NOTE ON HADWIGER–FINSLER'S INEQUALITIES

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(Communicated by S. Segura Gomis)

Abstract. In this article we present a new proof of the Finsler-Hadwiger inequality, we prove some consequences and one Finsler-Hadwiger type inequality. Finally we use the geometric inequalities that we obtain in this paper to present some algebraic applications.

1. Introduction

In [6] and [8], Finsler and Hadwiger proved the following inequality:

In any triangle ABC, the following inequalities hold:

$$4S\sqrt{3} + Q \leqslant a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + 3Q, \tag{1}$$

where S is the triangle area and $Q = (a-b)^2 + (b-c)^2 + (c-a)^2$. Equality occurs when triangle ABC is equilateral.

This result can be used to prove other known inequalities. We shall present a few such results and a new proof of the inequality (1). Finally we show another Finsler-Hadwiger type inequality and some algebraic applications.

2. Inequalities derived from Finsler-Hadwiger inequality

For next results, we shall consider a triangle with side lengths a, b, c and area S.

2.1. WEITZENBÖCK'S INEQUALITY (1919). [11] *In any triangle we have inequality*

$$a^2 + b^2 + c^2 \geqslant 4S\sqrt{3}.$$

Proof. We have $a^2 + b^2 + c^2 \geqslant 4S\sqrt{3} + Q \geqslant 4S\sqrt{3}$.

2.2. GORDON'S INEQUALITY(1966). [7] *In any triangle we have inequality*

$$ab + bc + ca \geqslant 4S\sqrt{3}.$$

Proof. $a^2 + b^2 + c^2 \geqslant 4S\sqrt{3} + Q$ is equivalent to $ab + bc + ca \geqslant 4S\sqrt{3} + a^2 + b^2 + c^2 - ab - bc - ca$. Using algebraic inequality $a^2 + b^2 + c^2 - ab - bc - ca \geqslant 0$, we conclude that $ab + bc + ca \geqslant 4S\sqrt{3}$.

2.3. TSINTSIFAS'S INEQUALITY(1986). [10] *Let m, n, p be real numbers with $m+n$, $n+p$ and $p+m$ strictly positive. In any triangle we have*

$$\frac{m}{n+p}a^2 + \frac{n}{m+p}b^2 + \frac{p}{m+n}c^2 \geqslant 2S\sqrt{3}.$$

Proof. Inequality in the statement is equivalent with

$$(m+n+p)\left(\frac{a^2}{n+p} + \frac{b^2}{m+p} + \frac{c^2}{m+n}\right) \geqslant 2S\sqrt{3} + a^2 + b^2 + c^2. \qquad (2)$$

But

$$\frac{a^2}{n+p} + \frac{b^2}{m+p} + \frac{c^2}{m+n} \geqslant \frac{(a+b+c)^2}{2(m+n+p)}$$

from Cauchy inequality. Also

$$a^2 + b^2 + c^2 - Q \geqslant 4S\sqrt{3}$$

is equivalent with

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \geqslant 4S\sqrt{3} + 2a^2 + 2b^2 + 2c^2.$$

Then

$$(m+n+p)\left(\frac{a^2}{n+p} + \frac{b^2}{m+p} + \frac{c^2}{m+n}\right) \geqslant (m+n+p)\frac{(a+b+c)^2}{2(m+n+p)}$$

$$= \frac{(a+b+c)^2}{2} \geqslant 2S\sqrt{3} + a^2 + b^2 + c^2,$$

which conclude (2).

2.4. CURRY'S INEQUALITY(1966). [3] *In any triangle, we have*

$$\frac{9abc}{a+b+c} \geqslant 4S\sqrt{3}.$$

Proof. For the first, we prove algebraic inequality

$$9xyz \geqslant (x+y+z)\left(2xy + 2yz + 2zx - x^2 - y^2 - z^2\right) \qquad (3)$$

for all $x, y, z > 0$

This is equivalent with

$$9xyz \geqslant x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 + 6xyz - x^3 - y^3 - z^3$$
$$\Leftrightarrow x^3 + y^3 + z^3 + 3xyz \geqslant x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2$$
$$\Leftrightarrow \left(x^3 - x^2y - x^2z + xyz\right) + \left(y^3 - y^2x - y^2z + xyz\right) + \left(z^3 - z^2y - z^2x + xyz\right) \geqslant 0$$
$$\Leftrightarrow x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) \geqslant 0$$

which is true from Schur inequality.

Now, (3) is equivalent with

$$\frac{9xyz}{x+y+z} \geqslant 2xy+2yz+2zx-x^2-y^2-z^2$$

$$\Leftrightarrow \frac{9xyz}{x+y+z} \geqslant x^2+y^2+z^2-(x-y)^2-(y-z)^2-(z-x)^2.$$

This result and Finsler-Hadwiger inequality solve the problem.

2.5. HADWIGER'S INEQUALITY(1939). [8] *In any triangle the following inequalities hold*

$$12S\sqrt{3}+2Q \leqslant (a+b+c)^2 \leqslant 12S\sqrt{3}+8Q.$$

Proof. It is an equivalent form of (1).

3. Proof of Finsler-Hadwiger's inequalities

Proof. If a,b,c are the side lengths of triangle ABC, then there exist three real numbers $x,y,z>0$ so that $a=y+z$, $b=x+z$ and $c=x+y$. With this notations, we have

$$S = \sqrt{xyz(x+y+z)}.$$

Inequality (1) is equivalent with

$$a^2+b^2+c^2-3Q \leqslant 4S\sqrt{3} \leqslant a^2+b^2+c^2-Q.$$

Then

$$a^2+b^2+c^2-3Q \leqslant 4S\sqrt{3}$$

$$\Leftrightarrow (y+z)^2+(x+z)^2+(x+y)^2-3(x-y)^2-3(y-z)^2-3(z-x)^2 \leqslant 4\sqrt{3xyz(x+y+z)}$$

$$\Leftrightarrow 2xy+2yz+2zx-x^2-y^2-z^2 \leqslant \sqrt{3xyz(x+y+z)}. \qquad (4)$$

But

$$2xy+2yz+2zx-x^2-y^2-z^2 \leqslant \frac{9xyz}{x+y+z}$$

from (3) and

$$\frac{9xyz}{x+y+z} \leqslant \sqrt{3xyz(x+y+z)}$$

because is equivalent with

$$27xyz \leqslant (x+y+z)^3$$

$$\Leftrightarrow \sqrt[3]{xyz} \leqslant \frac{x+y+z}{3}$$

which is Arithmetic and Geometric mean inequality. Thus, (4) is valid.

Next

$$4S\sqrt{3} \leqslant a^2+b^2+c^2-Q$$

$$\Leftrightarrow 4\sqrt{3xyz(x+y+z)} \leqslant (y+z)^2 + (x+z)^2 + (x+y)^2 - (x-y)^2 - (y-z)^2 - (z-x)^2$$

$$\Leftrightarrow \sqrt{3xyz(x+y+z)} \leqslant xy + yz + zx$$

$$\Leftrightarrow 3xyz(x+y+z) \leqslant (xy+yz+zx)^2$$

$$\Leftrightarrow (xy)(xz) + (xy)(yz) + (xz)(yz) \leqslant (xy)^2 + (xz)^2 + (zy)^2$$

which is true. Now, the proof is complete.

THEOREM 3.1. *Let* $u,v \in \mathbb{R}$. *If the inequalities*

$$4S\sqrt{3} + uQ \leqslant a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + vQ \tag{5}$$

are true for any triangle, then $u \leqslant 1$ *and* $v \geqslant 3$.

Proof. Let triangle ABC with $a = b = 1$ and $c = t \in (0,2)$. Then

$$4S\sqrt{3} + uQ \leqslant a^2 + b^2 + c^2$$

$$\Rightarrow t\sqrt{3(4-t^2)} + 2u(1-t)^2 \leqslant 2 + t^2.$$

With condition $t \to 0$ we obtain $2u \leqslant 2$, so $u \leqslant 1$.

From

$$a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + vQ,$$

we obtain

$$2 + t^2 \leqslant t\sqrt{3(4-t^2)} + 2v(1-t)^2.$$

Condition $t \to 2$ goes to $6 \leqslant 2v$, so $3 \leqslant v$.

Given the Finsler-Hadwiger inequality, we deduce that the maximum value of number u from (5) is 1 and the minimum value of v is 3 and these constants are best possible.

4. A Finsler-Hadwiger type inequality

Let the triangle ABC and set $M = (|a-b| + |b-c| + |c-a|)^2$. Then:

THEOREM 4.1. *In any triangle, we have:*

$$4S\sqrt{3} + \frac{1}{2}M \leqslant a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + \frac{3}{2}M. \tag{6}$$

Proof. In this proof we consider $a \geqslant b \geqslant c$. In these conditions we have $M = 4(a-c)^2$. For inequality $4S\sqrt{3} + \frac{1}{2}M \leqslant a^2 + b^2 + c^2$, let $x,y,z > 0$ with $a = y+z$, $b = x+z$ and $c = x+y$. Then inequality is equivalent with

$$4\sqrt{3xyz(x+y+z)} + 2(x-z)^2 \leqslant 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$$

$$\Leftrightarrow 2\sqrt{3xyz(x+y+z)} \leqslant y^2 + xy + yz + 3xz$$

$$\Leftrightarrow 2\sqrt{3xz \cdot y\,(x+y+z)} \leqslant y\,(x+y+z) + 3xz,$$

which is true because it is the Arithmetic and Geometric mean inequality.

For $a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + \frac{3}{2}M$ we prove the first inequality

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \leqslant \frac{1}{2}M. \tag{7}$$

This is equivalent with

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \leqslant 2\,(a-c)^2$$
$$\Leftrightarrow b^2 - ab - bc + ac \leqslant 0$$
$$\Leftrightarrow (b-a)\,(b-c) \leqslant 0,$$

which is true because $a \geqslant b \geqslant c$. Then

$$a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + 3\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \leqslant 4S\sqrt{3} + \frac{3}{2}M$$

which concludes the proof.

THEOREM 4.2. *Let $u, v \in \mathbb{R}$. If the inequalities*

$$4S\sqrt{3} + uM \leqslant a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + vM \tag{8}$$

hold for any triangle then $u \leqslant \frac{1}{2}$ and $v \geqslant \frac{3}{2}$.

Proof. Let the triangle ABC with $a = b = 1$ and $c = t \in (0,2)$. Then

$$4S\sqrt{3} + uM \leqslant a^2 + b^2 + c^2$$
$$\Rightarrow t\sqrt{3\,(4 - t^2)} + 4u\,(1 - t)^2 \leqslant 2 + t^2.$$

With $t \to 0$, we obtain $4u \leqslant 2$, so $u \leqslant \frac{1}{2}$.

Next

$$a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + vM$$
$$\Rightarrow 2 + t^2 \leqslant t\sqrt{3\,(4 - t^2)} + 4v\,(1 - t)^2.$$

Condition $t \to 2$ goes to $6 \leqslant 4v$ and $v \geqslant \frac{3}{2}$.

Given the inequality (6), we deduce that the maximum value of number u from (8) is $\frac{1}{2}$ and the minimum value of v is $\frac{3}{2}$, so this constants are best possible.

REMARK 1. Given the Finsler-Hadwiger's inequality, inequalities of proposition 4.1 and relation (7), we obtain following sequence of inequalities valid for any triangle,

$$4S\sqrt{3} + Q \leqslant 4S\sqrt{3} + \frac{1}{2}M \leqslant a^2 + b^2 + c^2 \leqslant 4S\sqrt{3} + 3Q \leqslant 4S\sqrt{3} + \frac{3}{2}M,$$

where $Q = (a-b)^2 + (b-c)^2 + (c-a)^2$ and $M = (|a-b| + |b-c| + |c-a|)^2$.

REMARK 2. We have equality in (1) if and only if triangle is equilateral. But the inequalities from 2.1, 2.2, 2.4 and 2.5 are consequences of the Finsler-Hadwiger's inequality and equality holds in same conditions. Supplementary, the equality from 2.3 holds with the conditions $m = n = p$. In same mode, the inequality from Theorem 4.1 is equality if and only if the triangle is equilateral.

5. Applications

Applications that we present below are algebraic inequalities. With the results included in the next lemmas, we can use some geometric arguments to prove these inequalities. These arguments are based on the existence of a triangle with given side lengths and on the inequalities for a triangle.

LEMMA 5.1. *For all real numbers $x, y, z > 0$, the following inequality holds*

$$\sqrt{x+y} < \sqrt{x+z} + \sqrt{y+z}.$$

Proof. $\sqrt{x+y} < \sqrt{x+z} + \sqrt{y+z}$
$\Leftrightarrow x+y < x+z+2\sqrt{(x+z)(y+z)}+y+z$
$\Leftrightarrow 0 < 2z+2\sqrt{(x+z)(y+z)}$ which is true.

LEMMA 5.2. *For all real numbers $x, y, z > 0$, there exists a triangle with side lengths $\sqrt{x+y}$, $\sqrt{x+z}$, $\sqrt{y+z}$ whose area is*

$$S = \frac{1}{2}\sqrt{xy+xz+yz}.$$

Proof. Lemma 5.1 ensures the existence of a triangle. Its area is

$$S = \frac{1}{2}\sqrt{x+y}\sqrt{x+z}\sin\alpha$$

where α is the angle of the sides of length $\sqrt{x+y}$ and $\sqrt{x+z}$. Then

$$\cos\alpha = \frac{x+y+x+z-y-z}{2\sqrt{(x+y)(x+z)}} = \frac{x}{\sqrt{(x+y)(x+z)}}$$

and

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{xy+xz+yz}}{\sqrt{(x+y)(x+z)}}.$$

Now, we obtain

$$S = \frac{1}{2}\sqrt{xy+xz+yz}.$$

APPLICATIONS 1. *For all real numbers $x, y, z > 0$, the following inequality holds*

$$\sum_{cyclic} \sqrt{(x+y)(x+z)} \geqslant x+y+z+\sqrt{3(xy+xz+yz)}.$$

Proof. We use 5.2 and inequality (1) for the triangle with side lengths $\sqrt{x+y}$, $\sqrt{x+z}$ and $\sqrt{y+z}$. We obtain

$$\sum_{cyclic} \left(\sqrt{x+y}\right)^2 - \sum_{cyclic} \left(\sqrt{x+y} - \sqrt{x+z}\right)^2 \geqslant 4\sqrt{3}\frac{1}{2}\sqrt{xy+xz+yz}$$

and

$$2\sum_{cyclic} \sqrt{(x+y)(x+z)} - 2(x+y+z) \geqslant 2\sqrt{3(xy+xz+yz)}$$

and this concludes the proof.

APPLICATIONS 2. *For all real numbers $x, y, z > 0$, the following inequality holds*

$$3 \sum_{cyclic} \sqrt{(x+y)(x+z)} \leqslant 5x + 5y + 5z + \sqrt{3(xy+xz+yz)}.$$

Proof. We use Lemma 5.2 and other part of inequality (1) for the triangle with side lengths $\sqrt{x+y}$, $\sqrt{x+z}$ and $\sqrt{y+z}$. We obtain

$$\sum_{cyclic} \left(\sqrt{x+y}\right)^2 - 3 \sum_{cyclic} \left(\sqrt{x+y} - \sqrt{x+z}\right)^2 \leqslant 4\sqrt{3} \frac{1}{2} \sqrt{xy+xz+yz}$$

which is equivalent with

$$6 \sum_{cyclic} \sqrt{(x+y)(x+z)} - 10(x+y+z) \leqslant 2\sqrt{3(xy+xz+yz)}$$

and this concludes the proof.

APPLICATIONS 3. (T. Andreescu, G. Dospinescu) [1] *For all real numbers $a, b, c > 0$ and $x, y, z > 0$, the following inequality holds*

$$\frac{a}{b+c}(x+y) + \frac{b}{a+c}(x+z) + \frac{c}{a+b}(y+z) \geqslant \sqrt{3(xy+xz+yz)}.$$

Proof. We apply Lemma 5.2 and inequality from 2.3 for triangle with side lengths $\sqrt{x+y}$, $\sqrt{x+z}$ and $\sqrt{y+z}$.

APPLICATIONS 4. *For all real numbers $a, b, c > 0$, the following inequality holds*

$$ab\frac{a+c}{b+c} + bc\frac{b+a}{c+a} + ac\frac{b+c}{a+b} \geqslant \sqrt{3abc(a+b+c)}.$$

Proof. The inequality is equivalent with

$$\frac{a}{b+c}(ab+bc) + \frac{b}{a+c}(ac+bc) + \frac{c}{a+b}(ab+ac) \geqslant \sqrt{3(ab \cdot ac + ab \cdot bc + ac \cdot bc)}.$$

This is true by using the Lemma 5.2 for the real numbers ab, ac and bc and the inequality from 2.3.

APPLICATIONS 5. *For all real numbers $x, y, z > 0$ satisfying $x + y + z = 1$, the following inequality holds*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geqslant \sqrt{\frac{3}{xyz}}.$$

Proof. We have $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} > 0$, so numbers $\sqrt{\frac{1}{x} + \frac{1}{y}}$, $\sqrt{\frac{1}{x} + \frac{1}{z}}$ and $\sqrt{\frac{1}{y} + \frac{1}{z}}$ could be the side lengths of a triangle. The area of this triangle is

$$\frac{1}{2}\sqrt{\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}} = \frac{1}{2}\sqrt{\frac{z+y+z}{xyz}} = \frac{1}{2\sqrt{xyz}}.$$

Apply inequality from (2.1) and obtain

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{x} + \frac{1}{z} + \frac{1}{y} + \frac{1}{z} \geqslant 2\sqrt{\frac{3}{xyz}}$$

and this concludes the proof.

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