1(r\bar{z}_i)\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b h_1(f(t))\, w(t)\,dt}{\int_0^1 h_2(r\bar{z}_i)\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b h_2(f(t))\, w(t)\,dt} \right). \tag{38}$$

By the inequality (37) we can consider

$$M_{t,s}(f,w) := \left(\frac{s(s-q)}{t(t-q)} \frac{\int_0^1 (r\bar{z}_i)^t\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^t(r)w(r)\,dr}{\int_0^1 (r\bar{z}_i)^s\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^s(r)w(r)\,dr} \right)^{\frac{1}{t-s}}, \tag{39}$$

for $0 < s \neq t \neq q \neq s$, as means in broader sense. Moreover we can extend these means in other cases. By taking the limit we get

$$\log M_{s,s}(f,w) = \frac{\int_0^1 (r\bar{z}_i)^s \log(r\bar{z}_i)\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^s(r)\log f(r)w(r)\,dr}{\int_0^1 (r\bar{z}_i)^s\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^s(r)w(r)\,dr} - \frac{2s-q}{s(s-q)}, \quad s \neq 0, q,$$

$$\log M_{0,0}(f,w) = \frac{\int_0^1 \log^2(r\bar{z}_i)\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b \log^2 f(r)w(r)\,dr}{2\left[\int_0^1 \log(r\bar{z}_i)\, w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b \log f(r)w(r)\,dr\right]} + \frac{1}{q},$$

$$\log M_{q,q}(f,w) = -\frac{1}{q} + \frac{\bar{z}_i^q \int_0^1 r^q \log^2(r\bar{z}_i) w[a(1-r)+br]\,dr - \frac{1}{b-a}\int_a^b f^q(r)\log^2 f(r)w(r)dr}{2\bar{z}_i^q \int_0^1 r^q \log(r\bar{z}_i) w[a(1-r)+br]\,dr - \frac{2}{b-a}\int_a^b f^q(r)\log f(r)w(r)dr}$$

Finally, we prove that this new mean is monotonic.

THEOREM 4.6. *Let $t \leqslant u,\ r \leqslant s$, then the following inequality is valid*

$$M_{t,r}(f,w) \leqslant M_{u,s}(f,w). \tag{40}$$

Proof. Since $\Gamma_s(f)$ is log-convex, by (22) we get (40). □

REMARK 4.7. If $w \equiv 1$, then the above means become

$$M_{t,s}(f,1) := \left(\frac{\frac{1}{\overline{z}} \int_0^{\overline{z}} \varphi_t(y)\,dy - \frac{1}{b-a} \int_a^b \varphi_t(f(x))\,dx}{\frac{1}{\overline{z}} \int_0^{\overline{z}} \varphi_s(y)\,dy - \frac{1}{b-a} \int_a^b \varphi_s(f(x))\,dx} \right)^{\frac{1}{t-s}},\ 0 < t \neq s,$$

$$\log M_{s,s}(f,1) = \frac{\alpha^{\frac{s}{q}} \log \alpha - \frac{q}{b-a} \int_a^b f^s(x)\,dx - \frac{q(s+1)}{b-a} \int_a^b f^s(x) \log f(x)\,dx}{\frac{s(s-q)}{q} \Upsilon_s^q(f)} - \frac{2s-q}{s(s-q)},\ s \neq 0, q,$$

$$\log M_{0,0}(f,1) = \frac{\log^2 \alpha - \frac{2q^2}{b-a} \int_a^b \log f(x)\,dx - \frac{q^2}{b-a} \int_a^b \log^2 f(x)\,dx}{2q \left[\log \alpha - q - \frac{q}{b-a} \int_a^b \log f(x)\,dx \right]} + \frac{1}{q},$$

$$\log M_{q,q}(f,1) = \frac{\alpha \log^2 \alpha - \frac{2q^2}{b-a} \int_a^b f^q(x) \log f(x)\,dx - \frac{q^2(q+1)}{b-a} \int_a^b f^q(x) \log^2 f(x)\,dx}{2q \left[\alpha \log \alpha - \frac{q}{b-a} \int_a^b f^q(x)\,dx - \frac{q(q+1)}{b-a} \int_a^b f^q(x) \log f(x)\,dx \right]} - \frac{1}{q},$$

where

$$\alpha = \frac{q+1}{b-a} \int_a^b f^q(x)\,dx.$$

In this way (40) for $w \equiv 1$ gives an extension of (14) (see Remark 3.6).

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