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GENERALIZED HILBERT OPERATORS ON WEIGHTED MORREY–HERZ SPACES

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Abstract. This paper give some necessary and sufficient conditions for the generalized Hilbert operators to be bounded on the weighted Morrey-Herz spaces. The corresponding new operator norm inequalities are obtained.

1. Introduction

Considerable attention has been given to the classical Hilbert operator T_0 defined by

$$T_0(f,x) = \int_0^\infty \frac{f(y)}{x+y}dy. \tag{1.1}$$

and the classical Hilbert inequality

$$\|T_0 f\|_p \leqslant \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_p, \ \text{ for } 0<p<\infty, \tag{1.2}$$

where $\|f\|_p = (\int_0^\infty |f(x)|^p dx)^{\frac{1}{p}}$ and the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best value [1]. In view of the mathematical importance and applications, considerable attention has also been given to various improvements, refinements and extensions of many inequalities by various authors (see e.g. [3, 4, 11] and the references cited therein). However, hardly any work was done on inequalities on the Morrey spaces and Herz spaces. It is well-known that the Herz spaces play an important role in characterizing the properties of functions and multipliers on the classical Hardy spaces, and the Morrey spaces have important applications in the theory of partial differential equations, in linear as well as in non-linear theory (see [5, 6]). In 2008, the author [8] obtained some new inequalities related to the generalized Hilbert operator

$$T(f,x) = \int_0^\infty K(x,y)\times f(y)dy \tag{1.3}$$

with the general kernel $K(x,y)$ on the Herz spaces. In 2009, the author [9] introduced the new weighted Morrey-Herz spaces. The aim of this paper is to generalize these results of the author [8] to the weighted Morrey-Herz spaces. We obtain some necessary and sufficient conditions for the generalized Hilbert operator T to be bounded on these spaces. The corresponding new operator norm inequalities are obtained.

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2. Definitions and statement of the main results

Let $k \in Z$, $B_k = \{x \in \mathbf{R}^n : |x| \leqslant 2^k\}$, $D_k = B_k - B_{k-1}$ and let $\varphi_k = \varphi_{D_k}$ denotes the characteristic function of the set D_k. Moreover, for a measurable function f on $\mathbf{R}^n$ and a non-negative weighted function $\omega(x)$, we write

$$\|f\|_{p,\omega} = \left(\int_{R^n} |f(x)|^p \omega(x)dx\right)^{1/p}.$$

In what follows, if $\omega \equiv 1$, then we will denote $L^p(R^n,\omega)$ (in brief $L^p(\omega)$) by $L^p(R^n)$.

DEFINITION 2.1. Let $\alpha \in \mathbf{R}^1$, $0 < p,q < \infty$ and $s \geqslant 0$. The Morrey spaces $M_q^s(R^n)$ is defined by [5] as follows:

$$M_q^s(R^n) = \left\{ f \in L_{loc}^q(R^n) : \sup_{r>0,x\in R^n} \frac{1}{r^s} \int_{|x-y|<r} |f(y)|^q dy < \infty \right\}, \tag{2.1}$$

and the homogeneous Herz space $\dot{K}_q^{\alpha,p}(R^n)$ is defined by [6] as follows:

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n - \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(R^n)} < \infty\}, \tag{2.2}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(R^n)} = \{\sum_{k\in Z} 2^{k\alpha p} \|f\varphi_k\|_q^p\}^{1/p}. \tag{2.3}$$

We can similarly define the non-homogeneous Herz space $K_q^{\alpha,p}(R^n)$.

In 2005, Lu and Xu [7] introduced the Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,s}(R^n)$ and $MK_{p,q}^{\alpha,s}(R^n)$. In 2009, author [9] introduced the following new weighted Morrey-Herz spaces:

DEFINITION 2.2. Let $\alpha \in \mathbf{R}^1$, $0 < p \leqslant \infty$, $0 < q < \infty$, $s \geqslant 0$ and ω_1 and ω_2 be non-negative weight functions. The homogeneous weighted Morrey-herz space $M\dot{K}_{p,q}^{\alpha,s}(\omega_1,\omega_2)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,s}(\omega_1,\omega_2) = \{f \in L_{loc}^q(R^n - \{0\},\omega_2) : \|f\|_{M\dot{K}_{p,q}^{\alpha,s}(\omega_1,\omega_2)} < \infty\}, \tag{2.4}$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,s}(\omega_1,\omega_2)} = \sup_{k_0\in Z} \omega_1(B_{k_0})^{-(\frac{s}{n})}\{\sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \|f\varphi_k\|_{q,\omega_2}^p\}^{1/p}. \tag{2.5}$$

We can similarly define the non-homogeneous weighted Morrey-Herz spaces $MK_{p,q}^{\alpha,s}(\omega_1,\omega_2)$. It is easy to see that when $\omega_1 = \omega_2 = 1$, we have $M\dot{K}_{p,q}^{\alpha,s}(1,1) = M\dot{K}_{p,q}^{\alpha,s}(R^n)$. $M\dot{K}_{p,q}^{\alpha,0}(R^n) = \dot{K}_q^{\alpha,p}(R^n)$, $M_q^s(R^n) \subset M\dot{K}_{q,q}^{0,s}(R^n)$, $\dot{K}_p^{(\alpha/p),p}(R^n) = L^p(|x|^\alpha dx)$, $\dot{K}_p^{0,p}(R^n) = L^p(R^n)$.

DEFINITION 2.3. [10] A non-negative weight function ω satisfies Muckenhoupt's A_∞ condition or $\omega \in A_\infty$, if there is a constant C independent of the cube Q in $\mathbf{R}^n$, such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\exp\left\{\frac{1}{|Q|}\int_Q \ln\left(\frac{1}{\omega(x)}\right)dx\right\} \leqslant C, \text{ all } Q\subset \mathbf{R}^n,$$

where $|Q|$ is the Lebesgue measure of Q.

Our main results are the following two Theorems:

THEOREM 2.1. *Let* $\alpha \in \mathbf{R}^1$, $0<p<\infty$, $\lambda, s>0$ *and* $1\leqslant q<\infty$, $\omega_1\in A_\infty$, *a non-negative weight function* ω_2 *satisfies*

$$\omega_2(tx) = t^{\beta}\omega_2(x), \ \ t>0, \ \ \beta\in\mathbf{R}^1, \tag{2.6}$$

$\omega_3(x) = x^{(1-\lambda)q}\omega_2(x)$. *Let* $K(x,y)$ *be a non negative measurable function on* $(0,\infty)\times(0,\infty)$ *and satisfying:*

(1) $K(tx,ty) = t^{-\lambda}K(x,y)$, *for all* $t>0$*;*

(2) $K(1,t)$ *has compact support on* $(0,\infty)$. *Let* $\|T\|$ *be the norm of the operator* T *is defined by (1.3):*

$$M\dot{K}^{\alpha,s}_{p,q}(\omega_1,\omega_3)\to M\dot{K}^{\alpha,s}_{p,q}(\omega_1,\omega_2).$$

(1) If $t^{\lambda-1-(\beta+1)/q}K(1,t)$ *is a concave function on* $(0,\infty)$, *and* $\int_0^\infty t^{(\lambda-1+(s-\alpha)\delta-(\beta+1)/q)}K(1,t)dt<\infty$, *then*

$$\|T\|\leqslant C(p,\alpha,\lambda)\int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt, \tag{2.7}$$

where

$$C(p,\alpha,\lambda) = \begin{cases} C_0^{(\alpha-s)}2^{(1/p)-2}(1+p)^{1/p}(1+2^{|s-\alpha|\delta}), & 0<p<1,\\ C_0^{(\alpha-s)}2^{1-(2/p)}(1+(1/p))(1+2^{|s-\alpha|\delta}), & 1\leqslant p<\infty. \end{cases} \tag{2.8}$$

(C_0 *and* δ *are the constants given in (3.5), see § 3 below).*

(2) If $\|T\|<\infty$, *then*

$$\int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt\leqslant \|T\|. \tag{2.9}$$

REMARK 1. ω_2 is an extension of the power weight $\omega_2(x)=|x|^{\beta}$, $(x\in\mathbf{R}^n)$. We use the following notation

$$MKF = \{f\in M\dot{K}^{\alpha,s}_{p,q}(\omega_1,\omega_3) : F(t) = \sup_{x\in(0,\infty)}|f(tx)|K(1,t) \text{ is a concave function on } (0,\infty)\}.$$

Then MKF is a subspace of the space $M\dot{K}^{\alpha,s)}_{p,q}(\omega_1,\omega_3)$.

THEOREM 2.2. *Let* $\alpha \in \mathbf{R}^1$, $0 < p < \infty$, $\lambda, s > 0$ *and* $0 < q < 1$, ω_1, ω_2, ω_3 *and* $K(x,y)$ *are as in Theorem 2.1.Let* $\|T\|$ *be the norm of the operator* T *is defined by (1.3):* $MKF \to M\dot{K}_{p,q}^{\alpha,s}(\omega_1,\omega_2)$.
(1) If $t^{\lambda-1-(\beta+1)/q}K(1,t)$ *is a concave function on* $(0,\infty)$, *and* $\int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}$ $K(1,t)dt < \infty$, *then*

$$\|T\| \leqslant C(p,q,\alpha,\lambda)\int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt, \tag{2.10}$$

where

$$C(p,q,\alpha,\lambda) = \begin{cases} C_0^{(\alpha-s)}2^{(1/p)-(1/q)-2}q^{-1/p}(p+q)^{1/p}(1+q)^{1/q}(1+2^{|s-\alpha)|\delta}), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < p \leqslant q < 1, \\ C_0^{(\alpha-s)}2^{(1/q)-2}(1+q)^{1/q}(1+2^{|s-\alpha|\delta}), \quad 0 < q \leqslant p < 1, \\ C_0^{(\alpha-s)}2^{(1/q)-(2/p)-1}(1+(1/p))(1+q)^{1/q}(1+2^{|s-\alpha|\delta}), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < q < 1 \leqslant p < \infty, \end{cases} \tag{2.11}$$

(C_0 and δ are the constants given in (3.5), see § 3 below.)
(2) If $\|T\| < \infty$, *then*

$$\int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt \leqslant \|T\|. \tag{2.12}$$

REMARK 2. If $\omega_1 = \omega_2 = 1$, $\omega(x) = \omega_3(x) = x^{(1-\lambda)q}$. (So, $C_0 = 1, \delta = 1$), and $s = 0$, then $M\dot{K}_{p,q}^{\alpha,0}(1,1) = \dot{K}_q^{\alpha,p}(R^n)$, $M\dot{K}_{p,q}^{\alpha,0}(1,\omega_3) = \dot{K}_q^{\alpha,p}(\omega)$, and so Theorem 2.1 reduces to Corollary 2 in [8]. Hence, our main results are significant generalization of [8].

REMARK 3. There are some similar results for the non-homogeneous weighted Morrey-Herz spaces. We omit the details here.

3. Proofs of Theorems

We require the following Lemmas to prove our results.

LEMMA 3.1. *Let* f *be a nonnegative measurable function on* $[0,b]$. *If* $1 \leqslant p < \infty$, *then*

$$\left(\int_0^b f(x)dx\right)^p \leqslant b^{(p-1)}\int_0^b f^p(x)dx \tag{3.1}$$

Lemma 1 is an immediate consequence of Hölder inequality.

LEMMA 3.2. (C_p inequality [3]) *Let* $a_1, a_2, \ldots, a_n$ *be arbitrary real (or complex)numbers, then for* $p > 0$,

$$\left(\sum_{k=1}^n |a_k|\right)^p \leqslant C_p\left(\sum_{k=1}^n |a_k|^p\right), \tag{3.2}$$

where

$$C_p = \begin{cases} 1, & 0 < p < 1, \\ n^{p-1}, & 1 \leqslant p < \infty. \end{cases}$$

LEMMA 3.3. ([2]) *Let f be a nonnegative measurable and concave function on $[a,b]$, $0 < \alpha \leqslant \beta < \infty$, then*

$$\left\{ \frac{\beta+1}{b-a} \int_a^b [f(x)]^\beta dx \right\}^{\frac{1}{\beta}} \leqslant \left\{ \frac{\alpha+1}{b-a} \int_a^b [f(x)]^\alpha dx \right\}^{\frac{1}{\alpha}}. \tag{3.3}$$

Setting $a = 0$, and for $\alpha = p$, $\beta = 1$, that is $0 < p \leqslant 1$, we obtain from (3.3) that

$$\left(\int_0^b f(x)dx \right)^p \leqslant \frac{p+1}{2^p} \times b^{p-1} \int_0^b f^p(x)dx. \tag{3.4}$$

By the properties of A_∞ weights, we have

LEMMA 3.4. ([10]) *If $\omega \in A_\infty$, then there exist δ_1, $\delta_2 > 0$, C_1, $C_2 > 0$, such that for each ball $\mathbf{B}$ in $\mathbf{R}^n$ and measurable subset E of $\mathbf{B}$,*

$$\frac{\omega(E)}{\omega(B)} \leqslant C_1 \left(\frac{|E|}{|B|} \right)^{\delta_1} \quad \text{and} \quad \left(\frac{|E|}{|B|} \right)^{\delta_2} \leqslant C_2 \frac{\omega(E)}{\omega(B)} \tag{3.5}$$

where $|E|$ is the Lebesgue measure of E and $\omega(E) = \int_E \omega(x)dx$. We set $C_0 = \max\{C_1, C_2\}$, $\delta = \min\{\delta_1, \delta_2\}$

In what follows, we shall write simply $M\dot{K}^{\alpha,s}_{p,q}(\omega_1,\omega_2) and M\dot{K}^{\alpha,s}_{p,q}(\omega_1,\omega_3)$ to denote $MK(2)$ and $MK(3)$, respectively.

Proof of Theorem 2.1. Since $K(1,t)$ has compact support on $(0,\infty)$, there exists $b > 0$, such that $\operatorname{supp} K(1,t) \subset [0,b]$. First, we prove (2.7). Using Minkowski's inequality for integrals and (2.6), and setting $u = tx$, we obtain

$$\begin{aligned} \|(Tf)\varphi_k\|_{q,\omega_2} &\leqslant \int_0^b \left\{ \int_{D_k} |f(tx)|^q x^{(1-\lambda)q} \omega_2(x)dx \right\}^{1/q} K(1,t)dt \\ &= \int_0^b \left\{ \int_{2^{k-1}t < |u| \leqslant 2^k t} |f(u)|^q \omega_3(u)du \right\}^{1/q} t^{\lambda-1-(\beta+1)/q} K(1,t)dt. \end{aligned}$$

For each $t \in (0,\infty)$, there exists an integer m such that $2^{m-1} < t \leqslant 2^m$. Setting

$$A_{k,m} = \{u \in (0,\infty) : 2^{k+m-1} < |u| \leqslant 2^{k+m}\},$$

By (3.2), we obtain

$$
\begin{aligned}
\|(Tf)\varphi_k\|_{q,\omega_2} &\leqslant \int_0^b \{\int_{A_{(k-1),m}} |f(u)|^q \omega_3(u)du \\
&\quad + \int_{A_{k,m}} |f(u)|^q \omega_3(u)du\}^{1/q} t^{\lambda-1-(\beta+1)/q} K(1,t)dt \\
&\leqslant \int_0^b (\|f\varphi_{k+m-1}\|_{q,\omega_3} + \|f\varphi_{k+m}\|_{q,\omega_3}) t^{\lambda-1-(\beta+1)/q} K(1,t)dt.
\end{aligned} \tag{3.6}
$$

It follows that

$$
\begin{aligned}
\|Tf\|_{MK(2)} &\leqslant \sup_{k_0\in Z} [\omega_1(B_{k_0})]^{-s} \Big\{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p} \\
&\quad \times \Big[\int_0^b (\|f\varphi_{k+m-1}\|_{q,\omega_3} + \|f\varphi_{k+m}\|_{q,\omega_3}) t^{\lambda-1-(\beta+1)/q} K(1,t)dt \Big]^p \Big\}^{1/p}.
\end{aligned} \tag{3.7}
$$

Now, we consider two cases for p:

Case 1. $0 < p < 1$. In this case, it follows from (3.7), (3.4) and (3.2) that

$$
\begin{aligned}
\|Tf\|_{MK(2)} &\leqslant \frac{(1+p)^{1/p}}{2 \times b^{\frac{1}{p}-1}} \sup_{k_0\in Z} [\omega_1(B_{k_0})]^{-s} \Big\{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p} \\
&\quad \times \int_0^b (\|f\varphi_{k+m-1}\|^p_{q,\omega_3} + \|f\varphi_{k+m}\|^p_{q,\omega_3}) t^{(\lambda-1-(\beta+1)/q)p} K^p(1,t)dt \Big\}^{1/p} \\
&\leqslant 2^{(1/p)-2}(1+p)^{1/p} b^{1-(\frac{1}{p})} \sup_{k_0\in Z} [\omega_1(B_{k_0})]^{-s} \\
&\quad \times \Big\{ \Big[\int_0^b \sum_{k=-\infty}^{k_0} \omega_1(B_{k+m-1})^{\alpha p} \|f\varphi_{k+m-1}\|^p_{q,\omega_3} \times \Big(\frac{\omega_1(B_k)}{\omega_1(B_{k+m-1})} \Big)^{\alpha p} \\
&\quad \times t^{(\lambda-1-(\beta+1)/q)p} K^p K(1,t)dt \Big]^{1/p} + \Big[\int_0^b \sum_{k=-\infty}^{k_0} \omega_1(B_{k+m})^{\alpha p} \|f\varphi_{k+m}\|^p_{q,\omega_3} \\
&\quad \times \Big(\frac{\omega_1(B_k)}{\omega_1(B_{k+m})} \Big)^{\alpha p} t^{(\lambda-1-(\beta+1)/q)p} K^p(1,t)dt \Big]^{1/p} \Big\}.
\end{aligned} \tag{3.8}
$$

By (3.5) and $|B_k| = 2^{k+1}$, we have

$$
\frac{\omega_1(B_k)}{\omega_1(B_{k+m-1})} \leqslant C_0 \Big(\frac{|B_k|}{|B_{k+m-1}|} \Big)^{\delta} = C_0 2^{-(m-1)\delta}; \quad \frac{\omega_1(B_k)}{\omega_1(B_{k+m})} \leqslant C_0 2^{-m\delta}. \tag{3.9}
$$

It follows from (3.8), (3.9) and (3.1) that

$$\begin{aligned}\|Tf\|_{MK(2)} &\leqslant C_0^{(\alpha-s)} 2^{(1/p)-2}(1+p)^{1/p}\|f\|_{MK(3)}\int_0^b (2^{(m-1)(s-\alpha)\delta} \\ &\quad + 2^{m(s-\alpha)\delta})t^{\lambda-1-(\beta+1)/q}K(1,t)dt \\ &\leqslant C_0^{(\alpha-s)} 2^{(1/p)-2}(1+p)^{1/p}(1+2^{|\alpha-s|\delta})\|f\|_{MK(3)} \\ &\quad \times \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt.\end{aligned} \tag{3.10}$$

Case 2. $1 \leqslant p < \infty$. In this case, it follows from (3.7), (3.1), (3.2), (3.4) and (3.9) that

$$\begin{aligned}\|Tf\|_{MK(2)} &\leqslant (2b)^{1-(1/p)} \sup_{k_0\in Z}[\omega_1(B_{k_0})]^{-s}\Big\{\sum_{k=-\infty}^{k_0}[\omega_1(B_k)]^{\alpha p} \\ &\quad \times \int_o^b (\|f\varphi_{k+m-1}\|_{q,\omega_3}^p + \|f\varphi_{k+m}\|_{q,\omega_3}^p)t^{(\lambda-1-(\beta+1)/q)p}K^p(1,t)dt\Big\}^{1/p} \\ &\leqslant (2b)^{1-(1/p)} \sup_{k_0\in Z}[\omega_1(B_{k_0})]^{-s}\Big\{\Big[\int_0^b \sum_{k=-\infty}^{k_0}[\omega_1(B_{k+m-1}]^{p\alpha}\|f\varphi_{k+m-1}\|_{q,\omega_3}^p \\ &\quad \times \left(\frac{\omega_1(B_k)}{\omega_1(B_{k+m-1})}\right)^{p\alpha} t^{(\lambda-1-(\beta+1)/q)p}K^p(1,t)dt\Big]^{1/p}\Big\} \\ &\quad + \Big[\int_0^b \sum_{k=-\infty}^{k_0}[\omega_1(B_{k+m})]^{p\alpha}\|f\varphi_{k+m}\|_{q,\omega_3}^p \\ &\quad \times \left(\frac{\omega_1(B_k)}{\omega_1(B_{k+m})}\right)^{p\alpha} t^{(\lambda-1-(\beta+1)/q)p}K^p(1,t)dt\Big]^{1/p}\Big\} \\ &\leqslant C_0^{\alpha-s}2^{1-(2/p)}(1+(1/p))\|f\|_{MK(3)} \\ &\quad \times \int_0^b (2^{(m-1)(s-\alpha)\delta} + 2^{m(s-\alpha)\delta})t^{\lambda-1-(\beta+1)/q}K(1,t)dt \\ &\leqslant C_0^{\alpha-s}2^{1-(2/p)}(1+(1/p))(1+2^{|\alpha-s|\delta})\|f\|_{MK(3)} \\ &\quad \times \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt.\end{aligned} \tag{3.11}$$

Hence, by (3.10) and (3.11), we get

$$\|T\| \leqslant C(p,\alpha,\lambda)\int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}K(1,t)dt, \tag{3.12}$$

where $C(p,\alpha,\lambda)$ is defined by (2.8).

To prove the opposite inequality (2.9), we take $\omega_1(B_k) = 2^{k\delta}$, $\omega_2(x) = x^\beta$, $x \in (0,\infty)$ and

$$f_0(x) = x^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q}, \quad x \in (0,\infty).$$

We need to consider two cases:

Case 1. $\alpha \neq s$. Then

$$\|f_0\varphi_k\|_{q,\omega_3} = \int_{2^{k-1}}^{2^k} x^{-(\beta+1)+(s-\alpha)q\delta+(\lambda-1)q+(1-\lambda)q+\beta} dx^{1/q} = C_3^{1/q} 2^{k(s-\alpha)\delta},$$

where

$$C_3 = \left| \frac{1-2^{(\alpha-s))q\delta}}{(s-\alpha)q\delta} \right|.$$

It follows that

$$\begin{aligned} \|f_0\|_{MK(3)} &= \sup_{k_0\in Z} [\omega_1(B_{k_0})]^{-s} \left\{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{p\alpha} \times (C_3^{1/q} 2^{k(s-\alpha)\delta})^p \right\}^{1/p} \\ &= C_3^{1/q} (2^{ps\delta}-1)^{-\frac{1}{p}}. \end{aligned} \tag{3.13}$$

Case 2. $\alpha = s$. Then

$$\|f_0\|_{MK(3)} = \left(\int_{2^{k-1}}^{2^k} x^{-1} dx \right)^{1/q} = (\ln 2)^{1/q}. \tag{3.14}$$

It follows from (3.13) and (3.14) that $f_0 \in MK(3)$. By (1.3), we obtain

$$T(f_0,x) = x^{(s-\alpha)\delta-(\beta+1)/q} \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q} K(1,t)dt$$

and

$$\|Tf_0\|_{MK(2)} = \|f_0\|_{MK(3)} \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q} K(1,t)dt.$$

Thus,

$$\|T\| \geqslant \frac{\|Tf_0\|_{MK(2)}}{\|f_0\|_{MK(3)}} = \int_0^\infty t^{\lambda-1+(s-\alpha)\delta-(\beta+1)/q} K(1,t)dt.$$

This completes the proof of Theorem 2.1. □

The idea of proof of Theorem 2.2 is similar to that of Theorem 2.1, we omit the details here.

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities* $(2^{rd}$ *ed*$)$, Cambridge University Press, London, 1952.

[2] E. F. BECKENBACH AND R. BELLMAN, *Inequalities* (2^{ed}), Springer-Verlag, New York, 1965.

[3] KUANG JICHANG, *Applied Inequalities* $(4^{rd}$ *ed*$)$, Shangdong Science Press, Jinan, 2010. (in Chinese)

[4] YANG BICHENG, *The norm of operator and Hilbert-type inequalities*, Science Press, Beijing, 2009. (in Chinese)

[5] CH. B. JR. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans.Amer. Math.Soc. **43** (1938), 126–166.

[6] C. S. HERZ, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech. **18** (1968/1969), 283–323.

[7] S. Z. LU AND L. F. XU, *Boundedness of rough singular integral operators on the homogeneous Morrey-Herz spaces*, Hokkaido Math. J. **34** (2005), 299–314.

[8] KUANG JICHANG, *Generalized Hilbert integral operators on the Herz spaces*, Tamkang J.Math. **40**, 2 (2009), 193–200.

[9] KUANG JICHANG, *Weighted Morrey-Herz spaces and applications*, Applied Math.E-Notes **10** (2010), 159–166.

[10] J. GARCÍA-CUERVA AND J.-L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Publishing, Amsterdam, 1985.

[11] BICHENG YANG, *Hilbert-type integral inequalities*, Bentham Science Publishers Ltd, 2009.

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