GENERALIZED HILBERT OPERATORS
ON WEIGHTED MORREY–HERZ SPACES

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Abstract. This paper gives some necessary and sufficient conditions for the generalized Hilbert operators to be bounded on the weighted Morrey-Herz spaces. The corresponding new operator norm inequalities are obtained.

1. Introduction

Considerable attention has been given to the classical Hilbert operator $T_0$ defined by

$$T_0(f,x) = \int_0^\infty \frac{f(y)}{x+y} dy,$$  \hspace{1cm} (1.1)

and the classical Hilbert inequality

$$\|T_0f\|_p \leq \frac{\pi}{\sin(\frac{\pi}{p})} \|f\|_p, \quad \text{for } 0 < p < \infty,$$  \hspace{1cm} (1.2)

where $\|f\|_p = \left( \int_0^\infty |f(x)|^p dx \right)^{\frac{1}{p}}$ and the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best value [1]. In view of the mathematical importance and applications, considerable attention has also been given to various improvements, refinements and extensions of many inequalities by various authors (see e.g. [3, 4, 11] and the references cited therein). However, hardly any work was done on inequalities on the Morrey spaces and Herz spaces. It is well-known that the Herz spaces play an important role in characterizing the properties of functions and multipliers on the classical Hardy spaces, and the Morrey spaces have important applications in the theory of partial differential equations, in linear as well as in non-linear theory (see [5, 6]). In 2008, the author [8] obtained some new inequalities related to the generalized Hilbert operator

$$T(f,x) = \int_0^\infty K(x,y) \times f(y) dy$$  \hspace{1cm} (1.3)

with the general kernel $K(x,y)$ on the Herz spaces. In 2009, the author [9] introduced the new weighted Morrey-Herz spaces. The aim of this paper is to generalize these results of the author [8] to the weighted Morrey-Herz spaces. We obtain some necessary and sufficient conditions for the generalized Hilbert operator $T$ to be bounded on these spaces. The corresponding new operator norm inequalities are obtained.

Keywords and phrases: Hilbert operator, Morrey-Herz spaces, norm inequality.
2. Definitions and statement of the main results

Let \( k \in \mathbb{Z} \), \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \), \( D_k = B_k - B_{k-1} \) and let \( \varphi_k = \varphi_{D_k} \) denotes the characteristic function of the set \( D_k \). Moreover, for a measurable function \( f \) on \( \mathbb{R}^n \) and a non-negative weighted function \( \omega(x) \), we write

\[
\| f \|_{p,\omega} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p}.
\]

In what follows, if \( \omega \equiv 1 \), then we will denote \( L^p(\mathbb{R}^n, \omega) \) (in brief \( L^p(\omega) \)) by \( L^p(\mathbb{R}^n) \).

**DEFINITION 2.1.** Let \( \alpha \in \mathbb{R}^1 \), \( 0 < p, q < \infty \) and \( s \geq 0 \). The Morrey spaces \( M^s_q(R^n) \) is defined by [5] as follows:

\[
M^s_q(R^n) = \left\{ f \in L^q_{\text{loc}}(R^n) : \sup_{r > 0, x \in R^n} \frac{1}{r^s} \int_{|x-y|<r} |f(y)|^q \, dy < \infty \right\},
\]

(2.1)

and the homogeneous Herz space \( \hat{K}^\alpha_{q,p}(R^n) \) is defined by [6] as follows:

\[
\hat{K}^\alpha_{q,p}(R^n) = \left\{ f \in L^q_{\text{loc}}(R^n - \{0\}) : \| f \|_{\hat{K}^\alpha_{q,p}(R^n)} < \infty \right\},
\]

(2.2)

where

\[
\| f \|_{\hat{K}^\alpha_{q,p}(R^n)} = \left\{ \sum_{k \in \mathbb{Z}} 2^{k \alpha p} \| f \varphi_k \|_p \right\}^{1/p}.
\]

(2.3)

We can similarly define the non-homogeneous Herz space \( K^\alpha_{q,p}(R^n) \).

In 2005, Lu and Xu [7] introduced the Morrey-Herz spaces \( M^\alpha_{p,q}(R^n) \) and \( MK^\alpha_{p,q}(R^n) \). In 2009, author [9] introduced the following new weighted Morrey-Herz spaces:

**DEFINITION 2.2.** Let \( \alpha \in \mathbb{R}^1 \), \( 0 < p \leq \infty \), \( 0 < q < \infty \), \( s \geq 0 \) and \( \omega_1 \) and \( \omega_2 \) be non-negative weight functions. The homogeneous weighted Morrey-herz space \( M^\alpha_{p,q}(\omega_1, \omega_2) \) is defined by

\[
M^\alpha_{p,q}(\omega_1, \omega_2) = \left\{ f \in L^q_{\text{loc}}(R^n - \{0\}, \omega_2) : \| f \|_{M^\alpha_{p,q}(\omega_1, \omega_2)} < \infty \right\},
\]

(2.4)

where

\[
\| f \|_{M^\alpha_{p,q}(\omega_1, \omega_2)} = \sup_{\omega_1(B_k)} \left\{ \sum_{k=0}^{k_0} \omega_1(B_k) \| f \omega_1 \|_{q, \omega_2} \right\}^{1/p}.
\]

(2.5)

We can similarly define the non-homogeneous weighted Morrey-Herz spaces \( MK^\alpha_{p,q}(\omega_1, \omega_2) \). It is easy to see that when \( \omega_1 = \omega_2 = 1 \), we have \( M^\alpha_{p,q}(1, 1) = MK^\alpha_{p,q}(R^n) \). \( \hat{K}^\alpha_{p,q}(R^n) = \hat{K}^\alpha_{p,q}(R^n), M^\alpha_{q}(R^n) \subset MK^\alpha_{q,p}(R^n) \), \( \hat{K}^\alpha_{p,q}(R^n) = L^p(\| x \| dx) \), \( \hat{K}^\alpha_{p,q}(R^n) = L^p(\mathbb{R}^n) \).
DEFINITION 2.3. [10] A non-negative weight function \( \omega \) satisfies Muckenhoupt’s \( A_\infty \) condition or \( \omega \in A_\infty \), if there is a constant \( C \) independent of the cube \( Q \) in \( \mathbb{R}^n \), such that
\[
\left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \exp \left\{ \frac{1}{|Q|} \int_Q \ln \left( \frac{1}{\omega(x)} \right) \, dx \right\} \leq C, \text{ all } Q \subset \mathbb{R}^n,
\]
where \( |Q| \) is the Lebesgue measure of \( Q \).

Our main results are the following two Theorems:

**THEOREM 2.1.** Let \( \alpha \in \mathbb{R}^1 \), \( 0 < p < \infty \), \( \lambda, s > 0 \) and \( 1 \leq q < \infty \), \( \omega_1 \in A_\infty \), a non-negative weight function \( \omega_2 \) satisfies
\[
\omega_2(tx) = t^\beta \omega_2(x), \quad t > 0, \quad \beta \in \mathbb{R}^1,
\]
\( \omega_3(x) = x^{(1-\lambda)q} \omega_2(x) \). Let \( K(x,y) \) be a non negative measurable function on \( (0,\infty) \times (0,\infty) \) and satisfying:

1. \( K(tx,ty) = t^{-\lambda}K(x,y) \), for all \( t > 0 \);
2. \( K(1,t) \) has compact support on \( (0,\infty) \). Let \( \|T\| \) be the norm of the operator \( T \) is defined by (1.3):
\[
MKF^{\alpha,s}_{p,q}(\omega_1,\omega_3) \rightarrow MKF^{\alpha,s}_{p,q}(\omega_1,\omega_2).
\]

(1) If \( t^{\lambda - 1 - (\beta+1)/q}K(1,t) \) is a concave function on \( (0,\infty) \), and \( \int_0^\infty t^{\lambda - 1 + (s-\alpha)\delta - (\beta+1)/q}K(1,t)dt < \infty \), then
\[
\|T\| \leq C(p,\alpha,\lambda) \int_0^\infty t^{\lambda - 1 + (s-\alpha)\delta - (\beta+1)/q}K(1,t)dt,
\]
where
\[
C(p,\alpha,\lambda) = \begin{cases} 
C_0^{(\alpha-s)2(1/p)-2(1+p)^{1/p}(1+2|s-\alpha|\delta)}, & 0 < p < 1, \\
C_0^{(\alpha-s)2-2(1/p)(1+(1/p))(1+2|s-\alpha|\delta)}, & 1 \leq p < \infty.
\end{cases}
\]

(\( C_0 \) and \( \delta \) are the constants given in (3.5), see § 3 below).

(2) If \( \|T\| < \infty \), then
\[
\int_0^\infty t^{\lambda - 1 + (s-\alpha)\delta - (\beta+1)/q}K(1,t)dt \leq \|T\|.
\]

REMARK 1. \( \omega_2 \) is an extension of the power weight \( \omega_2(x) = |x|^\beta, \ (x \in \mathbb{R}^n) \). We use the following notation
\[
MKF = \{ f \in MKF^{\alpha,s}_{p,q}(\omega_1,\omega_3) : F(t) = \sup_{x \in (0,\infty)} |f(tx)|K(1,t) \text{ is a concave function on } (0,\infty) \}. \]

Then \( MKF \) is a subspace of the space \( MKF^{\alpha,s}_{p,q}(\omega_1,\omega_3) \).
THEOREM 2.2. Let $\alpha \in \mathbb{R}^1$, $0 < p < \infty$, $\lambda, s > 0$ and $0 < q < 1$, $\omega_1$, $\omega_2$, $\omega_3$ and $K(x, y)$ are as in Theorem 2.1. Let $\|T\|$ be the norm of the operator $T$ is defined by (1.3): $MKF \to M\hat{K}^{\alpha_s}_p(q)(\omega_1, \omega_2)$.

(1) If $t^{\lambda - 1 - (\beta + 1)/q}K(1, t)$ is a concave function on $(0, \infty)$, and $\int_0^\infty t^{\lambda - 1 + (s - \alpha)\delta - (\beta + 1)/q}K(1, t)dt < \infty$, then

$$\|T\| \leq C(p, q, \alpha, \lambda) \int_0^\infty t^{\lambda - 1 + (s - \alpha)\delta - (\beta + 1)/q}K(1, t)dt, \quad (2.10)$$

where

$$C(p, q, \alpha, \lambda) = \begin{cases} 
C_0^{(\alpha-s)}2^{(1/p)-(1/q)-2}q^{-1/p}(p+q)^{1/p}(1+q)^{1/q}(1+2|s-\alpha|\delta), & 0 < p \leq q < 1, \\
C_0^{(\alpha-s)}2^{(1/q)-2(1+q)^{1/q}(1+2|s-\alpha|\delta)}, & 0 < q \leq p < 1, \\
C_0^{(\alpha-s)}2^{(1/q)-(2/p)-1(1+(1/p))(1+q)^{1/q}(1+2|s-\alpha|\delta)}, & 0 < q < 1 \leq p < \infty,
\end{cases} \quad (2.11)$$

($C_0$ and $\delta$ are the constants given in (3.5), see § 3 below.)

(2) If $\|T\| < \infty$, then

$$\int_0^\infty t^{\lambda - 1 + (s - \alpha)\delta - (\beta + 1)/q}K(1, t)dt \leq \|T\|. \quad (2.12)$$

REMARK 2. If $\omega_1 = \omega_2 = 1$, $\omega(x) = \omega_3(x) = x^{(1-\lambda)q}$. (So, $C_0 = 1$, $\delta = 1$), and $s = 0$, then $M\hat{K}^{\alpha_0}_p(1, 1) = \hat{K}^{\alpha_0}_q(R^n)$, $MK^{\alpha_0}_p(1, \omega_3) = \hat{K}^{\alpha_0}_q(\omega)$, and so Theorem 2.1 reduces to Corollary 2 in [8]. Hence, our main results are significant generalization of [8].

REMARK 3. There are some similar results for the non-homogeneous weighted Morrey-Herz spaces. We omit the details here.

3. Proofs of Theorems

We require the following Lemmas to prove our results.

**LEMMA 3.1.** Let $f$ be a nonnegative measurable function on $[0, b]$. If $1 \leq p < \infty$, then

$$\left( \int_0^b f(x)dx \right)^p \leq b^{(p-1)} \int_0^b f^p(x)dx \quad (3.1)$$

Lemma 1 is an immediate consequence of Hölder inequality.

**LEMMA 3.2.** ($C_p$ inequality [3]) Let $a_1, a_2, \ldots, a_n$ be arbitrary real (or complex) numbers, then for $p > 0$,

$$\left( \sum_{k=1}^n |a_k|^p \right)^p \leq C_p \left( \sum_{k=1}^n |a_k|^p \right), \quad (3.2)$$
where

\[ C_p = \begin{cases} 
1, & 0 < p < 1, \\
np^{-1}, & 1 \leq p < \infty.
\end{cases} \]

**Lemma 3.3.** ([2]) Let \( f \) be a nonnegative measurable and concave function on \([a, b]\), \( 0 < \alpha \leq \beta < \infty \), then

\[
\left\{ \frac{\beta + 1}{b - a} \int_a^b [f(x)]^\beta \, dx \right\}^{\frac{1}{p}} \leq \left\{ \frac{\alpha + 1}{b - a} \int_a^b [f(x)]^\alpha \, dx \right\}^{\frac{1}{q}}.
\] (3.3)

Setting \( a = 0 \), and for \( \alpha = p \), \( \beta = 1 \), that is \( 0 < p \leq 1 \), we obtain from (3.3) that

\[
\left( \int_0^b f(x) \, dx \right)^p \leq \frac{p + 1}{2p} \times b^{p-1} \int_0^b f^p(x) \, dx.
\] (3.4)

By the properties of \( A_\infty \) weights, we have

**Lemma 3.4.** ([10]) If \( \omega \in A_\infty \), then there exist \( \delta_1 \), \( \delta_2 > 0 \), \( C_1 \), \( C_2 > 0 \), such that for each ball \( B \) in \( \mathbb{R}^n \) and measurable subset \( E \) of \( B \),

\[
\frac{\omega(E)}{\omega(B)} \leq C_1 \left( \frac{|E|}{|B|} \right)^{\delta_1} \quad \text{and} \quad \left( \frac{|E|}{|B|} \right)^{\delta_2} \leq C_2 \frac{\omega(E)}{\omega(B)}
\] (3.5)

where \( |E| \) is the Lebesgue measure of \( E \) and \( \omega(E) = \int_E \omega(x) \, dx \). We set \( C_0 = \max \{ C_1, C_2 \} \), \( \delta = \min \{ \delta_1, \delta_2 \} \).

In what follows, we shall write simply \( M\dot{K}^\alpha_s(\omega_1, \omega_2) \) and \( M\dot{K}^\alpha_s(\omega_1, \omega_3) \) to denote \( MK(2) \) and \( MK(3) \), respectively.

**Proof of Theorem 2.1.** Since \( K(1, t) \) has compact support on \((0, \infty)\), there exists \( b > 0 \), such that \( \text{supp} K(1, t) \subset [0, b] \). First, we prove (2.7). Using Minkowski’s inequality for integrals and (2.6), and setting \( u = tx \), we obtain

\[
\| (T^* f) \varphi_k \|_{q, \omega_2} \leq \int_0^b \left\{ \int_{D_k} |f(tx)|^q x^{(1-\lambda)q} \omega_2(x) \, dx \right\}^{1/q} K(1, t) \, dt \\
= \int_0^b \left\{ \int_{2^{-k}t < |u| < 2^k t} |f(u)|^q \omega_3(u) \, du \right\}^{1/q} t^{\lambda - 1 - (\beta + 1)/q} K(1, t) \, dt.
\]

For each \( t \in (0, \infty) \), there exists an integer \( m \) such that \( 2^{m-1} < t \leq 2^m \). Setting

\[
A_{k,m} = \{ u \in (0, \infty) : 2^{k+m-1} < |u| \leq 2^{k+m} \},
\]
By (3.2), we obtain

\[
\|(Tf)\varphi_k\|_{q,\omega_2} \leq \int_0^b \left\{ \int_{A_{k,m}} |f(u)|^q \omega_2(u)du + \int_{A_{k,m}} |f(u)|^q \omega_3(u)du \right\}^{1/q} t^{\lambda-1-(\beta+1)/q} K(1,t)dt
\]

(3.6)

It follows that

\[
\|Tf\|_{MK(2)} \leq \sup_{k_0 \in \mathbb{Z}} [\omega_1(B_{k_0})]^{-s} \left\{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p} \times \left[ \int_0^b \left( \|f\varphi_{k+m-1}\|_{q,\omega_3} + \|f\varphi_{k+m}\|_{q,\omega_3} \right) t^{\lambda-1-(\beta+1)/q} K(1,t)dt \right]^{p/1/p} \right\}^{1/p}. (3.7)
\]

Now, we consider two cases for \(p\):

**Case 1.** \(0 < p < 1\). In this case, it follows from (3.7), (3.4) and (3.2) that

\[
\|Tf\|_{MK(2)} \leq \frac{(1+p)^{1/p}}{2 \times b^{\frac{1}{p}-1}} \sup_{k_0 \in \mathbb{Z}} [\omega_1(B_{k_0})]^{-s} \left\{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p} \times \left[ \int_0^b \left( \|f\varphi_{k+m-1}\|_{q,\omega_3} + \|f\varphi_{k+m}\|_{q,\omega_3} \right) t^{\lambda-1-(\beta+1)/q} K(1,t)dt \right]^{p/1/p} \right\}^{1/p}. (3.8)
\]

By (3.5) and \(|B_k| = 2^{k+1}\), we have

\[
\frac{\omega_1(B_k)}{\omega_1(B_{k+m-1})} \leq C_0 \left( \frac{|B_k|}{|B_{k+m-1}|} \right)^\delta = C_0 2^{-(m-1)\delta}, \quad \frac{\omega_1(B_k)}{\omega_1(B_{k+m})} \leq C_0 2^{-m\delta}. (3.9)
\]
It follows from (3.8), (3.9) and (3.1) that

\[
\|Tf\|_{MK(2)} \leq C_0^{(s-s)} 2^{(1/p)-2}(1 + p)^{1/p} \|f\|_{MK(3)} \int_0^b (2^{\gamma - (1 - (s) \delta)) + 2m(s - \alpha) \delta t^{\lambda - 1 + (1 - (s) \delta)} dt
\]

\[
\leq C_0^{(s-s)} 2^{(1/p)-2}(1 + p)^{1/p} (1 + 2^{\alpha - s}) \|f\|_{MK(3)} \times \int_0^\infty t^{\lambda - 1 + (s) \delta - (1 + (1 - (s) \delta)) dt.}
\]

\[(3.10)\]

**Case 2.** \(1 \leq p < \infty\). In this case, it follows from (3.7), (3.1), (3.2), (3.4) and (3.9) that

\[
\|Tf\|_{MK(2)} \leq (2b)^{1-1/(p)} \sup_{k_0 \in Z} [\omega_1(B_k)]^{-s} \left\{\sum_{k=1}^{k_0} [\omega_1(B_k)]^{\alpha p} \times \left(\int_0^b (\|f\|_{MK(3)} \|\|f\|_{q,\omega_3} + \|f\|_{MK(3)} \|p\|_{q,\omega_3} f^{(\lambda - 1 - (1 - (s) \delta))} dt\right)^{1/p} \right\}
\]

\[
\leq (2b)^{1-1/(p)} \sup_{k_0 \in Z} [\omega_1(B_k)]^{-s} \left\{\left[\int_0^b \sum_{k=1}^{k_0} [\omega_1(B_k)]^{\alpha p} \|f\|_{MK(3)} \|\|f\|_{q,\omega_3} \times \left(\left(\frac{\omega_1(B_k)}{\omega_1(B_{k+1-m})}\right)^{\alpha p} t^{(\lambda - 1 + (1 - (s) \delta))} dt\right)^{1/p} \right\}
\]

\[
\leq C_0^{(s-s)} 2^{1-1/(p)} (1 + (1/p)) \|f\|_{MK(3)}
\]

\[
\times \left(\int_0^b (2^{\gamma - (1 - (s) \delta)) + 2m(s - \alpha) \delta t^{\lambda - 1 + (1 - (s) \delta)} dt\right)
\]

\[
\leq C_0^{(s-s)} 2^{1-1/(p)} (1 + (1/p)) (1 + 2^{\alpha - s}) \|f\|_{MK(3)} \times \int_0^\infty t^{\lambda - 1 + (s) \delta - (1 + (1 - (s) \delta)) dt.}
\]

\[(3.11)\]

Hence, by (3.10) and (3.11), we get

\[
\|T\| \leq C(p, \alpha, \lambda) \int_0^\infty t^{\lambda - 1 + (s) \delta - (1 + (1 - (s) \delta)) dt.}
\]

\[(3.12)\]

where \(C(p, \alpha, \lambda)\) is defined by (2.8).

To prove the opposite inequality (2.9), we take \(\omega_1(B_k) = 2^k\), \(\omega_2(x) = x^{\beta}\), \(x \in (0, \infty)\) and

\[
f_0(x) = x^{\lambda - 1 + (s) \delta - (1 + (1 - (s) \delta))}, \quad x \in (0, \infty).
\]
We need to consider two cases:

**Case 1.** $\alpha \neq s$. Then
\[
\|f_0\|_{Q,MK(3)} = \int_{2^{k-1}}^{2^k} x^{-(1+\beta)+s(\lambda-1)+((1-\lambda)q)+\beta} dx \quad = C_3^{1/q} 2^{k(s-\alpha)} \delta,
\]

where
\[
C_3 = \left| \frac{1 - 2(\alpha-s)q}{(s-\alpha)q\delta} \right|.
\]

It follows that
\[
\|f_0\|_{MK(3)} = \sup_{k_0 \in \mathbb{Z}} \left\{ \left( \int_{2^{k_0}}^{2} x^{-1} dx \right)^{1/q} \right\}^{1/p} = C_3^{1/q} (2pn\delta - 1)^{-\frac{1}{p}}. \quad (3.13)
\]

**Case 2.** $\alpha = s$. Then
\[
\|f_0\|_{MK(3)} = \left( \int_{2^{k-1}}^{2^k} x^{-1} dx \right)^{1/q} = (\ln 2)^{1/q}. \quad (3.14)
\]

It follows from (3.13) and (3.14) that $f_0 \in MK(3)$. By (1.3), we obtain
\[
T(f_0, x) = \chi^{(s-\alpha)\delta/(\beta+1)/q} \int_0^\infty t^{\lambda-1+(s-\alpha)\delta/(\beta+1)/q} K(1,t) dt
\]
and
\[
\|Tf_0\|_{MK(2)} = \|f_0\|_{MK(3)} \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q} K(1,t) dt.
\]

Thus,
\[
\|T\| \geq \frac{\|Tf_0\|_{MK(2)}}{\|f_0\|_{MK(3)}} = \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q} K(1,t) dt.
\]

This completes the proof of Theorem 2.1. \hfill \Box

The idea of proof of Theorem 2.2 is similar to that of Theorem 2.1, we omit the details here.

**REFERENCES**


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