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RELATIVE CONVEXITY AND QUADRATURE RULES FOR THE RIEMANN–STIELTJES INTEGRAL

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Abstract. We develop Trapezoid, Midpoint, and Simpson's rules for the Riemann-Stieltjes integral, the latter two being new. These rules are completely natural when the notion of relative convexity is used.

1. For f continuous (say) on $[a,b]$, the classical Midpoint Rule is

$$\int_a^b f(x)dx \cong f(\tfrac{a+b}{2})[b-a], \tag{M}$$

and the classical Trapezoid Rule is

$$\int_a^b f(x)dx \cong \tfrac{f(a)+f(b)}{2}[b-a]. \tag{T}$$

A useful relationship between these is the well-known

HADAMARD'S INEQUALITY. *If f is a convex function on $[a,b]$ then*

$$f(\tfrac{a+b}{2})[b-a] \leqslant \int_a^b f(x)\,dx \leqslant \tfrac{f(a)+f(b)}{2}[b-a]. \tag{H}$$

Moreover, if f'' is continuous then the errors can be expressed as follows [1]: *There are α_1 and $\alpha_2 \in (a,b)$ such that*

$$f(\tfrac{a+b}{2})[b-a]+\tfrac{1}{24}f''(\alpha_1)(b-a)^3 = \int_a^b f(x)\,dx = \tfrac{f(a)+f(b)}{2}[b-a]-\tfrac{1}{12}f''(\alpha_2)(b-a)^3. \tag{E}$$

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Now for f continuous (say), and g increasing (say) on $[a,b]$, a possible Riemann-Stieltjes Midpoint Rule is

$$\int_a^b f dg \cong f(\tfrac{a+b}{2})[g(b)-g(a)] \tag{RSM}$$

and a possible Riemann-Stieltjes Trapezoid Rule is

$$\int_a^b f dg \cong \tfrac{f(a)+f(b)}{2}[g(b)-g(a)]\,. \tag{RST}$$

We write *possible* to mean that each reduces to its corresponding classical quadrature rule when $g(x)=x$.

These RS quadrature rules have been used and studied by a good number of authors (e.g. [2–7]). In [8] it is shown that for f convex, the obvious analog of (H) for RSM and RST does not hold. In that paper alternatives to RSM and RSM are offered, for which (H) does hold. Error terms are also obtained.

In this investigation we take a different approach — this time adjusting, in a natural way, the notion of convexity. The very nature of the adjustment ensures that the right hand side of the obvious analog for (H) holds and it forces us to rethink the left hand side, thus engendering an alternative to the RSM above. Error terms analogous to (E) are obtained as well as a RS Simpson's rule.

2. Let f be defined on $[a,b]$ and get $g:[a,b]\to[a,b]$, with g increasing. We say f *is convex with respect to* g if $f\circ g^{-1}$ is convex [9]. As g is increasing we may write

$$\int_a^b f dg(x) = \int_{g(a)}^{g(b)} f\circ g^{-1}(x)dx.$$

Then by applying (M) and (T) and using (H), we immediately obtain the following.

THEOREM 1. *Let f and g be defined on $[a,b]$ and suppose that f is convex with respect to g there. Then*

$$f\circ g^{-1}(\tfrac{g(a)+g(b)}{2})[g(b)-g(a)] \leqslant \int_a^b f dg(x) \leqslant \tfrac{f(a)+f(b)}{2}[g(b)-g(a)]\,.$$

We can obtain error terms for these quadrature rules which are perfectly analogous to those in (E), as follows.

THEOREM 2. *Suppose that f and g are defined on $[a,b]$, that f is convex with respect to g there, and that f and g are twice continuously differentiable. Then there are ξ_1 and $\xi_2\in(a,b)$ such that*

$$\int_a^b f\,dg = f\circ g^{-1}(\tfrac{g(a)+g(b)}{2})[g(b)-g(a)] + \frac{d^2f}{dg^2}(\xi_1)\frac{(g(b)-g(a))^3}{24}.$$

and

$$\int_a^b f\,dg = \tfrac{f(a)+f(b)}{2}[g(b)-g(a)] - \frac{d^2f}{dg^2}(\xi_2)\frac{(g(b)-g(a))^3}{12}.$$

Proof. It is a routine matter to verify that

$$(f\circ g^{-1})''(x) = \frac{g'(g^{-1}(x))f''(g^{-1}(x)) - f'(g^{-1}(x))g''(g^{-1}(x))}{(\,g'(g^{-1}(x))\,)^3},$$

so we may apply (E) to obtain

$$\int_a^b f\,dg - f\circ g^{-1}(\tfrac{g(a)+g(b)}{2})(g(b)-g(a)) = \left[\frac{g'(\xi_1)f''(\xi_1)-f'(\xi_1)g''(\xi_1)}{(g'(\xi_1))^3}\right]\frac{(g(b)-g(a))^3}{24}$$

and

$$\int_a^b f\,dg\; - \tfrac{f(a)+f(b)}{2}(g(b)-g(a)) = -\left[\frac{g'(\xi_2)f''(\xi_2)-f'(\xi_2)g''(\xi_2)}{(\,g'(\xi_2)\,)^3}\right]\frac{(g(b)-g(a))^3}{12},$$

where $\xi_1 = g^{-1}(\alpha_1)$ and $\xi_2 = g^{-1}(\alpha_2)$. Now f and g are each defined for $t\in[a,b]$, so

$$\frac{df}{dg} = \frac{\dot f}{\dot g} \qquad \text{and} \qquad \frac{d^2f}{dg^2}\frac{dg}{dt} = \frac{\dot g\ddot f - \dot f\ddot g}{(\dot g)^2}.$$

Therefore

$$\frac{d^2f}{dg^2} = \frac{\dot g\ddot f - \dot f\ddot g}{(\dot g)^3},$$

and the proof is complete. □

We close with two observations. It is a useful fact that Simpson's Rule $=\frac{1}{3}($Trapezoid Rule$)+\frac{2}{3}($Midpoint Rule$)$. Using this in the RS context, and following the same idea as in Theorem 2, we obtain the following. (Another possible RS Simpson's Rule – for g only of bounded variation – can be found in [5].)

COROLLARY 3. *Let f and g be defined on $[a,b]$, that f is convex with respect to g there, and that f and g are four times continuously differentiable. Then there is $\xi_3\in(a,b)$ such that*

$$\int_a^b f\,dg = \left(\tfrac{2}{3}f\circ g^{-1}(\tfrac{g(a)+g(b)}{2}) + \tfrac{f(a)+f(b)}{6}\right)[g(b)-g(a)] - \frac{d^4f}{dg^4}(\xi_3)\frac{(g(b)-g(a))^5}{2880}.$$

Also, it is a fact that ([9])

$$f \text{ is convex with respect to } g \;\Leftrightarrow\; \frac{f''}{f'} \geqslant \frac{g''}{g'}\,,$$

and this ensures that each of the terms $[\cdot]$ in the proof of Theorem 2 is nonnegative. Therefore, we have

COROLLARY 4. *Let f and g be defined on $[a,b]$, and let f and g be twice continuously differentiable. Then*

$$f \text{ is convex with respect to } g \iff \frac{d^2 f}{dg^2} \geqslant 0\,.$$

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